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ON NORMAL LATTICES AND SEPARATION AND SEMI-SEPARATION OF LATTICES

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ABSTRACT. This present paper is concerned with two main conditions,that of normality of a lattice, and separation and semi-separation of two lattices,both looked at using measure theoretic techniques.We look at each property using {0,1} two valued measures and associated {0,1} valued set functions.

For normal lattices we look at consequences of normality in terms of properties of their measures and closely allied set functions.For separation and semi-separation of two lattices,we investigate the realtionship between regular measures of both lattices, define the notion of weak going up and look at this notion in terms of separation and semi-separation.We then give necessary and sufficent conditions for semi-separation in terms of equality of two set fuctions, derived from regular measures on the smaller lattice on the larger lattice.

KEY WORDS AND PHRASES. Normal lattices,countable compactness, Almost countable compactness, countable paracompactness, disjunctiveness, complement generated, separation, semi-separation, strongly normal, two valued measures, regular measures, sigma-smooth measures,weak going up property.
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1. INTRODUCTION

In this paper we consider necessary and sufficent conditions for a lattice of subsets of an abstract set to be normal,in terms of measure theoretic conditions.We also consider conditions when two lattices separate or semi-separate each other,again using measure theoretic methods.

In the first part of the paper,we consider consequences of a lattice $\mathbf{L}$ of subsets of an abstract set X being normal.This is is equivalent as is well known ,(and which we prove), to each element of $\mu\varepsilon I(\mathbf{L})$,the set of non-trivial finitely additive{0,1} two valued measures having a unique regular extension $\nu\varepsilon IR(\mathbf{L})$ st $\nu\geq\mu$ $(\mathbf{L})$.We then extend this work to look at relations with various classes of measures I\$($\mathbf{L}$),IW($\mathbf{L}$) , set functions μ',μ'',and side conditions on the lattice such as cg,, and look at necessary and sufficent conditions that a lattice of subsets have the normal property.

In the second part of the paper we investigate when two lattices $\mathbf{L}_1,\mathbf{L}_2$ of an abstract set X $\mathbf{L}_2\supseteq\mathbf{L}_1$,$\mathbf{L}1$ either separates or semi-separates $\mathbf{L}_2$,as well as consequences of separation or semiiseparation of two lattices.We again, investigate these properties in some detail in a measure theoretic setting,where they are equivalent to the existence and uniqueness of extensions or restrictions of regular measures on the two lattices.

We also include a section on notation,terminology ,basic backround,and references for the readers convenience.In addition other notions are introduced as needed in the sections in which they occur.

2. BACKROUND AND NOTATION

We begin by reviewing some notation and terminology which is fairly standard (see,for example, Alexsandroff [1], Camacho [2], Grassi [3], and Szeto [4]).We supply some backround and notation for the readers convenience.

Let X be an abstract set and **L** a lattice of subsets of X st $\emptyset$,Xε**L**.A delta lattice is one that is closed under countable intersections,and the delta lattice genereated by **L** is denoted δ(**L**) .A lattice is complement generated iff for every Lε**L** there exists a sequence of subsets $A_n \varepsilon$L n=1,2,... such that L=$\cap A_n{}'$(' denotes complement). **L** is countably paracompact if for every sequence $L_n\varepsilon$**L** and $L_n\downarrow\emptyset$ then there exists $L_n{}^{\sim}\varepsilon$**L** st $L_n{}^{\sim\prime}\supseteq L_n$ and $L_n{}^{\sim\prime}\downarrow\emptyset$.A tau lattice is one that is closed under arbitrary intersections,and the tau lattice generated by **L** is denoted τ**L**.

Let I(**L**) denote the set of non-trivial two valued {0,1} fintely additive measures on the algebra A(**L**) generated by {**L**}.Also let $\mu\varepsilon$I(σ*,**L**) denote those elements of I(**L**) that are sigma-smooth on L,i.e. $L_n\varepsilon$**L** $L_n\downarrow\emptyset$,$\mu\varepsilon$I(σ*,**L**) then lim$\mu(L_n)$=0.I\$(**L**) denotes those elements of I(σ*,**L**) such that if $L_n\varepsilon$**L** $\mu\varepsilon$I\$(**L**),$L_n\downarrow$, and $\cap L_n$=Lε**L** then μ(L)=lim$\mu(L_n)$. I(σ,**L**) will denote those measures that are sigma-smooth on A(**L**),i.e. if $A_n\varepsilon$A(**L**) $A_n\downarrow\emptyset$ then lim$\mu(A_n)$=0. Note that this is equivalent to countable additivity.IR(**L**) will stand for those measures on A(**L**) that are **L**-regular on A(**L**) ,i.e. $\mu\varepsilon$IR(**L**) then for AεA(**L**) μ(A)=sup{μ(L): Lε**L**,A$\supseteq$L}.IR(σ,**L**) denotes those measures in I(σ,**L**) that are **L** regular.The obvious relations hold I(**L**)$\supseteq$I(σ*,**L**)$\supseteq$I\$(**L**)$\supseteq$I(σ,**L**)$\supseteq$IR(σ,**L**) and I(L)$\supseteq$IR(**L**).

A lattice is said to be disjunctive if for any xεX and Lε**L** such that x$\notin$ L then there exists a $L_1\varepsilon$**L** st xεL_1 and L$\cap L_1$=$\emptyset$.A lattice is said to be normal if for $L_1,L_2\varepsilon$**L** and $L_1\cap L_2$=$\emptyset$,there exists $L_3,L_4\varepsilon$**L** such that $L_3{}'\supseteq L_1$ $L_4{}'\supseteq L_2$ and $L_3{}'\cap L_4{}'$=$\emptyset$.A lattice is said to be T_2 if for x,yεX there exists $L_1,L_2\varepsilon$**L** such that x$\varepsilon L_1{}'$,y$\varepsilon L_2{}'$ and $L_1{}'\cap L_2{}'$=$\emptyset$.

A fact we will use throughout this paper is that there exists a 1-1 correspondence between prime **L**-filters and elements of I(**L**),and a one to one correspondence between **L**-ultrafilters and elements of IR(**L**).This correspondence is set up by letting $\mu\varepsilon$I(**L**) and H={Lε**L** I μ(L)=1}.Then H is a prime **L**-filter and conversely if H is a prime **L**-filter there exists a measure associated with H such that if LεH μ(L)=1. A similiar correspondence holds for H and $\mu\varepsilon$IR(**L**) in which case H is an **L**-ultrafilter.

We define $\mu\leq\nu$ (**L**) for $\nu,\mu\varepsilon$I(**L**) if μ(L)$\leq\nu$(L) for all Lε**L**.We now prove two results that will be useful in the sequel:

THEOREM 2.1: Let **L** be normal and countably paracompact,then if $\mu\varepsilon$I(σ*,**L**) there exists a unique $\mu_1\varepsilon$IR(σ,**L**) such that $\mu\leq\mu_1$ (**L**).

Proof: Let $\mu\varepsilon$I(σ*,**L**) and $\mu_1\varepsilon$IR(**L**) such that $\mu\leq\mu_1$ (**L**).Then we must prove that $\mu_1\varepsilon$IR(σ,L).Let {A_n}εL $A_n\downarrow\emptyset$.Since **L** is countably paracompact there exists {$B_n{}'$}such that $B_n{}'\downarrow\emptyset$,$B_n\varepsilon$**L**,and $B_n{}'\supseteq A_n$ for every n.Since $B_n{}'\supseteq A_n$ and **L** is normal and $A_n\cap B_n$=$\emptyset$,there exists $C_n,D_n\varepsilon$**L** such that $C_n{}'\supseteq A_n$ $D_n{}'\supseteq B_n$ and $D_n{}'\cap C_n{}'$=$\emptyset$.Then $B_n{}'\supseteq D_n\supseteq C_n{}'\supseteq A_n$ and we can assume without loss of generality that these inclusions hold with $D_n\downarrow\emptyset$.Then $\mu_1(A_n)\leq\mu_1(C_n{}')\leq\mu(C_n{}')\leq\mu(D_n)$,and since $B_n{}'\downarrow\emptyset$ $D_n\downarrow\emptyset$ plus the fact $\mu\varepsilon$I(σ*,**L**) imply that lim$\mu(D_n)$=0 as n$\rightarrow\infty$.Then $\mu_1(A_n)$=0 as n$\rightarrow\infty$ and $\mu_1\varepsilon$IR(σ,**L**).Uniqueness follows from normality.

