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A DISCRETE STOCHASTIC KOROVKIN THEOREM

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ABSTRACT. In this article we give a sufficient condition for the pointwise -- in the first mean Korovkin property on $B_{\underline{0}}(P)$, the space of stochastic processes with real state space and countable index set P and bounded first moments.

KEY WORDS AND PHRASES. Positive linear operator, stochastic processes, pointwise - in the first mean convergence.
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1. INTRODUCTION.

Let $(\underline{0},A,\tau)$ be a probability space and let P denote a fixed countable set. Consider stochastic processes X with real state space and the expectation operator $E(X)(t) = \int_{\underline{0}} X(t,\omega)\tau(d\omega)$, $t \in P$. Define $B_{\underline{0}}(P) = \{X: \sup_{t \in P} E|X|(t) < \infty\}$. Let T_n: $B_{\underline{0}}(P) \to B_{\underline{0}}(P)$ be any sequence of positive linear operators such that $ET_n = T_nE$, all $n = 1,2,\ldots$. In Theorem 1, under Korovkin type assumptions, we give a sufficient condition such that for each $X \in B_{\underline{0}}(P)$,

$$\lim_{n\to\infty} E[(T_nX)(t,\omega) - X(t,\omega)] = 0, \text{ for each } t \in P.$$

In [3], see Theorem 3.2, was treated the continuous case, that is, when P is an uncountable compact space. There the sufficient condition is similar to ours, however, it is produced under the additional assumption that T_n is a stochastically simple operator.

Our result has as follows:

THEOREM 1. Let $(\underline{0},A,\tau)$ be a probability space and $P = \{t_1,\ldots,t_j,\ldots\}$ be a countable set of cardinality ≥ 2. Consider the space of stochastic processes with real state space

$$B_{\underline{0}}(P) = \{X: \sup_{t \in P} \int_{\underline{0}} |X(t,\omega)|\tau(d\omega) < \infty\}$$

and the space

$$B(P) = \{f: P \to \mathbb{R} \mid \|f\|_\infty < \infty\},$$

where

$$\|f\|_\infty = \sup_{t \in P} |f(t)|; \quad B(P) \subset B_{\underline{0}}(P).$$

Let T_n: $B_{\underline{0}}(P) \to B_{\underline{0}}(P)$ be a sequence of positive linear operators that are E-commutative, i.e.

$$(E(T_nX))(t,\omega) = (T_n(EX))(t,\omega), \quad \text{for all } (t,\omega) \in P \times \underline{0}$$

where

$$(EX)(t) := E(X(t,\omega)) := \int_{\underline{0}} X(t,\omega)\tau(d\omega)$$

is the expectation.

Also assume that $(T_n 1)(t,\omega) = 1$, for all $(t,\omega) \in P \times \underline{0}$. For

$$\{X_1(t,\omega),\ldots,X_k(t,\omega)\} \subset B_{\underline{0}}(P)$$

assume that

$$\lim_{n\to\infty} E[(T_n X_i)(t_j,\omega) - X_i(t_j,\omega)] = 0,$$

for all $t_j \in P$ and all $i = 1,\ldots,k$. (I.e.

$$\lim_{n\to\infty} [(T_n(EX_i))(t_j) - (EX_i)(t_j)] = 0,$$

for all $t_j \in P$ and $i = 1,\ldots,k$.)

In order that

$$\lim_{n\to\infty} E[(T_n X)(t_j,\omega) - X(t_j,\omega)] = 0,$$

for all $t_j \in P$ and all $X \in B_{\underline{0}}(P)$, it is enough to assume that each $t_j \in P$ there are real constants $\beta_1,\ldots,\beta_k$ such that

$$\sum_{i=1}^{k} \beta_i E[X_i(t,\omega) - X_i(t_j,\omega)] \geq 1, \text{ for all } t \in P - \{t_j\}.$$

