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THE NUCLEI AND OTHER PROPERTIES OF p-PRIMITIVE SEMIFIELD PLANES

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ABSTRACT. In this article the nuclei of p-primitive semifield planes are studied. The behavior of this class of planes under the operations of derivation, transposition and dualization is also analyzed.

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1. INTRODUCTION.

A p-primitive semifield plane is a semifield plane of order p^4, where p is an odd prime, and kernel $K \cong GF(p^2)$ that admits a p-primitive Baer collineation α (i.e. α is a Baer collineation whose order is a p-primitive divisor of p^2-1 that is, $|\alpha| \mid p^2-1$ but $|\alpha| \nmid p-1$).

p-primitive semifield planes have been studied in [1], [2], [3] and indirectly in [4] as this is precisely the class of planes obtained when the construction method in [3] is applied to the Desarguesian plane of order p^2.

In this article the nuclei of p-primitive semifield planes are studied. It is proved, (2.1), that there are only two possibilities for the nuclei: either $\mathcal{N}_m = \mathcal{N}_r = \mathcal{N}_l \cong GF(p^2)$ or $\mathcal{N}_m = \mathcal{N}_r \cong GF(p)$. The behavior of p-primitive semifield planes under the operations of derivation, transposition and dualization is also analyzed. The notation is that of [1].

2. THE NUCLEI OF p-PRIMITIVE SEMIFIELD PLANES.

THEOREM 2.1. Let $\pi(f_0, f_1)$ be a p-primitive semifield plane and let $\mathcal{N}_m, \mathcal{N}_r, \mathcal{N}_l$ be its middle, right and left nucleus, respectively. Then exactly one of the following holds:

(i) $\mathcal{N}_m = \mathcal{N}_l = \mathcal{N}_r \cong GF(p^2)$

or

(ii) $\mathcal{N}_m = \mathcal{N}_r \cong GF(p)$.

Moreover, we have

(i) holds $\Leftrightarrow f_0 = 0$

(ii) holds $\Leftrightarrow f_0 \neq 0$.

PROOF. The proof is divided into two steps (2.2) and (2.3)

LEMMA 2.2. Let $\pi(f)$ be a p-primitive semifield plane and let $\mathcal{N}_l$ be its left nucleus. Then

$$\mathcal{N}_l = \{(a,0): a \in GF(p^2)\}$$

PROOF. Let $a \in GF(p^2)$. By direct computation we show that $(a,0) \in \mathcal{N}_l$.

$$((a,0)(x_1,x_2))(y_1,y_2) = \left[(a,0)\begin{bmatrix} x_1 & x_2 \\ f(x_2) & x_1^p \end{bmatrix}\right](y_1,y_2)$$

$$= (ax_1, ax_2)\begin{bmatrix} y_1 & y_2 \\ f(y_2) & y_1^p \end{bmatrix}$$
$$= (ax_1y_1 + ax_2f(y_2), ax_1y_2 + ax_2y_1^p)$$

On the other hand

$$(a,0)((x_1,x_2)(y_1,y_2)) = (a,0)\begin{bmatrix} x_1y_1 + x_2f(y_2) & x_1y_2 + x_2y_1^p \\ f(x_1y_2 + x_2y_1^p) & (x_1y_1 + x_2f(y_2))^p \end{bmatrix}$$
$$= (ax_1y_1 + ax_2f(y), ax_1y_2 + ax_2y_1^p).$$

Thus $((a,0)(x_1,x_2))(y_1,y_2) = (a,0)((x_1,x_2)(y_1,y_2))$ for every $x_1,x_2,y_1,y_2 \in GG(P)$ and therefore,

$$\{(a,0): a \in GF(p^2)\} \subseteq \mathcal{N}_\ell.$$

Since $|\mathcal{N}_\ell| = p^2$ we have the result.