THEOREM 2.2: If the lattice $\mathbf{L}$ is complement generated, it is countably paracompact.
Proof: Let $\{A_n\}\downarrow\emptyset$ $A_n\epsilon\mathbf{L}$ and $A_n=\cap L_{n_i}'$ $=1,2\ldots,\infty$ $L_{n_i}\epsilon\mathbf{L}$. Since $A_n\downarrow\emptyset$ $B_n'=\cap\cap L_{m_i}'$ both i and m go from 1 to n, $B_n'\downarrow\emptyset$ $B_n'\epsilon\mathbf{L}'$ $B_n'\supseteq A_n$. Thus $\mathbf{L}$ is countably paracompact.

Now consider various sets of measures defined on the algebra generated by the lattice $\mathbf{L}$. For example consider $I(\mathbf{L})$, $I(\sigma^*,\mathbf{L})$, $IR(\mathbf{L})$ and $IR(\sigma,\mathbf{L})$. Denote such sets by I. Also consider the collection of sets $H(\mathbf{L})$ where $H(\mathbf{L})=\{H(L)\mid L\epsilon\mathbf{L}\}$ and $H(L)=\{\mu\epsilon I\mid\mu(L)=1\}$. Then the following hold: a) $H(A\cup B)=H(A)\cup H(B)$ $A,B\epsilon\mathbf{L}$. b) $H(A\cap B)=H(A)\cap H(B)$ $A,B\epsilon\mathbf{L}$. c) $H(A')=H(A)'$ $A\epsilon\mathbf{L}$. d) If $A\supseteq B$ then $H(A)\supseteq H(B)$ $A,B\epsilon\mathbf{L}$. e) If $\mathbf{L}$ is disjunctive (if necessary) and $H(A)\supseteq H(B)$ $A,B\epsilon\mathbf{L}$ then $A\supseteq B$. f) The collection $H(\mathbf{L})$ is a lattice and $H(A(\mathbf{L}))=A(H(\mathbf{L}))$.

We will assume in discussing $H(\mathbf{L})$ for convenience, that $\mathbf{L}$ is disjunctive, although it will be clear that this assumption is not always necessary.

If $\mu\epsilon I$ then define a measure on $A(H(\mathbf{L}))$ $\mu^\wedge\epsilon I(H(\mathbf{L}))$ by $\mu^\wedge(H(A))=\mu(A)$ for $A\epsilon A(\mathbf{L})$. Conversely if for $\mu^\wedge\epsilon I(H(\mathbf{L}))$ define a measure on $A(\mathbf{L})$ $\mu\epsilon I$ by $\mu(A)=\mu^\wedge(H(A))$ $H(A)\epsilon A(H(\mathbf{L}))$. Then the following hold:

THEOREM 2.3: If $\mathbf{L}$ is disjunctive (if necessary) then there exists a 1-1 correspondence between the sets I and $I(H(\mathbf{L}))$ given by $\mu\leftrightarrow\mu^\wedge$. Further $\mu\epsilon I$ is σ-smooth or $\mathbf{L}$-regular iff $\mu^\wedge\epsilon I(H(\mathbf{L}))$ is σ-smooth or $H(\mathbf{L})$-regular.

If $I=IR(\mathbf{L})$ we let $H(\mathbf{L})=W(\mathbf{L})$.

If $I=I(\mathbf{L})$ we let $H(\mathbf{L})=V(\mathbf{L})$.

If $I=I(\sigma^*,\mathbf{L})$ we let $H(\mathbf{L})=V(\sigma,\mathbf{L})$.

If $I=IR(\sigma,\mathbf{L})$ we let $H(\mathbf{L})=W(\sigma,\mathbf{L})$.

3. ON NORMAL LATTICES

In this section we extend the work of Eid [5].and Huerta [6],and consider further consequences of a lattice being normal as well as new equivalent characterizations of normality. First we have the following measure theoretic characterization of normality:

THEOREM 3.1: A lattice $\mathbf{L}$ is normal iff for $\mu\epsilon I(\mathbf{L})$ and $\nu_1,\nu_2\epsilon IR(\mathbf{L})$ st $\mu\leq\nu_1$ $(\mathbf{L})$ $\mu\leq\nu_2$ $(\mathbf{L})$ implies that $\nu_1=\nu_2$.

Proof: Let $\mathbf{L}$ be normal. Assume that for $\mu\epsilon I(\mathbf{L})$ there exists $\nu_1,\nu_2\epsilon IR(\mathbf{L})$ st $\mu\leq\nu_1$ $(\mathbf{L})$, $\mu\leq\nu_2$ $(\mathbf{L})$ and $\nu_1\neq\nu_2$. Then there exists $L_1\epsilon\mathbf{L}$ $\nu_1(L_1)=1$ $\nu_2(L_1)=0$. Since $\nu_2\epsilon IR(\mathbf{L})$ there exists $L_2\epsilon\mathbf{L}$ $L_1'\supseteq L_2$ and $\nu_2(L_2)=\nu_2(L_1')=1$ and $L_1\cap L_2=\emptyset$. Since $\mathbf{L}$ is normal there exists $L_3,L_4\epsilon\mathbf{L}$ st $L_3'\supseteq L_1$, $L_4'\supseteq L_2$ and $L_3'\cap L_4'=\emptyset$. Since $\nu_1(L_1)=1$ this implies that $\nu_1(L_3')=1$, and $\nu_2(L_2)=1$ implies $\nu_2(L_4')=1$. Thus $\mu(L_3')=\mu(L_4')=1$ since $\mu\geq\nu_1$ $(\mathbf{L}')$ and $\mu\geq\nu_2$ $(\mathbf{L}')$. Then $\mu(L_3'\cap L_4')=1$, but $L_3'\cap L_4'=\emptyset$ implies that $\mu(L_3'\cap L_4')=0$, a contradiction. Therefore $\nu_1=\nu_2$.

Conversely let $\mu\epsilon I(\mathbf{L})$ $\nu_1,\nu_2\epsilon IR(\mathbf{L})$ $\mu\leq\nu_1$ $(\mathbf{L})$, $\mu\leq\nu_2$ $(\mathbf{L})$ imply that $\nu_1=\nu_2$, and assume that $\mathbf{L}$ is not normal. Then there exists $L_1,L_2\epsilon\mathbf{L}$ st $L_1\cap L_2=\emptyset$ and any $L_3'\supseteq L_1$ $L_4'\supseteq L_2$ $L_3,L_4\epsilon\mathbf{L}$ imply that $L_3'\cap L_4'\neq\emptyset$. Let $H=\{L'\mid L'\supseteq L_1$ or $L'\supseteq L_2\}$. Since H has the finite intersection property and forms a filter base there exists a prime $\mathbf{L}$-filter containing H and an associated measure $\mu\epsilon I(\mathbf{L}')$ st $\mu(L')=1$ $L'\epsilon H$. Look at $\mu(L_5)=1$ $L_5\epsilon\mathbf{L}$ then $\mu(L_5')=0$ and L_5' does not contain L_1 thus $L_1\cap L_5\neq\emptyset$. Since the collection of all such L_5 's has the fip there exists a measure $\mu_1\epsilon IR(\mathbf{L})$ st $\mu\leq\mu_1$ $(\mathbf{L})$ and $\mu_1(L_1)=1$. By similiar reasoning there exists a $\mu_2\epsilon IR(\mathbf{L})$ st $\mu\leq\mu_2$ $(\mathbf{L})$ and $\mu_2(L_2)=1$. By hypothesis $\nu_1=\nu_2$. But then $\nu_1(L_1)=\nu_2(L_2)=1$ or $\nu_1(L_1\cap L_2)=1$. But $L_1\cap L_2=\emptyset$, thus $\nu_1(L_1\cap L_2)=0$, a contradiction. $\mathbf{L}$ must be normal.

