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SEMICONTINUOUS GROUPS AND SEPARATION PROPERTIES

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In 1948, Samuel [2] pointed out that the intersection of two group topologies need not be a group topology. However, a number of properties that hold for a group topology still hold for a topological space that is an intersection of group topologies. In order to study these properties, we shall describe a class of topologies that can be placed on a group which we call semicontinuous topologies. (We point out here that Fuchs [1] calls these spaces semitopological groups).

One important attribute of topological groups is separation. In particular, a topological group is Hausdorff if and only if the identity is a closed subset. While this is not true for semicontinuous groups, we shall see that an interesting "echo" of this property is true.

For each group G we have a bijection inv: $G \to G$ defined by inv $(x) = x^{-1}$. Also for any fixed $a \in G$ we have bijections $1_a: G \to G$ defined by $1_a(x) = ax$ and $r_a: G \to G$ defined by $r_a(x) = xa$.

DEFINITION. A semicontinuous group is a group G and a topology τ on G making inv, 1_a and r_a continuous for $a \in G$.

Clearly a semicontinuous group is a homogeneous space. Thus a great deal can be determined by considering a basis for the topology at the identity. In a manner analogous to that found in the theory of topological groups, one can demonstrate the following:

PROPOSITION 1. If (G,τ) is a semicontinuous group and $\mathcal{I}$ is a neighborhood base at the identity, then $\mathcal{I}$ satisfies

(i) If $U, V \in \mathcal{I}$, then there exists $W \in \mathcal{I}$ such that $W \subset U \cap U$.

(ii) If $a \in U$ and $U \in \mathcal{I}$, then there exists $V \in \mathcal{I}$ such that $Va \subset U$.

(iii) If $U \in \mathcal{I}$ then there exists $V \in \mathcal{I}$ such that $V^{-1} \subset U$.

(iv) If $U \in \mathcal{I}$ and $x \in G$ then there exists $V \in \mathcal{I}$ such that $xV_x{}^{-1} \subset U$.

Furthermore, if $\mathcal{I}$ is any collection of subsets of G, each containing the identity, and $\mathcal{I}$ satisfies (i)-(iv) above, then there exists a unique semicontinuous topology τ on G for which $\mathcal{I}$ is a neighborhood base at the identity.

Any collection of subsets $\mathcal{S}$ satisfying (i)-(iv) is called a semifundamental system. Let $V = (-1,1) - \{x \mid x = r + \sqrt{2}$ and $r \in Q\} \subseteq R$ and let W be the collection of all translation sets $a + V$ such that $0 \in a + V$. Finally let $\mathcal{S}$ be the collection of all finite intersections of elements of W.

A moment's reflection shows that $\mathcal{S}$ is a semifundamental system that generates a topology τ which is finer than the usual topology on R. The set Q is closed in (R, τ). Yet the quotient topology generated on R/Q by projection from (R, τ) is the finite complement topology. Therefore the separation properties for semicontinuous groups are clearly different from those found in topological groups.

Another interesting example of a semicontinuous topology can be described as follows; let B_n be the open ball of radius $1/n$ centered at the origin of the plane, and let $V_n = B_n - \{(x,y) \mid 0 < \frac{1}{n}x \leq y \leq nx\}$. The collection of sets $\{V_n\}_{n=2}^{\infty}$ forms a semifundamental system for the group $(R^2, +)$. The relative topology on $(Q^2, +)$ is an example of a second countable metric space that cannot be a topological group since no square of an open set can be placed inside V_n.

Let (G, t) be a semicontinuous group and $m: G \times G \to G$ the multiplication map. We let $q(t)$ denote the quotient topology on G generated by m when the product topology $t \times t$ is placed on $G \times G$. If N is a normal subgroup of G and (G, t) is a semicontinuous group, we shall denote the quotient topology on G/N generated by the natural map $\pi: G \to G/N$, by $\pi(t)$.

