Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 319-322

THE ZEROS OF $az^2 J_\nu''(z) + bzJ_\nu'(z) + cJ_\nu(z)$ AS FUNCTIONS OF ORDER

A. McD. MERCER
Department of Mathematics and Statistics
University of Guelph
Ontario, N1G 2W1

(Received July 2, 1991 and in revised form October 3, 1991)

ABSTRACT. If $j''_{\nu k}$ denotes the k^{th} positive zero of the Bessel function $J''_\nu(x)$, it has been shown recently by Lorch and Szego [2] that $j''_{\nu 1}$ increases with ν in $\nu > 0$ and that (with k fixed in $2,3,\cdots$) $j''_{\nu k}$ increases in $0 < \nu \leq$ 3838. Furthermore, Wong and Lang have now extended the latter result, as well, to the range $\nu > 0$. The present paper, by using a different kind of analysis, re-obtains these conclusions as a special case of a more general result concerning the positive zeros of the function $az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$. Here, the constants a, b and c are subject to certain mild restrictions.

KEY WORDS AND PHRASES. Bessel functions, zeros, eigenvalues, boundary-value problems, ordinary differential equations.
1991 AMS SUBJECT CLASSIFICATION CODE. 33A40.

1. INTRODUCTION.

Let $J_\nu(x)$ be the Bessel function of the first kind and let $j_{\nu k}$, $j'_{\nu k}$ and $j''_{\nu k}$ denote the k^{th} positive zeros of J_ν, J'_ν and J''_ν respectively. It is well-known that $j_{\nu k}$ and $j'_{\nu k}$ (with k fixed in $1,2,\cdots$) are increasing functions of ν in $\nu > 0$ [1]. Recently, in [2], it has been shown that $j''_{\nu 1}$ increases in $\nu > 0$ and that (with k fixed in $2,3,\cdots$) $j''_{\nu k}$ increases in $0 < \nu \leq$ 3838. Also, in [3], Wong and Lang have extended these results to conclude that (with k fixed in $1,2,\cdots$) $j''_{\nu k}$ increases in $\nu > 0$. Each of the papers [2] and [3] contains some very delicate analysis.

In another recent paper [4] it has been proved that the k^{th} positive zero of $\mu J_\nu(x) + xJ'_\nu(x)$ increases with ν in $\nu > 0$. Of course, this last result provides no information about $j''_{\nu k}$.

It is the purpose of the present note to show the following: if $\lambda_{\nu k}$ denotes the k^{th} positive zero of $az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$ then (with k fixed in $1,2,\cdots$) $\lambda_{\nu k}$ is an increasing function of ν in $0 < \nu < \infty$, provided suitable restrictions are placed on the parameters a, b and c. These restrictions will allow the case of $j''_{\nu k}$ to be included and so some results appearing in [2] and [3] will be re-obtained, as special cases, by the use of some comparatively simple analysis. The method we shall use is based on a study of [4].

2. THE BOUNDARY-VALUE PROBLEM.

We shall consider the problem:

$$(x,y') = \frac{\nu^2}{x}\, y - \lambda^2 x y \tag{2.1}$$

subject to $y(0)$ is bounded and

$$ay''(1) + by'(1) + cy(1) = 0$$

Since the differential equation gives $y''(1) + y'(1) = (\nu^2 - \lambda^2)\, y(1)$, the second boundary condition here can also be written as

$$\Big[a(\nu^2-\lambda^2)+c\Big]y(1)+qy'(1)=0 \qquad \text{(where } q\equiv b-a) \tag{2.2}$$

There is clearly no loss of generality if we take $c \geq 0$. Also, since the case $q=0$ is essentially the case of studying the zeros $j_{\nu k}$, which is classical, we shall henceforth assume that $q \neq 0$. In the analysis to follow, certain restrictions will be placed on these parameters (except in 5(a) and Theorem 2) so now, at the outset, we state these once and for all and refer to them as "Condition $\mathcal{P}$". They are

$$(c=0 \text{ and } q\neq 0) \quad \text{or} \quad (c>0 \text{ and } q>0) \tag{$\mathcal{P}$}$$

It is a trivial matter to solve the above boundary-value problem and we see that its solution is

$$y(z)\equiv AJ_\nu(\lambda x)$$

where A is an arbitrary constant and λ is a zero of

$$az^2J''_\nu(z)+bzJ'_\nu(z)+cJ_\nu(z)\equiv\Big[a(\nu^2-z^2)+c\Big]J_\nu(z)+qzJ'_\nu(z)$$

Clearly, this function possesses an infinity of real positive zeros which we enumerate as $\lambda_{\nu k}$ $(k=1,2,\cdots)$.