DEFINITION 3.1: A lattice $\mathbf{L}$ is said to be countably compact (cc) if for any countable collection of elements in the lattice $\{L_n\}\epsilon\mathbf{L}$ and $\cap L_n=\emptyset$ $n=1,2,\ldots$, then there exists a finite subindexing st $\cap L_{n_i}=\emptyset$ $i=1,2,\ldots N$. This is equivalent measure theoretically to the condition that if $\mu\epsilon I(\mathbf{L})$ then $\mu\epsilon I(\sigma^*,\mathbf{L})$.

Definition 3.2: A lattice L is almost countably compact (acc) if $\mu\varepsilon IR(\mathbf{L}')$ implies that $\mu\varepsilon I(\sigma^*,\mathbf{L})$.

We then have the following theorem.

THEOREM 3.2: If $\mathbf{L}$ is normal and cp then $\mathbf{L}$ is cc iff $\mathbf{L}$ acc.

Proof: Assume $\mathbf{L}$ is cc,then let $\mu\varepsilon IR(\mathbf{L}')$ which implies that $\mu\varepsilon I(\mathbf{L})$ .But since $\mathbf{L}$ is cc this implies that $\mu\varepsilon I(\sigma^*,\mathbf{L})$.(Note $\mathbf{L}$ cc implies $\mathbf{L}$ acc without any other conditions on the lattice).Conversely let $\mathbf{L}$ be normal cp and acc.Then let $\mu\varepsilon I(\mathbf{L})$.This implies that $\mu\varepsilon I(\mathbf{L}')$ and since every filter is contained in an ultrafilter ,there exists an associated $\nu\varepsilon IR(\mathbf{L}')$ st $\mu\leq\nu$ $(\mathbf{L}')$ or $\mu\geq\nu$ $(\mathbf{L})$.Since $\mathbf{L}$ is acc $\nu\varepsilon I(\sigma^*,\mathbf{L})$,and also since $\mathbf{L}$ is normal and cp there exists a $\nu_1\varepsilon IR(\sigma,\mathbf{L})$ st $\nu\leq\nu_1$ $(\mathbf{L})$. Thus because $\mathbf{L}$ is normal this implies that $\nu\leq\mu\leq\nu_1$ $(\mathbf{L})$,$\mu\varepsilon I(\sigma^*,\mathbf{L})$ and $\mathbf{L}$ is cc.

THEOREM 3.3: If $\mathbf{L}$ is normal,and if $\mu\varepsilon I(\sigma^*,\mathbf{L})$ $\nu\varepsilon IR(\mathbf{L})$,$\mu\leq\nu$ $(\mathbf{L})$ then $\nu\varepsilon I(\sigma^*,\mathbf{L}')$.

Proof: Assume not then there exists $A_n\varepsilon\mathbf{L}$ $\{A_n'\}\downarrow\emptyset$ and $\nu(A_n')=1$all n.Since $\nu\varepsilon IR(\mathbf{L})$ there exists $B_n\varepsilon\mathbf{L}$ st $A_n'\supseteq B_n$ and $\nu(B_n)=1$ all n.Without loss of generality we can assume that $\{B_n\}\downarrow\emptyset$ since $\{A_n'\}\downarrow\emptyset$ and $A_n'\supseteq B_n$ all n.Since $\mathbf{L}$ is normal there exists $C_n,D_n\varepsilon\mathbf{L}$ st $C_n'\supseteq B_n$ $D_n'\supseteq A_n$ and $C_n'\cap D_n'=\emptyset$ all n. $\nu(B_n)=1$ all n , $\mu(B_n)=0$ n>N because $\mu\varepsilon I(\sigma^*,\mathbf{L})$. $\nu(C_n')=\mu(C_n')=1$ since $C_n'\supseteq B_n$ $\nu(B_n)=1$ all n and $\mu\geq\nu$ $(\mathbf{L}')$.Now $A_n'\supseteq D_n\supseteq C_n'\supseteq B_n$ and since $\{B_n\}\downarrow\emptyset$ $\{A_n'\}\downarrow\emptyset$ then $\{D_n\}\downarrow\emptyset$ and because $\mu\varepsilon I(\sigma^*,\mathbf{L})$,$\mu(D_n)=0$ for n>M.Then since $D_n\supseteq C_n'\mu(C_n')=0$ n>M,a contradiction.Then $\nu\varepsilon I(\sigma^*,\mathbf{L}')$.

THEOREM 3.4: Let $\mathbf{L}$ be cg and normal,and $\mu\varepsilon I\$(\mathbf{L})$ then $\mu\varepsilon IR(\mathbf{L})$.

Proof: Suppose $\mu\varepsilon I\$(\mathbf{L})$ and $\mathbf{L}$ cg normal.Let $\nu\varepsilon IR(\mathbf{L})$ be such that $\mu\leq\nu$ $(\mathbf{L})$.If $\mu\neq\nu$ there exists $A\varepsilon\mathbf{L}$ st $\mu(A)=0$ $\nu(A)=1.A=\cap A_n'$ n=1,2..., $A_n\varepsilon\mathbf{L}$ by cg property.But $\mathbf{L}$ is normal and $A_n\cap A=\emptyset$.Therefore there exists $C_n,B_n\varepsilon\mathbf{L}$ st $C_n'\supseteq A$ $B_n'\supseteq A_n$ and $C_n'\cap B_n'=\emptyset$ all n.$\nu(A_n')=1$ all n since $A_n'\supseteq A$.Also $\mu(A_n')=1$ all n since $\nu\leq\mu$ $(\mathbf{L}')$ .Now $\mu(B_n)=1$ all n since $C_n'\supseteq A$ all n ,$\nu(A)=1$,thus $\nu(C_n')=1$ all n,$\mu\geq\nu$ $(\mathbf{L}')$ and $B_n\supseteq C_n'$ all n.But $A_n'\supseteq B_n\supseteq C_n'\supseteq A$ which implies $A=\cap B_n$ n=1,2,..., and since $\mu\varepsilon I\$(\mathbf{L})$ $\mu(A)=1$,a contradiction.$\mu\varepsilon IR(\mathbf{L})$ and $IR(\mathbf{L})\supseteq I\$(\mathbf{L})$.

THEOREM 3.5: Let $\mathbf{L}$ be cg,and $\mu\varepsilon I(\sigma^*,\mathbf{L}')$ then $\mu\varepsilon IR(\mathbf{L})$.

Proof: Let $\mu\varepsilon I(\sigma^*,\mathbf{L}')$ and let $\mu(L')=1$ $L\varepsilon\mathbf{L}$.Since $\mathbf{L}$ is cg $L=\cap L_i'$ i=1,2,...,$L'=\cup L_i$.Now $\emptyset=L'\cap L=L'\cap(\cap L_i')$ and thus $A_n'=L'\cap(\cap L_i')$ i=1,2...n $A_n'\varepsilon\mathbf{L}'$ $\{A_n'\}\downarrow\emptyset$. Since $\mu\varepsilon I(\sigma^*,\mathbf{L}')$ $\lim\mu(A_n')=0$ or $\mu(A_n')=0$ for n>N or $\mu(A_n)=1$ n>N.$A_n=L\cup(\cup L_i)$ i=1,2,...n $\mu(L)=0$, which implies that $\mu(\cup L_i)=1$, $\cup L_i\varepsilon L$ for i=1,2,...,n.$L'\supseteq\cup L_i$ i=1,2,...n ,thus $\mu\varepsilon IR(\mathbf{L})$.

