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SEMI-SIMPLICITY OF A PROPER WEAK H^*-ALGEBRA

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ABSTRACT. A weak right $H^\star$-algebra is a Banach algebra A which is a Hilbert space and which has a dense subset D_r with the property that for each x in D_r there exists x^r such that $(yx, z) = (y, zx^r)$ for all y, z in A. It is shown that a **proper** (each x^r is unique) weak right $H^\star$-algebra is semi-simple. Also there is an example of weak right $H^\star$-algebra which is not a left $H^\star$-algebra.

KEY WORDS AND PHRASES. Hilbert algebra, $H^\star$-algebra, weak right $H^\star$-algebra, weak left $H^\star$-algebra, complemented algebra, right complemented algebra, left complemented algebra.
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1. INTRODUCTION.

Assumption of semi-simplicity plays an important role in the theory of complemented algebras. It was noted in the author's last paper (Saworotnow [1]) that it is rather difficult to deduct semi-simplicity from axioms of a (proper) weak right $H^\star$-algebra. However, there is a different story for the case of a two-sided (weak) $H^\star$-algebra. Here it is not too difficult to show that each closed two-sided ideal has an idempotent which, in turn, implies semi-simplicity. But it was established in Saworotnow [1] that each proper weak right $H^\star$-algebra is also a weak left $H^\star$-algebra. It follows that each proper right $H^\star$-algebra is semi-simple (Theorem 2 below). This is the central result of this paper. We included also important consequences of it and an example of an algebra which is a right $H^\star$-algebra but not a left $H^\star$-algebra. The algebra in the example is also an example of a weak right $H^\star$-algebra which is not a weak left $H^\star$-algebra

2. PRELIMINARIES.

A **weak right** $H^\star$-algebra (Saworotnow [1]) is a Hilbert algebra A (a Banach algebra which is a Hilbert space) which has a dense subset D_r with the property that for each $a \in D_r$ there is a member a^r of D_r such that $(xa, y) = (x, ya^r)$ for all $x, y \in A$; a^r is called the right adjoint of a. It is said to be proper if a^r is unique for every a in D_r; this is equivalent to the condition that the right annihilator $r(A) = \{x \in A : Ax = 0\}$ of A consists of zero alone (A is proper if and only if $r(A) = (0)$).

We define weak left H^*-algebra in a similar way. Weak two-sided H^*-algebra is a weak right H^*-algebra which is also a (weak) left H^*-algebra.

THEOREM 1. Every weak right H^*-algebra is a right complemented algebra (Saworotnow [2]), i.e., the orthogonal complement R^p of any right ideal R in A is also a right ideal.

PROOF. If $x \in R$ and $a \in A$, then $(xa, y) = lim(xa_n, y) = lim(x, ya_n^r) = 0$ for some sequence $\{a_n\} \subset D_r$ converging to a and each $y \in R$. This implies that R^p is also a right ideal.

PROPOSITION 1. The orthogonal complement I^p of each two-sided I in a weak right H^*-algebra A is again a weak right H^*- algebra. (Note that we do not allege I itself to be a weak right H^*-algebra.)

PROOF. First note that $I^pI \subset I^p \cap I = (0)$, i.e., $xy = 0$ for all $x \in I^p, y \in I$.

Now consider $a \in I^p$ and let $\epsilon > 0$ be arbitrary. Take $b \in D_r$ so that $\| a - b \| < \epsilon$ and write $b = b_1 + b_2$, $b^r = c_1 + c_2$ with $b_1, c_1 \in I^p$ and $b_2, c_2 \in I$. Then $\| a - b_1 \| < \epsilon$ and we have for each $x, y \in I^p$:

$$(xb_1, y) = (xb_1 + xb_2, y) = (xb, y) = (x, yb^r) = (x, yc_1 + yc_2) = (x, yc_1), \tag{2.1}$$

which simply means that c_1 is a right adjoint of b_1. Thus: every neighborhood of a contains a vector having a right adjoint.

PROPOSITION 2. Each closed two-sided ideal I in a **proper** weak right H^*-algebra A is a proper weak right H^*-algebra. In fact, it is also a weak left H^*-algebra.

PROOF. It was shown in Saworotnow [1] that A is also a proper weak left H^*-algebra. This means that I^p is also a left ideal (we can use here the proof of Theorem 1 above). Thus: I is the orthogonal complement of a two-sided ideal. Proposition 2 now follows from Proposition 1 (I is the orthogonal complement of the two-sided ideal I^p); the fact that I is proper is also easy to establish.

3. MAIN THEOREM.

Now we can prove our main result.

THEOREM 2. Every **proper** weak right H^*-algebra A is semi-simple.

PROOF. Proposition 2 implies that the radical (Jacobson [3]) R of A is a right H^*-algebra. Hence it contains a non-zero vector a having a (unique) right adjoint $a^r \neq 0$. Then $aa^r \neq 0$ (otherwise $\| xa \|^2 = (x, xaa^r) = 0$ for each $x \in A$) and as in 27A of Loomis [4] one can show that, for some scalar λ, the sequence $\{\lambda aa^r\}^{2n}$ converges to some idempotent $e \in R$. This is impossible since every member of R is a generalized nilpotent (Theorem 16, page 309 in Jacobson [3]).

An important consequence of this theorem is the fact that we can now apply to the algebra A the theory of complemented algebras developed in Saworotnow [2] and Saworotnow [5] (more specifically: Theorem 1 in Saworotnow [2] and Theorem 3 in Saworotnow [5]). We summarize it as follows:

THEOREM 3. Every proper weak right H^*- algebra is a direct sum of simple weak right H^*- algebras, each of which is a semi-simple.

THEOREM 4. For each proper simple weak right H^*-algebra A there is a Hilbert space H and a positive self-adjoint norm-increasing operator α on H such that A is isomorphic and isometric to the algebra of all Hilbert Schmidt operators a on H such that $a\alpha$ is also of Hilbert Schmidt type.

This means that each simple proper weak right (as well as left) H^*-algebra is of the type described in the Example on page 56 of Saworotnow [5].

4. AN EXAMPLE.

To conclude the paper, we give an example of a right H^*-algebra which is not a weak left H^*-algebra. This example shows that our assumption of an algebra to be proper is rather essential.

EXAMPLE. Let A be the algebra of all 2 x 2 matrices and let

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

Consider the subalgebra A_0 of A generated by e_1 and e_{21}, $A_0 = \{\lambda e_1 + \mu e_{21} : \lambda, \mu \text{ complex}\}$. Then A_0 is a right (as well as a weak right) H^*-algebra (note that $\overline{\lambda} e_1$ is a right adjoint of $\lambda e_1 + \mu e_{21}$). But A_0 could not be a left weak H^*-algebra since the orthogonal complement $L^p = \{e_1\}$ of the left ideal $L = \{e_{21}\}$ is not a left ideal (here $\{x\}$ denotes the 1-dimensional subspace of A generated by x). Note that $r(A_0) = (0)$ and $\ell(A_0) = L$ (here $\ell(A_0)$ denotes the left annihilator of A_0, $\ell(A_0) = \{x : xA = 0\}$).

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