3. AN EXPRESSION FOR $\wedge_{\nu k}$.

We shall write $\wedge_{\nu k}\equiv\lambda^2_{\nu k}$ and, whenever there is no risk of confusion, simply write $\wedge$ and $\wedge_{\nu k}$. The eigenfunction corresponding to $\wedge_{\nu k}$ will be $y_{\nu k}$ or simply y.

Let $\wedge$ and y be an eigen-solution of (2.1). Multiply (2.1) by y and integrate over (0,1). We get

$$y'(1)y(1)-\int_0^1 x\big[y'(x)\big]^2dx=\nu^2\int_0^1\tfrac{1}{x}\,y^2(x)\,dx-\wedge\int_0^1 xy^2(x)\,dx$$

To deal with the first term here, multiply (2.2) and $y(1)$ and use this to replace the first term by $-\frac{1}{q}\Big[a(\nu^2-\wedge)+c\Big]y^2(1)$. After simplification we get

$$\wedge\, Q=\nu^2P+\int_0^1 x\big[y'(x)\big]^2dx+\tfrac{c}{q}\,y^2(1) \tag{3.1}$$

where

$$P\equiv\int_0^1\tfrac{1}{x}\,y^2(x)\,dx+\tfrac{a}{q}\,y^2(1) \tag{3.2}$$

and

$$Q\equiv\int_0^1 xy^2(x)\,dx+\tfrac{a}{q}\,y^2(1) \tag{3.3}$$

4. AN EXPRESSION FOR $\dfrac{\partial\wedge_{\nu k}}{\partial\nu}$.

We follow the technique of [4]. But here the eigenvalue appears in the boundary condition as well as in the differential equation. For a fixed k we let the variable ν change to $\nu+\varepsilon$. Again we shall avoid the use of suffices by letting the 'before and after' values of $\wedge_{\nu k}$ and $y_{\nu k}(x)$ be denoted by $\wedge$ and Γ and by y and v respectively.

From (2.1) we get

$$(xy')'=\frac{\nu^2}{x}\,y-\wedge\, xy \quad \text{and} \quad (xv')'=\frac{(\nu+\varepsilon)^2}{x}\,v-\Gamma xv$$

Multiplying the former of these by v, the latter by y, subtracting and integrating over (0,1), we get

$$y'(1)v(1)-v'(1)y(1)=\Big[\nu^2-(\nu+\varepsilon)^2\Big]\int_0^1\tfrac{1}{x}yv\,dx-(\wedge-\Gamma)\int_0^1 xyv\,dx \tag{4.1}$$

Now from (2.2) we also have

$$\Big[a(\nu^2-\wedge)+c\Big]y(1)+q\,y'(1)=0$$

and

$$\Big[a((\nu+\varepsilon)^2-\Gamma)+c\Big]v(1)+q\,v'(1)=0$$

We now use these to remove $y'(1)$ and $v'(1)$ from the first term in (4.1) and, after simplifying, we get

$$-\tfrac{1}{q}v(1)y(1)\left\{a(\nu^2-(\nu+\varepsilon)^2)-a(\Lambda-\Gamma)\right\}=\left\{\nu^2-(\nu+\varepsilon)^2\right\}\int_0^1 \tfrac{1}{x}yv\,dx-(\Lambda-\Gamma)\int_0^1 xyv\,dx$$

Now as $\varepsilon \to 0$

$$\frac{\nu^2-(\nu+\varepsilon)^2}{-\varepsilon}\to 2\nu \quad \text{and} \quad \frac{\Lambda-\Gamma}{-\varepsilon}\to\frac{\partial\Lambda}{\partial\nu}$$

so dividing this last equation by $-\varepsilon$ and letting $\varepsilon \to 0$ and simplifying, we get

$$2\nu\, P=\frac{\partial\Lambda}{\partial\nu}\,Q \tag{4.2}$$

where P and Q are given by (3.2) and (3.3) above.