If L is cg and normal then $I\$(\mathbf{L})\supseteq IR(\sigma,\mathbf{L})\supseteq I\$(\mathbf{L})$ by theorem 3.4 and $I\$(\mathbf{L})=IR(\sigma,\mathbf{L})$. $\mathbf{L}$ cg implies that L is cp so $I(\sigma^*,\mathbf{L})\supseteq I(\sigma^*,\mathbf{L}')$ holds by theorem 2.2.In addition from theorem 3.3,if $\mathbf{L}$ is also normal $I(\sigma^*,\mathbf{L})\supseteq I(\sigma^*,\mathbf{L}')\supseteq IR(\sigma,\mathbf{L})$,clearly $I(\sigma,\mathbf{L}')\supseteq IR(\sigma,\mathbf{L})$. Also by theorem 3.5 if $\mathbf{L}$ is normal and cg $IR(\mathbf{L})\supseteq I(\sigma^*,\mathbf{L}')$ or $IR(\sigma,\mathbf{L})\supseteq I(\sigma,\mathbf{L}')$.Thus if $\mathbf{L}$ is cg and normal $I(\sigma,\mathbf{L}')=IR(\sigma,\mathbf{L})=I(\sigma,\mathbf{L})$.

DEFINITION 3.3: Let $\mu\varepsilon I(\mathbf{L})$ $X\supseteq E$ then $\mu'(E)=\inf\{\mu(L')\mid L'\supseteq E\}$.

DEFINITION 3.4: $IW(\mathbf{L})$ consists of those $\mu\varepsilon I(\mathbf{L})$ st $\mu(L')=1$ implies that $L'\supseteq L_1$,where $L_1\varepsilon\mathbf{L}$ and $\mu'(L_1)=1$.

THEOREM 3.6: Let $\mathbf{L}$ be normal then $IR(\mathbf{L})=IW(\mathbf{L})$.

Proof: First it is clear that $IW(\mathbf{L})\supseteq IR(\mathbf{L})$ thus only need to prove $IR(\mathbf{L})\supseteq IW(\mathbf{L})$.

Let $\mu\varepsilon IW(\mathbf{L})$ and $\mu(L')=\mu'(L')=1$ $L\varepsilon\mathbf{L}$,then there exists a $L_3\varepsilon\mathbf{L}$ st $L'\supseteq L_3$ and $\mu'(L_3)=1$.Since $\mathbf{L}$ is normal and $L_3\cap L=\emptyset$ there exists $L_1,L_2\varepsilon\mathbf{L}$ st $L_1'\supseteq L,L_2'\supseteq L_3$ and $L_1'\cap L_2'=\emptyset$.This implies that $X=L_1\cup L_2$.Assume that $\mu(L_2)=1$ then $\mu(L_2)=\mu'(L_2)=1$.Thus $\mu(L_2')=\mu'(L_2')=0$.But $L_2'\supseteq L_3$ and $\mu'(L_3)=1$,a contradiction.Therefore $\mu(L_2)=0$ and $\mu(L_1)=1$,and $L'\supseteq L_1$.Thus one must have $\mu\varepsilon IR(\mathbf{L})$,$IR(\mathbf{L})\supseteq IW(\mathbf{L})$,and $IR(\mathbf{L})=IW(\mathbf{L})$ if $\mathbf{L}$ is normal.

DEFINITION 3.5: Let $\mu\varepsilon I(\sigma^*,\mathbf{L})$,E st $X\supseteq E$ then $\mu''(E)=\inf\Sigma\mu(L_i')$ i=1,2,..., st $\cup L_i'\supseteq E$ and $L_i\varepsilon\mathbf{L}$.

Note that μ'' is an outer measure.

THEOREM 3.7: Let $\mu\varepsilon I(\sigma^*,\mathbf{L})$, then $\mu'=\mu''$ on $\mathbf{L}'$ iff $\mu\varepsilon I\$(\mathbf{L})$.

Proof: Let $\mu\varepsilon I(\sigma^*,\mathbf{L})$ and $\mu'=\mu''$ on $\mathbf{L}'$. Also let $\cap A_n\downarrow A\varepsilon\mathbf{L}$ $A_n\varepsilon\mathbf{L}$ $n=1,2,\ldots,\infty$. Assume $\mu\notin I\$(\mathbf{L})$ and let the above sequence $\cap A_n\downarrow A$ be such that $\mu(A_n)=1$ all n and $\mu(A)=0$. Then $\mu(A')=1$ and $\mu(A')=\mu'(A')=\mu''(A')=1$ by hypothesis. But $\mu''(A')=\mu''(\cup A_n')\le\Sigma\mu(A_n')=0$ since $\mu(A_n')=0$ all n, a contradiction. $\mu\varepsilon I\$(\mathbf{L})$.

Conversely let $\mu\varepsilon I\$(\mathbf{L})$. Clearly $\mu''\le\mu'$ on $\mathbf{L}'$. Let $\mu''(L')=0$ $L\varepsilon\mathbf{L}$ then there exists $\cup L_i'$ $L_i\varepsilon\mathbf{L}$ $i=1,2,\ldots,\infty$ st $\Sigma\mu(\cup L_i')=0$ or $\mu(L_i')=0$ all i, or $\mu(L_i)=1$ and $L\supseteq\cap L_i$ $i=1,2,\ldots,\infty$. Thus one has that $L=\cap(L\cup L_i)$ $L\cup L_i\varepsilon L$ and $L_n=\cap(L\cup L_i)$ $i=1,2,\ldots,n$ $L_n\varepsilon\mathbf{L}$ and $L_n\downarrow L$. This implies that $\mu(L)=\inf\mu(L\cup L_i)=\inf 1=1$ since $\mu\varepsilon I\$(\mathbf{L})$. Then $\mu(L')=\mu'(L')=\mu''(L')=0$ and $\mu'=\mu''$ on $\mathbf{L}'$.

THEOREM 3.8: If $\mu\varepsilon I\$(\mathbf{L})$, and if $\mathbf{L}$ is cg then $\mu\varepsilon IW(\mathbf{L})$.

Proof: Suppose that $L\varepsilon\mathbf{L}$ and $\mu(L')=\mu''(L')=1$. Then from the previous theorem 3.7 $\mu''(L')=1$. Since $\mathbf{L}$ is cg then $L'=\cup L_i$ $L_i\varepsilon\mathbf{L}$ $i=1,2,\ldots,\infty$ and $1=\mu''(\cup L_i)\le\Sigma\mu''(L_i)$. Thus $\mu''(L_i)=1$ for some i and since $\mu\le\mu''\le\mu'$ on $\mathbf{L}$ $\mu'(L_i)=1$ $L'\supseteq L_i$ thus $\mu\varepsilon IW(\mathbf{L})$.

From theorems 3.6, 3.7 and 3.8 we have that $IR(\mathbf{L})=IW(\mathbf{L})\supseteq I\$(\mathbf{L})$ or $I\$(\mathbf{L})=IR(\sigma,\mathbf{L})$ if $\mathbf{L}$ is cg and normal. This gives a second proof of this fact.

THEOREM 3.9: If $\mathbf{L}$ is normal and if $\mu\le\nu$ on $\mathbf{L}$ $\mu\varepsilon I(\mathbf{L})$ $\nu\varepsilon IR(\mathbf{L})$ then $\nu(L')=1$ $L\varepsilon\mathbf{L}$ implies there exist $L^{\sim}\varepsilon\mathbf{L}$ $L'\supseteq L^{\sim}$ and $\mu(L^{\sim})=1$. Conversely this condition implies that $\mathbf{L}$ is normal.