LEMMA 2.3. Let $\pi(f_0, f_1)$ be a p-primitive semifield plane and let $\mathcal{N}_m$ and $\mathcal{N}_r$ be its middle and right nucleus, respectively. Then

$$\mathcal{N}_m = \{(n,0): f_0n^p = nf_0\} = \mathcal{N}_r.$$

PROOF. Let $(n,m) \in \mathcal{N}_m$. Then for every $x_1, x_2, y_1, y_2 \in GF(p^2)$ we must have

$$((x_1,x_2)(n,m))(y_1,y_2) = (x_1,x_2)((n,m)(y_1,y_2))$$

$$(x_1,x_2)\begin{bmatrix} n & m \\ f(m) & n^p \end{bmatrix} \cdot \begin{bmatrix} y_1 & y_2 \\ f(y_2) & y_1^p \end{bmatrix} = (x_1,x_2)(n,m)\begin{bmatrix} y_1 & y_2 \\ f(y_2) & y_1^p \end{bmatrix}$$

$$= (x_1,x_2)\begin{bmatrix} ny_1 + mf(y_2) & ny_2 + my_1^p \\ y_1f(m) + f(y_2)n^p & f(m)y_2 + n^py_1^p \end{bmatrix}$$

$$= (x_1,x_2)\begin{bmatrix} ny_1 + mf(y_2) & ny_2 + my_1^p \\ f(ny_2 + my_1^p) & (ny_1 + mf(y_2))^p \end{bmatrix}$$

Therefore the following two equations must hold for every $y_1, y_2 \in GF(p)$.

(1) $y_1f(m) + f(y_2)n^p = f(ny_2 + my_1^p)$

(2) $y_2f(m) + y_1^pn^p = (ny_1 + mf(y_2))^p$

From (2) we have

$$y_2f(m) = f(y_2)m)^p$$

Suppose that $m \neq 0$. Then we have $f(y_2) = \frac{f(m)^p}{m} \cdot y_2^p$ and hence $f_0 = 0$. Also, from (2) with $f_0 = 0$ we have

$$y_2^pf_1^pm = mf_1y_2^p.$$

Thus $f_1^p = f_1$ and hence $f_1 \in GF(p)$. But if $f_0 = 0$ then $f_1 \notin GF(p)$ [1, (2.1)]. Therefore we must have $m = 0$.

Now (1) becomes

$$f(y_2)n^p = f(ny_2)$$

i.e., $f_0 y_2 n^p + f_1 y_2^p n^p = f_0 n y_2 + f_1 n^p y_2^p$. So, $f_0 y_2 n^p = f_0 n y_2$ for every $y_2 \in GF(p)$. In particular if $y_2 \neq 0$ we must have

$$f_0 n^p = f_0 n.$$

Conversely, if n satisfies $f_0 n^p = f_0 n$ then

$$((x_1,x_2)(n,0))(y_1,y_2) = \left[(x_1,x_2)\begin{bmatrix} n & 0 \\ 0 & n^p \end{bmatrix}\right](y_1,y_2)$$

$$= (x_1 n, x_2 n^p)\begin{bmatrix} y_1 & y_2 \\ f(y_2) & y_1^p \end{bmatrix}$$

$$= (x_1 n y_1 + x_2 n^p f(y_2), x_1 n y_2 + x_2 n^p y_1^p)$$

$$= (x_1 n y_1 + x_2 f_0 n^p y_2 + x_2 f_1 n^p y_2^p, x_1 n y_2 + x_2 n^p y_1^p)$$

$$= (x_1 n y_1 + x_2 f(n y_2), x_1 n y_2 + x_2 (n y_1)^p)$$

$$= (x_1,x_2)\begin{bmatrix} n y_1 & n y_2 \\ f(n y_2) & (n y_1)^p \end{bmatrix}$$

$$= (x_1,x_2)((n,0)(y_1,y_2))$$

and therefore $(n,0) \in \mathcal{N}_m$. Therefore, we have

$$\mathcal{N}_m = \{(n,0) \colon f_0 n^p = f_0 n\}.$$

By similar computations, we obtain

$$\mathcal{N}_r = \{(n,0) \colon f_0 n^p = f_0 n\}.$$

Now, by (2.3), $(n,m) \in \mathcal{N}_m$ if and only if $m = 0$ and $n^p f_0 = n f_0$. Thus $\mathcal{N}_m \cong \{(n,0) \colon n \in GF(p)\} \cong Gf(p)$ if and only if $f_0 \neq 0$, and from (2.2) we get $\mathcal{N}_m = \mathcal{N}_\ell$ if and only if $f_0 = 0$. The results for $\mathcal{N}_r$ are obtained in a similar way.