Before continuing, a word concerning the limiting processes which have taken place here, in passing from (4.1) to (4.2), seems to be in order. If the zero $\lambda_{\mu k}$ (for a fixed k) is known to be a continuous function of μ, it will follow that so is $J_\mu(\lambda_{\mu k}x)$, for each x in (0,1). The inequality

$$|\,xJ_{\nu k}(\lambda_{\nu k}x)J_{\mu k}(\lambda_{\mu k}x)\,|\le|\,xJ_{\nu k}(\lambda_{\nu k}x)\,|$$

along with Lebesgue's theorem of dominated convergence will then allow us to conclude that

$$\int_0^1 xyv\,dx \to \int_0^1 xy^2\,dx \quad \text{as} \quad \mu\to\nu \quad (\mu\equiv\nu+\varepsilon)\,,$$

for example. The other terms in (4.1) can be dealt with similarly. The fact that $\lambda_{\mu k}$ is indeed a continuous function of μ can be established by analysis analogous to that in [1] (p. 246). It is interesting to note that, in the present case, it is merely the continuity and not the differentiability of $\lambda_{\mu k}$ with respect to μ which is needed at the outset.

5. CONCLUSIONS.

For ease of reference, let us write out equations (3.1) and (4.2) again. They are

$$\Lambda\; Q=\nu^2\,P+\int_0^1 x[y'(x)]^2\,x+\tfrac{c}{q}\,y^2(1) \tag{3.1}$$

and

$$2\nu\, P=\frac{\partial\Lambda}{\partial\nu}\,Q \tag{4.2}$$

in which $\Lambda\equiv\Lambda_{\nu k}$, $y\equiv y_{\nu k}$ and P and Q are given by equations (3.2) and (3.3). In particular we note that

$$P-Q=\int_0^1\left(\tfrac{1}{x}-x\right)y^2(x)\,dx \quad \text{so that} \quad P-Q>0 \tag{5.1}$$

This inequality, along with (4.2), shows that Q is never zero. For a given ν and eigen-solution $\lambda_{\nu k}$, $y_{\nu k}$ it will, in general, not be known whether Q is positive or negative. So let us examine the consequences of each possibility.

(a) The case of $Q>0$.

Since Q is homogeneous in y, we can choose the real arbitrary constant A of the solution to make $Q=1$. Then from (4.2) and (5.1) we get

$$\frac{\partial\Lambda}{\partial\nu}=2\nu\,P=2\nu\left\{1+\int_0^1\left(\tfrac{1}{x}-x\right)y^2(x)\,dx\right\}$$

and since the integral here is positive we find that $\frac{\partial}{\partial\nu}(\Lambda-\nu^2)>0$ for such an eigen-solution.

(b) The case of $Q<0$.

Let us now require that 'Condition $\mathcal{P}$' applies to the parameters. We eliminate P from (3.1) and (4.2) which gives

$$\left\{2\nu\Lambda-\nu^2\frac{\partial\Lambda}{\partial\nu}\right\}Q=2\nu\left\{\int_0^1 x[y'(x)]^2\,dx+\tfrac{c}{q}\,y^2(1)\right\}$$

By 'Condition $\mathcal{P}$' the right hand side is positive. And since Q is negative, it must be the case that

$$\frac{\partial\Lambda}{\partial\nu}-\frac{2\Lambda}{\nu}>0$$

so that $\frac{\partial}{\partial \nu}\left(\frac{\Lambda}{\nu^2}\right) > 0$ for such an eigen-solution.