Proof: Let $\mathbf{L}$ be normal, $\mu\le\nu$ $(\mathbf{L})$ $\mu\varepsilon I(\mathbf{L})$ $\nu\varepsilon IR(\mathbf{L})$ and let $\nu(L')=1$ for $L\varepsilon\mathbf{L}$. Assume that for $L'\supseteq L_1$, $L_1\varepsilon L$ $\mu(L_1)=0$ for all such L_1. Then look at $H=\{L_1' \mid L_1'\supseteq L\}$ then for all such $L_1'\varepsilon H$ $\mu(L_1')=1, L_1\varepsilon\mathbf{L}$. Then if $\mu(L_1)=1$ then $\mu(L_1')=0$ and thus L_1' does not contain L so that $L_1\cap L\neq\emptyset$. The collection of all such L_1 has the fip, and thus there exists a ultrafilter and its associated measure $\nu_2\varepsilon IR(\mathbf{L})$, st $\mu\le\nu_2$ $(\mathbf{L})$. Since $\mathbf{L}$ is normal $\nu=\nu_2$ and since $\nu(L')=1$ $\nu(L)=0$. But because ν_2 is an ultrafilter containing all such L_1 st $\mu(L_1)=1$ which is a filterbase and all such L_1 have non-empty intetsection with L $\nu(L)=1$, a contradiction. Thus there must exist a L_1 st $L'\supseteq L_1$ $\mu(L_1)=1$ $L_1\varepsilon\mathbf{L}$ when $\nu(L')=1$.

Conversely suppose $\mathbf{L}$ is not normal then there exists $L_1,L_2\varepsilon\mathbf{L}$ st $L_1\cap L_2=\emptyset$ but there does not exist $L_3,L_4\varepsilon\mathbf{L}$ st $L_3'\supseteq L_1$, $L_4'\supseteq L_2$ and $L_3'\cap L_4'=\emptyset$. Then $H=\{L' \mid L'\supseteq L_1 \text{ or } L'\supseteq L2\}$ has the fip and thus there exists a prime $\mathbf{L}$-filter containing H and an associated measure $\mu\varepsilon I(\mathbf{L}')$ st $\mu(L')=1$ $L'\varepsilon H$. Look at at $\mu(L_5)=1$ $L_5\varepsilon\mathbf{L}$ then $\mu(L_5')=0$ and L_5' does not contain L_1 thus $L_1\cap L_5\neq\emptyset$. Since the collection of all such L_5 has the fip there exists a $\mu_1\varepsilon I(\mathbf{L})$ st $\mu\le\mu_1$ $(\mathbf{L})$ and $\mu_1(L_1)=1$. By similiar reasoning there exists a $\mu_2\varepsilon I(\mathbf{L})$ st $\mu\le\mu_2$ $(\mathbf{L})$ and $\mu_2(L_2)=1$. But since every filter is contained in an ultrafilter there exists $\nu_1,\nu_2\varepsilon IR(\mathbf{L})$ st $\mu\le\mu_1\le\nu_1$ and $\mu\le\mu_2\le\nu_2$ $(\mathbf{L})$. Now $L_1'\supseteq L_2$ $L_2'\supseteq L_1$ therefore $\nu_2(L_1')=1$ and $\nu_1(L_2')=1$. By hypothesis there exists $L_5,L_6\varepsilon\mathbf{L}$ st $L_1'\supseteq L_5, L_2'\supseteq L_6$ st $\mu(L_5)=\mu(L_6)=1$, thus $\mu(L_5\cap L_6)=1$. In addition $L_1'\supseteq L_5\cap L_6$ and $L_2'\supseteq L_5\cap L_6$. But since $\mu\le\nu_1$ $(\mathbf{L})$ and $\mu\le\nu_2$ $(\mathbf{L})$, $\nu_1(L_5\cap L_6)=\nu_2(L_5\cap L_6)=1$. Now $\nu_1(L_1)=1$ so $\nu_1(L_1\cap L_5\cap L_6)=1$. But $L_1'\supseteq L_5\cap L_6$ thus $L_5\cap L_6\cap L_1=\emptyset$ thus $\nu_1(L_1\cap L_5\cap L_6)=0$, a contradiction. $\mathbf{L}$ must be normal.

Finally, we prove one further result that holds for normal lattices.

THEOREM 3.10: If $\mathbf{L}$ is normal and $\mu\varepsilon I(\mathbf{L})$, $\nu\varepsilon IR(\mathbf{L})$, and $\mu\le\nu$ $(\mathbf{L})$ then $\mu'=\nu$ $(\mathbf{L})$.

Proof: Since by definition $\mu'(L)=\inf\mu(L_4')$ $L_1'\supseteq L$ $L,L_4\varepsilon\mathbf{L}$, and since $\mu\le\nu$ $(\mathbf{L})$ or $\nu\le\mu$ $(\mathbf{L}')$, then $\mu\le\nu\le\mu'$ $(\mathbf{L})$.

Assume that $\nu\neq\mu'$ $(\mathbf{L})$ then there exists $L\varepsilon\mathbf{L}$ st $\nu(L)=0$ and $\mu'(L)=1$. Thus $\nu(L')=1$ and since $\nu\varepsilon IR(\mathbf{L})$ there exists $L_3\varepsilon\mathbf{L}$ st $L'\supseteq L_3$ and $\nu(L_3)=1$. Since $\mathbf{L}$ is normal and $L_3\cap L=\emptyset$, there exists $L_1,L_2\varepsilon L$ st $L_1'\supseteq L$ and $L_2'\supseteq L_3$ and $L_1'\cap L_2'=\emptyset$. Thus since $L_2'\supseteq L_3$ and $\nu(L_3)=1$ and $\nu\le\mu$ $(\mathbf{L}')$, $\mu(L_2')=1$ which implies $\mu(L_2)=0$. Also since $L_2\supseteq L_1'$ $\mu(L_1')=0$ and $L_1'\supseteq L$. But $\mu'(L)=\inf\mu(L')$ $L'\supseteq L$ thus $\mu'(L)=0$, a contradiction. If L is normal $\mu'=\nu$ $(\mathbf{L})$.

4. LATTICE SEPARATION

In this section we study and characterize separation and semi-separation between pairs of lattice in a measure theoretic fashion,and give some applications of these results .We first give some definitions.

DEFINITION 4.1: Let $\mathbf{L}_1,\mathbf{L}_2$ be lattices st $\mathbf{L}_2\supseteq\mathbf{L}_1$.Then $\mathbf{L}_1$ is said to semi-separate $\mathbf{L}_2$ if for $L_1\varepsilon\mathbf{L}_1$ and $L_2\varepsilon\mathbf{L}_2$ and $L_1\cap L_2=\emptyset$,there exists a $L_1^{\sim}\varepsilon\mathbf{L}_1$ st $L_1^{\sim}\supseteq L_2$ and $L_1\cap L_1^{\sim}=\emptyset$.

DEFINITION 4.2: Let $\mathbf{L}_1,\mathbf{L}_2$ be lattices such that $\mathbf{L}_2\supseteq\mathbf{L}_1$ then $\mathbf{L}_1$ is said to separate $\mathbf{L}_2$ if for $L_2,L_2^{\sim}\varepsilon\mathbf{L}_2$ and $L_2\cap L_2^{\sim}=\emptyset$,then there exists $L_1,L_1^{\sim}\varepsilon\mathbf{L}_1$ st $L_1\supseteq L_2$ $L_1^{\sim}\supseteq L_2^{\sim}$ and $L_1\cap L_1^{\sim}=\emptyset$.

DEFINITION 4.3: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be lattices such that $\mathbf{L}_2\supseteq\mathbf{L}_1$, then if $\mu\varepsilon I(\mathbf{L}_2)$ the restriction of μ to $A(\mathbf{L}_1)$ will be noted by $\mu|$,and $\mu|\varepsilon I(\mathbf{L}_1)$.