LEMMA 2. If (G, t) is a semicontinuous group, then both m and π are open maps and both G/N and $(G, q(t))$ are semicontinuous groups.

PROOF. Let $U \times V$ be a basic open set in $t \times t$. Then $m^{-1}(m(U \times V)) = \bigcup_{g \in G} (Ug \times g^{-1}V)$. Therefore m is an open map. Likewise $\pi^{-1}(\pi(u)) = UN$ which is open in (G, t) whenever $U \in t$. Thus π is an open map.

Since $1_a \times$ id: $(G \times G, t \times t) \to (G \times G, t \times t)$ is continuous and $q(t)$ is a quotient topology, $1_a: (G, q(t)) \to (G, q(t))$ is continuous. Similar arguments show that the maps $r_a: (G, q(t)) \to (G, q(t))$ and inv: $(G, q(t)) \to (G, q(t))$ are continuous. The proof that the quotient topology on G/N is semicontinuous is done in the same fashion.

LEMMA 3. If $S \subset G$ then $\bar{S} = \bigcap_{V \in \mathcal{S}} VS$.

PROOF. $x \notin \bigcap_{V \in \mathcal{S}} VS$ iff there exists $W \in \mathcal{S}$ with $x \notin WS$ iff $W^{-1}x \cap S = \phi$.

THEOREM 4. G/N is Hausdorff iff $N = \bigcap_{V \in \mathcal{S}} V^2 N$.

PROOF. We consider the following commutative diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{\pi \times \pi} & G/N \times G/N \\ \downarrow m & & \downarrow \bar{m} \\ G & \xrightarrow{\pi} & G/N \end{array}$$

We have that $\{V^2 \mid V \in \mathcal{S}\}$ is a semifundamental system for $q(t)$ whenever $\mathcal{S}$ is a semifundamental system for t. The identity element in $(G/N, \pi(q(t)))$ will be closed if and only if $N = \bigcap_{V \in \mathcal{S}} V^2 N$. The identity element in $(G/N, q(\pi(t)))$ will be closed if and only if the diagonal is closed in $G/N \times G/N$. However $\pi(q(t)) = q(\pi(t))$ since the maps are open.

COROLLARY 5. (G, t) is Hausdorff if and only if $\bigcap_{V \in \mathcal{S}} V^2 = \{e\}$.

COROLLARY 6. If (G,t) is a minimal Hausdorff semicontinuous group then (G,t) is topological group if and only if $\bigcap_{V \in \mathcal{V}} V^4 = \{e\}$.

We can define an equivalence relation on (G,t) by defining $x \sim y$ if and only if there does not exist $V \in \mathcal{V}$ such that $xV \cap yV = \phi$. Let K denote the equivalence class of e under this equivalence relation. We call K the Hausdorff Kernel of (G,t).

THEOREM 7. $K = \bigcap_{V \in \mathcal{V}} V^2$ and K is the minimum normal subgroup with the property that G/K is Hausdorff.

PROOF. We note by Lemma 3 that $\bigcap_{V \in \mathcal{V}} V^2$ is the closure of $\{e\}$ in $(G,q(t))$. Therefore by an argument similar to that for topological groups, $\bigcap_{V \in \mathcal{V}} V^2$ is a normal subgroup of G. Since we can without loss of generality assume that V is symmetric, it is clear the $K = \bigcap_{V \in \mathcal{V}} V^2$. The proof of Theorem 4 shows that G/K is Hausdorff if and only if K is closed in $(G,q(t))$. But K is the smallest closed normal subgroup in $(G,q(t))$.

In a like manner we can define an equivalence relation on (G,t) by declaring $x \sim y$ if and only if there does not exists a continuous function $\phi: G \to R$ with $\phi(x) \neq \phi(y)$. The equivalence class of e under this relation will also be a closed normal subgroup that we call the completely Hausdorff kernel of (G,t).

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