Since we shall usually not know which of these cases we are in, let us impose 'Condition $\mathfrak{P}$' generally. On replacing Λ by λ^2 in the above conclusions we then obtain the following result:

THEOREM 1. If 'Condition $\mathfrak{P}$' applies to the parameters c and q ($\equiv b-a$) then (with k fixed in $1,2,\cdots$) the positive zero $\lambda_{\nu k}$ of $az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu(z)$ is an increasing function of ν in $\nu>0$. In particular, on taking $a=1$, $b=c=0$, we find that $j_{\nu k}''$ increases with ν in $\nu>0$.

There is one set of circumstances in which we can be sure that Q is positive for all eigen-solutions; that is when $\frac{a}{q}\geq 0$ (see equation (3.3)). In this case there is no need to invoke 'Condition $\mathfrak{P}$'. The result is:

THEOREM 2. Subject only to the restriction that $\frac{a}{q}(\equiv a/(b-a))\geq 0$, $\lambda_{\nu k}^2-\nu^2$ is an increasing function of ν for $\nu>0$. Here, again, $\lambda_{\nu k}$ denotes the k^{th} positive zero of $az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu(z)$ and k is fixed in the range $1,2,\cdots$.

We shall conclude with the following note. In the case in which $a=1$, and $b=c=0$(and so $q=-1$), the result (4.2) specializes to

$$\frac{\partial j''}{\partial \nu}=\frac{\nu\left\{\int_0^1 \frac{1}{x}J_\nu^2(j''x)\,dx-J_\nu^2(j'')\right\}}{j''\left\{\int_0^1 xJ_\nu^2(j''x)\,dx-J_\nu^2(j'')\right\}} \tag{5.2}$$

(Here and in what follows we shall, for brevity, write j'' to mean j_ν''). Now from [5] (p. 135, eq. (11))) we obtain

$$2\int_0^1 x\,J_\nu^2(j''x)\,dx=J_\nu^2(j'')-J_{\nu-1}(j'')\,J_{\nu+1}(j'')$$

and so, using this and treating the numerator in an obvious way, (5.2) becomes

$$\frac{\partial j''}{\partial \nu}=\frac{-2\nu\left\{\int_0^{j''}\frac{1}{x}J_\nu^2(x)\,dx-J_\nu^2(j'')\right\}}{j''\left\{J_\nu^2(j'')+J_{\nu-1}(j'')\,J_{\nu+1}(j'')\right\}}$$

Now in [2] another expression for $\frac{\partial j''}{\partial \nu}$ was given namely,

$$\frac{\partial j''}{\partial \nu}=\frac{2\nu\left\{\int_0^{j''}\frac{1}{x}J_\nu^2(x)\,dx-J_\nu^2(j'')\right\}}{(j'')^2\,J_\nu(j'')\,J_\nu'''(j'')}$$

so, comparing these, we conclude that

$$J_\nu^2(j'')+J_{\nu-1}(j'')\,J_{\nu+1}(j'')=-j''J_\nu(j'')\,J_\nu'''(j'')$$

Now it was also proved in [2] that if $\nu>0$ ($k=2,3,\cdots$) or if $0<\nu\leq 1$ ($k=1$) then the right hand side here is negative. Then, so too, must be the left hand side in these cases.

In conclusion, I wish to acknowledge the helpful suggestions of the referee.

REFERENCES

1. OLVER, F.W.J., Asymptotics and special functions, Academic Press, New York and London, (1974).

2. LORCH L. and SZEGO, P., On the points of inflection of Bessel functions of positive order. I. Can. Jour. Math. Vol. XLIV, No. 5, 1990, pp. 933-948.

3. WONG, R. and LANG, T., On the points of inflection of Bessel functions of positive order. II. Can. Jour. Math. Vol. XLIV, No. 3, 1991, pp. 628-651.

4. HACIK, M. and MICHALIKOVA, E., A note on monotonicity of zeros of Bessel functions as functions of order, Prace a Studie Vysokej Skoly Dopravy a Spojov v Ziline, Seria Mate. - Fyzik. Rok, 1989, pp. 7-13.

5. WATSON, G.N., A treatise on the Theory of Bessel Functions. 2nd Ed., Cambridge University Press, (1966).