We now proceed to look at what separation and semi-separation implies about the relationship between $IR(\mathbf{L}_1)$ and $IR(\mathbf{L}_2)$.

THEOREM 4.1: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be lattices such that $\mathbf{L}_2\supseteq\mathbf{L}_1$ and $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$. Then if $\nu\varepsilon IR(\mathbf{L}_2)$ we have that $\mu=\nu\,|\,(\mathbf{L}_1)$ and $\mu\varepsilon IR(\mathbf{L}_1)$.

Proof: Let $\nu\varepsilon IR(\mathbf{L}_2)$ and let $\mu=\nu\,|\,(\mathbf{L}_1)$ then $\mu\varepsilon I(\mathbf{L}_1)$.Assume that $\mu(L_1')=\nu(L_1')=1$,then since $\mathbf{L}_2\supseteq\mathbf{L}_1$ and $\nu\varepsilon IR(\mathbf{L}_2)$ there exists a $L_2\varepsilon\mathbf{L}_2$ st $L_1'\supseteq L_2$ and $\nu(L2)=1$,also $L_1\cap L_2=\emptyset$.But $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$,then there exists $L_1^{\sim}\varepsilon\mathbf{L}1$ st $L_1^{\sim}\supseteq L_2$ and $L_1^{\sim}\cap L_1=\emptyset$. This implies that $L_1'\supseteq L_1^{\sim}$ and $\nu(L_1^{\sim})|=\mu(L_1^{\sim})=1$ $(\mathbf{L}_1)$ .Thus $\mu\varepsilon IR(\mathbf{L}_1)$.

THEOREM 4.2: Let $\mathbf{L}_1,\mathbf{L}_2$ be lattices such that $\mathbf{L}_2\supseteq\mathbf{L}_1$ and let $\mathbf{L}_1$ separate $\mathbf{L}_2$.Then there exists a one to one correspondence between $IR(\mathbf{L}_1)$ and $IR(\mathbf{L}_2)$.

Proof: Since separation implies semi-separation we know from theorem 4.1 that if $\mu\varepsilon IR(\mathbf{L}_2)$ then $\mu|=\nu$ $(\mathbf{L}_1)$ then $\nu\varepsilon IR(L_1)$ .Thus we need only prove if $\mu\varepsilon IR(\mathbf{L}_1)$ there exists a unique $\nu\varepsilon IR(\mathbf{L}_2)$ st $\nu|=\mu$ $(\mathbf{L}_1)$ .Assume that this is not true and thus there exists a $\mu\varepsilon IR(\mathbf{L}_1)$ and $\nu_1,\nu_2\varepsilon IR(\mathbf{L}_2)$ st $\nu_1|=\mu=\nu_2|$ $(\mathbf{L}_1)$ and $\nu_1\neq\nu_2$ .Then there exists a $L_2\varepsilon\mathbf{L}_2$ st $\nu_1(L_2)=1$ and $\nu_2(L_2)=0$ say.But $\nu_2\varepsilon IR(\mathbf{L}_2)$ therefore there exists $L_2^{\sim}\varepsilon\mathbf{L}_2$ $L_2'\supseteq L_2^{\sim}$ and $\nu_2(L_2^{\sim})=1$,and $L_2\cap L_2^{\sim}=\emptyset$.Since $\mathbf{L}_1$ separates $\mathbf{L}_2$ there exists $L_1,L_1^{\sim}\varepsilon\mathbf{L}_1$ st $L_1\supseteq L_2$ and $L_1^{\sim}\supseteq L_2^{\sim}$ and $L_1\cap L_1^{\sim}=\emptyset$.Also $\nu_1(L_1)=1$ $\nu_2(L_1^{\sim})=1$ thus $\mu(L_1)=\nu_1(L_1)=1$ and $\mu(L_1^{\sim})=\nu_2(L_1^{\sim})=1$ which implies $\mu(L_1\cap L_1^{\sim})=1$.But $L_1\cap L_1^{\sim}=\emptyset$ so $\mu(L_1\cap L_1^{\sim})=0$,a contradiction.$\nu_1=\nu_2$ (L_2) and thus there exists a one to one correspondence between $IR(\mathbf{L}_1)$ and $IR(\mathbf{L}_2)$ if $\mathbf{L}_1$ separates $\mathbf{L}_2$.

THEOREM 4.3: Let $\mathbf{L}_2\supseteq\mathbf{L}_1$,and $\mathbf{L}_1$ separate $\mathbf{L}_2$ then $\mathbf{L}_1$ is normal iff $\mathbf{L}_2$ is normal.

Proof: Assume that $\mathbf{L}_1$ is normal and let $L_2,L_2^{\sim}\varepsilon\mathbf{L}_2$ st $L_2\cap L_2^{\sim}=\emptyset$.Since $\mathbf{L}_1$ separates $\mathbf{L}_2$ there exists $L_1,L_1^{\sim}\varepsilon\mathbf{L}_1$ st $L_1\supseteq L_2$ $L_1^{\sim}\supseteq L_2^{\sim}$ and $L_1\cap L_1^{\sim}=\emptyset$.Now since $\mathbf{L}_1$ is normal there exists $L_3,L_4\varepsilon\mathbf{L}_1$ st $L_3'\supseteq L_1$ $L_4'\supseteq L_1$~.But $\mathbf{L}_2\supseteq\mathbf{L}_1$ and $L_3'\supseteq L_1\supseteq L_2$ and $L_4'\supseteq L_1^{\sim}\supseteq L_2^{\sim}$,and thus this implies that $\mathbf{L}_2$ is normal.

Conversely assume $\mathbf{L}_2$ is normal and let $\mu\varepsilon I(\mathbf{L}_1)$ and $\nu_1,\nu_2\varepsilon IR(\mathbf{L}_1)$ st $\mu\leq\nu_1$ $(\mathbf{L}_1)$ and $\mu\leq\nu2$ $(\mathbf{L}_1)$.Extend $\mu\varepsilon I(\mathbf{L}_1)$ to $\nu\varepsilon I(\mathbf{L}_2)$ .We know by theorem 4.2 that since $\mathbf{L}_1$ separates $\mathbf{L}_2$ there exists a one to one correspondence between $IR(\mathbf{L}_1)$ and $IR(\mathbf{L}_2)$.Thus projecting $\nu_1,\nu_2\varepsilon IR(\mathbf{L}1)$ up onto unique elements $\nu_3,\nu_4\varepsilon IR(L2)$ st $\nu_1=\nu_3|(\mathbf{L}_1)$ and $\nu_2=\nu_4|(\mathbf{L}_1)$.Also since $\mathbf{L}_1$ separates $\mathbf{L}_2$ $\nu\leq\nu_3$ and $\nu\leq\nu_4$ $(\mathbf{L}_2)$ (see theorem 4.6).Further since $\mathbf{L}_2$ is normal $\nu_3=\nu_4$ $(\mathbf{L}_2)$,then $\nu_1=\nu_2=\nu_3|=\nu_4|$ $(\mathbf{L}_1)$.This implies that $\mathbf{L}_1$ is normal.

THEOREM 4.4: Let $\mathbf{L}_1,\mathbf{L}_2$ be lattices such that $\mathbf{L}_1$ separates $\mathbf{L}_2$ then $\nu\varepsilon IR(\mathbf{L}_2)$ is $\mathbf{L}_1$ regular on $\mathbf{L}_2'$.Conversely if $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$ and the above condition holds for all such $\nu\varepsilon IR(\mathbf{L}_2)$, then $\mathbf{L}_1$ separates $\mathbf{L}_2$.