PROOF. If there exists $X \in B_{\underline{0}}(P)$ and $t_{j_0} \in P$ such that

$$E[(T_n X)(t_{j_0},\omega) - X(t_{j_0},\omega)] \not\to 0,$$

then there exist a subsequence T_{λ_n} and an $\varepsilon > 0$ such that

$$|(E(T_{\lambda_n} X))(t_{j_0}) - (EX)(t_{j_0})| > \varepsilon, \text{ for all } n \geq 1.$$

By E-commutativity of T_{λ_n} we get

$$|(T_{\lambda_n}(EX))(t_{j_0}) - (EX)(t_{j_0})| > \varepsilon, \text{ for all } n \geq 1.$$

Let μ be a positive finite measure on P with $\mu(\{t\}) > 0$, for all $t \in P$. Here $B(P) \subset L_p(P,\mu)$, $1 \leq p < \infty$.

Let $f \in B(P)$, then $E(f) = f$. Hence $T_n(f) = T_n(Ef) = ET_n(f)$ and $T_n(f) \in B(P)$, i.e. T_n maps $B(P)$ into itself. Because each positive linear functional $T_n(\cdot,t_j)$ on $B(P)$ is bounded, by Riesz representation theorem, for the specific $j = j_0$, there exists $g_{t_{j_0},n} \in L_q(P,\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$(T_n(f))(t_{j_0}) = \int_P f(t) g_{t_{j_0},n}(t)\mu(dt), \text{ for all } f \in B(P).$$

By $T_n(1) = 1$ and the positivity of $T_n(\cdot,t_{j_0})$ one obtains

$$\int_P g_{t_{j_0},n}(t)\mu(dt) = 1 \quad \text{and} \quad g_{t_{j_0},n}(t) \geq 0, \text{ for all } t \in P.$$

Since $EX \in B(P)$, we have

$$(T_{\lambda_n}(EX))(t_{j_0}) = \int_P (EX)(t)\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt).$$

Thus

$$\varepsilon < |(T_{\lambda_n}(EX))(t_{j_0}) - (EX)(t_{j_0})| = \Big|\int_P (EX)(t)\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt) - \int_P (EX)(t_{j_0})\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt)\Big|$$

$$= \Big|\int_{P-\{t_{j_0}\}} [(EX)(t) - (EX)(t_{j_0})]\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt)\Big|$$

$$\leq ||EX - (EX)(t_{j_0})||_\infty \cdot \Big(\int_{P-\{t_{j_0}\}} g_{t_{j_0},\lambda_n}(t)\mu(dt)\Big),$$

so that

$$\int_{P-\{t_{j_0}\}} g_{t_{j_0},\lambda_n}(t)\mu(dt) > \frac{\varepsilon}{||EX - (EX)(t_{j_0})||_\infty} =: \delta > 0, \text{ for all } n \geq 1.$$

There cannot be real constants $\beta_1,\ldots,\beta_k$ with

$$\sum_{i=1}^{k} \beta_i E[X_i(t,\omega) - X_i(t_{j_0},\omega)] \geq 1, \text{ for all } t \in P - \{t_{j_0}\}.$$

Since, otherwise, we would have

$$\sum_{i=1}^{k} \beta_i E[X_i(t,\omega) - X_i(t_{j_0},\omega)]\cdot g_{t_{j_0},\lambda_n}(t) \geq g_{t_{j_0},\lambda_n}(t), \text{ for all } t \in P - \{t_{j_0}\}$$

and therefore

$$\sum_{i=1}^{k} \beta_i \cdot \int_{P-\{t_{j_0}\}} [(EX_i)(t) - (EX_i)(t_{j_0})]\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt) \geq \int_{P-\{t_{j_0}\}} g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt) > \delta.$$

(Note that

$$(T_{\lambda_n}(EX_i))(t_{j_0}) = \int_P (EX_i)(t)\cdot g_{t_{j_0},\lambda_n}(t)\cdot\mu(dt), \ i = 1,\ldots,k.)$$

However from the assumptions of the theorem, we have

$$\lim_{n\to\infty} (T_{\lambda_n}(EX_i))(t_{j_0}) = (EX_i)(t_{j_0}), \text{ all } i = 1,\ldots,k.$$