3. OPERATIONS IN p-PRIMITIVE SEMIFIELD PLANES.

DERIVATION. The first general geometric process discovered for constructing new affine planes from given ones is the process of derivation which was invented by T.G. Ostrom. In derivation, a collection of lines in a given plane π, "the derivable net," is replaced by a suitable collection of Baer subplanes in π to form a new affine plane [5].

Let $\pi = \pi(f)$ be a p-primitive semifield plane. Then Hiramine et. al., [3], showed that π is derivable. In particular the set of components of π contains the derivable net

$$x = 0,\ y = x\begin{bmatrix} u & 0 \\ 0 & u^p \end{bmatrix}$$

for every $u \in GF(p^2)$. The translation plane derived from replacing this net is not p-primitive; it has order p^4 and *kernel* $GF(p)$ [3].

TRANSPOSITION. Another construction technique which was investigated by Knuth [6] is the following:

Let π be a translation plane with matrix spread set $\mathcal{M} = \{\mathcal{M}_i\}$. Then taking the transpose of each matrix $\mathcal{M}_i$ in $\mathcal{M}$, we obtain a matrix spread set $\mathcal{M}^t = \{\mathcal{M}_i^t\}$ which gives a translation plane, π^t, called the transposed plane [6].

On the transpose of a p-primitive semifield plane we prove the following:

THEOREM 3.1. *Let $\pi = \pi(f)$ be a p-primitive semifield plane and let π^t denote the transpose plane. Then $\pi^t \cong \pi$.*

PROOF. Let

$$M(u,v) = \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} \text{ for } u,v \in GF(p^2).$$

Then $\mathcal{S} = \{M(u,v): u,v \in GF(p^2)\}$ is a matrix spread set for π and

$$\mathcal{S}^t = \{M(u,v)^t: u,v \in GF(p^2)\}$$

is a matrix spread set for π^t.

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ induces an isomorphism between the two planes, since

$$A^{-1}M(u,v)A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} u^p & f(v) \\ v & u^p \end{bmatrix}$$

$$= \begin{bmatrix} z & f(v) \\ v & z^p \end{bmatrix} = M(z,v)^t.$$

DUALIZATION. If π is a semifield plane with associated semifield $(S, +, \cdot)$ then the dual plane, π^D has associated semifield $(S, +, *)$ where

$$a*b = b \cdot a$$

for every $a,b \in S$. It follows that

$$\mathcal{N}_\ell(\pi) = \mathcal{N}_r(\pi^D)$$

and

$$\mathcal{N}_r(\pi) = \mathcal{N}_\ell(\pi^D).$$

THEOREM 3.2. Let $\pi = \pi(f_0, f_1)$ be a p-primitive semifield plane. If the dual plane π^D is also p-primitive then π is a Hughes-Kleinfeld semifield plane.

PROOF. Let π be a p-primitive. Then $\mathcal{N}_\ell(\pi) \cong GF(p^2)$ and if π^D is also p-primitive then $\mathcal{N}_l(\pi^D) \cong GF(p^2)$. Since $\mathcal{N}_\ell(\pi^D) = \mathcal{N}_r(\pi)$ we have $\mathcal{N}_r(\pi) \cong GF(p^2)$. Thus, $\mathcal{N}_\ell(\pi) \cong \mathcal{N}_m(\pi) \cong \mathcal{N}_r(\pi)$ by (2.1) and $f_0 = 0$. Therefore π is a Hughes-Kleinfeld semifield plane (by [1, 3.3]).

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