Proof: Let $\mathbf{L}_1$ separate $\mathbf{L}_2$ and let $\nu\varepsilon IR(\mathbf{L}_2)$ and let $L_2\varepsilon\mathbf{L}_2$ st $\nu(L_2')=1$.Since $\nu\varepsilon IR(\mathbf{L}_2)$ there exists $L_2^{\sim}\varepsilon\mathbf{L}_2$ st $L_2'\supseteq L_2^{\sim}$ st $\nu(L_2^{\sim})=1$ and $L_2\cap L_2^{\sim}=\emptyset$.Since $\mathbf{L}_1$ separates $\mathbf{L}_2$ there exists $L_1,L_1^{\sim}\varepsilon\mathbf{L}_1$ st $L_1\supseteq L_2,L_1^{\sim}\supseteq L_2^{\sim}$,and $L_1\cap L_1^{\sim}=\emptyset$.Since there exists a 1-1 correspondence between $IR(\mathbf{L}_1)$ and $IR(\mathbf{L}_2)$ there exists a unique $\mu\varepsilon IR(\mathbf{L}_1)$ st $\nu|=\mu$ $(\mathbf{L}_1)$. Since

$\nu(L_2{}^{\sim})=1,\nu(L_1{}^{\sim})=1$ and $L_1{}'\supseteq L_1{}^{\sim}$ implies that $\mu(L_1{}')=1$.But $L_2{}'\supseteq L_1{}'$ and since $\mu\varepsilon IR(\mathbf{L}_1)$ there exists $L\varepsilon\mathbf{L}_1$ st $L_2{}'\supseteq L_1{}'\supseteq L$ and $\mu(L)=\nu(L)|$.Therefore $\nu\varepsilon IR(\mathbf{L}_2)$ is $\mathbf{L}_1$ regular on $\mathbf{L}_2{}'$.

Conversely let $\mathbf{L}_1$ semi-separate $\mathbf{L}_2$ and let all $\nu\varepsilon IR(\mathbf{L}_2)$ be $\mathbf{L}_1$ regular on $\mathbf{L}_2{}'$.Assume that $\mathbf{L}_1$ does not separate $\mathbf{L}_2$.Then there exists $L_2,L_2{}^{\sim}\varepsilon\mathbf{L}_2$ st $L_2\cap L_2{}^{\sim}=\emptyset$,but $L_1\supseteq L_2,L_1{}^{\sim}\supseteq L_{2\sim}$ has that $L_1\cap L_1{}^{\sim}\neq\emptyset$ for all such $L_1,L_1{}^{\sim}$.Then $H=\{L \mid L\supseteq L_2 \text{ or } L\supseteq L_{2\sim}\ L\varepsilon\mathbf{L}_1\}$ has the fip and there exists a associated measure and thus a regular measure on $\mathbf{L}_1$ st $\mu(L)=1$ for $L\varepsilon H$ and $\mu\varepsilon IR(\mathbf{L}_1)$ .Since $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$,$L\cap L_2\neq\emptyset$ and $L\cap L_2{}^{\sim}\neq\emptyset$ for all $L\varepsilon H$.Therefore we can extend μ to measures $\nu_1,\nu_2\varepsilon IR(\mathbf{L}_2)$ such that $\nu_1(L_2)=1$ and $\nu_2(L_2{}^{\sim})=1$.Therfore $\nu_1(L_2{}^{\sim})=\nu_2(L_2)=0$ and hence $\nu_1(L_2{}^{\sim\prime})=\nu_2(L_2{}')=1$.Since ν_1 and ν_2 are $\mathbf{L}_1$ regular on $\mathbf{L}_2{}'$,there exists $L_3,L_4\varepsilon\mathbf{L}_1$ such that $L_2{}'\supseteq L_3$,$L_2{}^{\sim\prime}\supseteq L_4$ and $\nu_2(L_3)=\nu_1(L_4)=1$.Therefore $\mu(L_3)=\mu(L_4)=1$.Thus $\mu(L_3\cap L_4)=\nu_1(L_3\cap L_4)=1$,a contadiction since $L_2{}'\supseteq L_3\cap L_4$ and $\nu_1(L_2{}')=0$.

We next define the notion for two lattices of the weak going up property.

DEFINITION 4.4: Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two lattices st $\mathbf{L}_2\supseteq\mathbf{L}_1$ and let $\mu_1\varepsilon I(\mathbf{L}_1),\mu_2\varepsilon IR(\mathbf{L}_1)$,$\nu_1\varepsilon I(\mathbf{L}_2)$ with $\mu_1\leq\mu_2$ $(\mathbf{L}_1)$ and ν_1 an extension on $\mathbf{L}_2$ of μ_1 on $\mathbf{L}_1$,i.e. $\nu_1|=\mu_1$ $(\mathbf{L}_1)$. Then the weak going up property holds if there exists $\nu_2\varepsilon IR(\mathbf{L}_2)$ st $\nu_1\leq\nu_2$ $(\mathbf{L}_2)$,and $\mu_2=\nu_2|$.

THEOREM 4.5: Let $\mathbf{L}_1$ semi-separate $\mathbf{L}_2$ $(\mathbf{L}_2\supseteq\mathbf{L}_1)$ and let $\mathbf{L}_1$ be normal,then the weak going up property holds.

Proof: Let $\mu_1\varepsilon I(\mathbf{L}_1),\mu_2\varepsilon IR(\mathbf{L}_1)$ and $\nu_1\varepsilon I(\mathbf{L}_2)$ st $\mu_1\leq\mu_2$ $(\mathbf{L}_1)$ and ν_1 is an extension of μ_1 $\nu_1|=\mu_1$.Let $\nu_2\varepsilon IR(\mathbf{L}_2)$ be an element such that $\nu_1\leq\nu_2$ $(\mathbf{L}_2)$.Then since $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$ $\nu_2|=\mu$ $(\mathbf{L}_1)$ and $\mu\varepsilon IR(\mathbf{L}_1)$ and $\mu_1\leq\mu$ $(\mathbf{L}_1)$. Since $\mathbf{L}_1$ is normal and $\mu_1\leq\mu$ $(\mathbf{L}_1)$ and $\mu_1\leq\mu_2$ $(\mathbf{L}_1)$ we have $\mu_2=\nu_2|=\mu\varepsilon IR(\mathbf{L}_1)$ and ν_2 extends μ_2 and the weak going up property holds.

THEOREM 4.6: If $\mathbf{L}_1$ separates $\mathbf{L}_2$ then the weak going up property holds.

Proof: Suppose not and let $\mu_1\varepsilon I(\mathbf{L}_1),\mu_2\varepsilon IR(\mathbf{L}_1),\nu_1\varepsilon I(\mathbf{L}_2)$ and $\mu_1\leq\mu_2$ (L_1) and $\mu_1=\nu_1|$ $(\mathbf{L}_1)$. Also, let $\nu_2\varepsilon IR(\mathbf{L}_2)$ be st $\nu_2\varepsilon IR(\mathbf{L}_2)$ st $\nu_2|=\mu_2$ $(\mathbf{L}_1)$ and $\nu_1\leq\nu_2$ $(\mathbf{L}_2)$ does not hold. Then there exists $L_2\varepsilon\mathbf{L}_2$ st $\nu_1(L_2)=1$,$\nu_2(L_2)=0$ say or $\nu_2(L_2{}')=1$.Since $\nu_2\varepsilon IR(\mathbf{L}_2)$ there exists a $L_2{}^{\sim}\varepsilon\mathbf{L}_2$ st $\nu_2(L_2{}^{\sim})=1$ and $L_2{}'\supseteq L_2{}^{\sim}$.Also since $\mathbf{L}_1$ separates $\mathbf{L}_2$ there exists $L_1,L_1{}^{\sim}\varepsilon\mathbf{L}_1$ st $L_1\supseteq L_2,L_1{}^{\sim}\supseteq L_2{}^{\sim}$ and $L_1\cap L_1{}^{\sim}=\emptyset$.Then $\mu_1(L_1)=1$ and thus $\mu_2(L_1)=1$ since $\mu_1\leq\mu_2$ $(\mathbf{L}_1)$ .In addition $L_1{}^{\sim}\supseteq L_2{}^{\sim}$ therefore $\mu_2(L_1{}^{\sim})=1$,a contradiction.$\nu_1\leq\nu_2$ $(\mathbf{L}_2)$. Thus the weak going up property holds.