Hence

$$0 = \lim_{n\to\infty} \Big(\sum_{i=1}^{k} \beta_i[(T_{\lambda_n}(EX_i))(t_{j_0}) - (EX_i)(t_{j_0})]\Big) > \delta.$$

Thus $\delta < 0$, contradicting $\delta > 0$. □

To show that the assumptions of Theorem 1 are not empty and they are powerful, we present

EXAMPLE 2. (i) Consider the probability space $([-a,a], \mathcal{B}, \frac{\lambda}{2a})$, where $a > 0$, $\mathcal{B}$ the Borel σ-algebra on $[-a,a]$, λ the Lebesgue measure on $[-a,a]$. Since $\frac{\lambda}{2a}([-a,a]) = 1$, $\frac{\lambda}{2a}$ is a probability measure on $[-a,a]$. Let also $P = \{\pm 1, \pm 2, \ldots, \pm T\}$ be a finite set of integers. That is here $\omega \in \underline{0} = [-a,a]$ and $t \in P$.

Consider the sequence of operators

$$T_n: \underline{B_0}(P) \to \underline{B_0}(P)$$

such that

$$(T_nX)(t,\omega) = X(t,\omega)(1 - e^{-n|t|}) + X(-t,\omega)e^{-n|t|}, \text{ for all } n \geq 1.$$

If $X \geq 0$ then $T_nX \geq 0$, that is T_n is a positive operator, furthermore $T_n(1) = 1$, for all $n \geq 1$. It is obvious that T_n is linear.

Observe that

$$(E(T_nX))(t,\omega) = (EX)(t)\cdot(1 - e^{-n|t|}) + (EX)(-t)\cdot e^{-n|t|} = (T_n(EX))(t,\omega),$$

i.e., $ET_n = T_nE$, that is T_n is E-commutative for all $n \geq 1$. Therefore T_n fulfills the assumptions of Theorem 1.

From

$$(E(T_nX))(t) = (EX)(t)\cdot(1 - e^{-n|t|}) + (EX)(-t)\cdot e^{-n|t|},$$

it is clear that

$$\lim_{n\to\infty} E[(T_nX)(t,\omega) - X(t,\omega)] = 0,$$

for all $t \in P$ and all $X \in \underline{B_0}(P)$. Thus T_n fulfills the conclusion of Theorem 1.

(ii) Continuing in the setting of part (i): Let $X_1(t,\omega) = 1$, $X_2(t,\omega) = 2t|\omega|/a$ and $X_3(t,\omega) = 3t^2\omega^2/a^2$. Then $(EX_1)(t) = 1$, $(EX_2)(t) = t$ and $(EX_3)(t) = t^2$. It is obvious that $X_1,X_2,X_3 \in \underline{B_0}(P)$. We would like to find β_1,β_2,β_3 such that

$$\sum_{i=1}^{3} \beta_i[(EX_i)(t) - (EX_i)(t_j)] \geq 1, \text{ for all } t \in P - \{t_j\}.$$

For that we can pick β_1 an arbitrary real number, $\beta_2 = -2t_j$ and $\beta_3 = 1$. We have

$$\beta_1(1 - 1) + (-2t_j)(t - t_j) + (t^2 - t_j^2) = (t - t_j)^2 \geq 1,$$

for all $t \in P - \{t_j\}$. Hence X_i, $i = 1,2,3$ fulfill the sufficient condition of Theorem 1.

Trivially $T_nX_i = X_i$, giving us $ET_nX_i = EX_i$, for $i = 1,3$. And

$$(T_nX_2)(t,\omega) = X_2(t,\omega)(1 - e^{-n|t|}) + X_2(-t,\omega)\cdot e^{-n|t|},$$

implying

$$(E(T_nX_2))(t) = t(1 - 2e^{-n|t|}).$$

Clearly

$$\lim_{n\to\infty} (E(T_nX_2))(t) = (EX_2)(t).$$

We have seen how X_i, $i = 1,2,3$ fulfill the assumptions of Theorem 1.

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