We have from theorem 4.2 that if $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$ then $\psi:IR(\mathbf{L}_2)\rightarrow IR(\mathbf{L}_1)$ the restriction map is defined .A converse holds for special lattices in the next theorem.

THEOREM 4.7: Let $\mathbf{L}_1,\mathbf{L}_2$ be lattices such that $\mathbf{L}_2\supseteq\mathbf{L}_1$,$\mathbf{L}_2$ is disjunctive and $\mathbf{L}_1$ is normal. Also suppose that $\psi:IR(\mathbf{L}_2)\rightarrow IR(\mathbf{L}_1)$ is defined where $IR(\mathbf{L}_1),IR(\mathbf{L}_2)$ have the wallman topology ,i.e. $\tau W_1(\mathbf{L}_1),\tau W_2(\mathbf{L}_2)$ are the respective lattices which define a topology on $IR(\mathbf{L}_1),IR(\mathbf{L}_2)$. Then $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$.

Proof: Suppose $L_1\varepsilon\mathbf{L}_1$ and $L_2\varepsilon\mathbf{L}_2$ and $L_1\cap L_2=\emptyset$.Then $W_2(L_1)\cap W_2(L_2)=\emptyset$,and also $\psi(W_2(L_2))\cap W_1(L_1)=\emptyset$.For if $\mu=\psi(\nu)$ where $\nu\varepsilon W_2(L_2)$ and $\nu(L_2)=1$ and $\nu(L_1)=\mu(L_1)=1$,a contradiction.Thus $\psi(W_2(L_2))\cap W_1(L_1)=\emptyset$.Second, $\psi(W_2(L_2))=\cap W_1(L_{1i})$ $i\varepsilon I$ an arbitrary index set ,and $L_{1i}\supseteq L_2$.This hold since $W_2(L_2)$ is closed and thus compact since the space $W_2(X)$ is compact and $W_2(X)\supseteq W_2(L_2)$.In addition ψ is continous since $\psi^{-1}(W_1(L_1))=W_2(L_1)$,$\mathbf{L}_1$ is normal which is equivalent to $W_1(\mathbf{L}_1)$ normal and thus T_2 by a known result .Therefore since $W_2(\mathbf{L}_2)$ is compact and since ψ is continuous then $\psi(W_2(L_2))$ is compact and since $W_1(\mathbf{L}_1)$ is T_2,$\psi(W_2(L_2))$ is closed and thus $\psi(W_2(L_2))=\cap W_1(L_{1i})$ $i\varepsilon I$ an arbitrary index set.Since $\mathbf{L}_2$ is disjunctive and since $\psi;IR(\mathbf{L}_2)\rightarrow IR(\mathbf{L}_1)$ is defined,$\mathbf{L}_1$ is disjunctive.But this implies $L_{1i}\supseteq L_2$.Thus $\psi(W_2(L_2))=\cap W_1(L_{1i})$,$i\varepsilon I$ and $L_{1i}\supseteq L_2$.

Now look at $\psi(W_2(L_2))\cap W_1(L1)=(\cap W_1(L_{1i})\cap W_1(L_1)=\emptyset$.Then by compactness $(\cap W_1(L_{1\alpha})\cap W_1(L_1)=\emptyset,\alpha=1,2,...N$.Since $\mathbf{L}_1$ is disjunctive,this implies that $\cap L_{1\alpha}\supseteq L_2$,$L_1{}^{\sim}=\cap L_{1\alpha},L_1{}^{\sim}\varepsilon\mathbf{L}1$ and $L_1\cap L_1{}^{\sim}=\emptyset$.Thus $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$.

DEFINITION 4.5: Let $\mu\varepsilon I(\mathbf{L})$ and define for E,st $X\supseteq E$, $\mu^{\sim}(E)=\inf\mu(L_1)$ where $L_1\varepsilon\mathbf{L}_1$.

We now state and prove a theorem giving necessary and sufficent conditions for semi-separation of lattices $\mathbf{L}_2\supseteq\mathbf{L}_1$.

THEOREM 4.8: $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$ iff $\mu'=\mu^{\sim}$ on $\mathbf{L}_2$ where $\mu\varepsilon IR(\mathbf{L}_1)$.

Proof: Look at $\mu'(L_2)=\inf\mu(L_1')$ $L_1'\supseteq L_2$.Then since $L_1\cap L_2=\emptyset$,and $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$ there exists a $L_1{}^{\sim}\varepsilon\mathbf{L}_1$ st $L_1{}^{\sim}\supseteq L_2$ and $L_1{}^{\sim}\cap L_1=\emptyset$.$L_1'\supseteq L_1{}^{\sim}$ thus $\inf\mu(L_1')\geq\inf\mu(L_1{}^{\sim})$ $\mu'\geq\mu^{\sim}$ on $\mathbf{L}2$.Now look at $\mu^{\sim}(L_2)$ assume that $\mu^{\sim}(L_2)=0$.Then there exists a $L_1{}^{\sim}\varepsilon\mathbf{L}_1$ st $L_1{}^{\sim}\supseteq L_2$ and $\mu(L_1{}^{\sim})=0$ or $\mu(L_1{}^{\sim\prime})=1$.Since $\mu\varepsilon IR(\mathbf{L}_1)$ there exists a $L_3\varepsilon\mathbf{L}_1$ st $L_1{}^{\sim\prime}\supseteq L_3$ $\mu(L_3)=1$ or $\mu(L_3')=0$ and $L_3'\supseteq L_1{}^{\sim}\supseteq L_2$ or $\mu'(L_2)=\mu^{\sim}(L_2)=0$.Thus $\mu'=\mu^{\sim}$ on $\mathbf{L}_2$.

Conversely assume that $\mathbf{L}_1$ does not semi-separate $\mathbf{L}_2$ then there exists $L_1\varepsilon\mathbf{L}_1$,and $L_2\varepsilon\mathbf{L}_2$ st $L_1\cap L_2=\emptyset$ and $L_1\cap L_1{}^{\sim}\neq\emptyset$ $L_1{}^{\sim}\supseteq L_2$ and $L_1{}^{\sim}\varepsilon\mathbf{L}_1$.Look at $H=\{L_1{}^{\sim} \mid L_1{}^{\sim}\supseteq L_2,L_1{}^{\sim}\varepsilon\mathbf{L}_1\}$. Then H has the fip and there exists a filter and thus an ultrfilter and its associated measure $\mu\varepsilon IR(\mathbf{L}_1)$ st $\mu(L_1{}^{\sim})=1,L_1{}^{\sim}\varepsilon H$ and since $L_1\cap L_1{}^{\sim}\neq\emptyset,\mu(L_1)=1$.Now look at $\mu'(L_2)$.Since $L_1\cap L_2=\emptyset$ then $L_1'\supseteq L_2$ and since $\mu(L_1)=1,\mu(L_1')=0$,and thus $\mu'(L_2)=\inf\mu(L_3')=0$ $L_3'\supseteq L_2$,and $L_3\varepsilon\mathbf{L}_1$.Now look at $\mu^{\sim}(L_2)=\inf\mu(L_4)$ $L_4\supseteq L_2,L_4\varepsilon\mathbf{L}_1$ then every such L_4 is a member of H and thus $\mu^{\sim}(L_2)=\inf\mu(L_4)=1$,a contradiction.Thus $\mathbf{L}_1$ must semi-separate $\mathbf{L}_2$.

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