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OSCILLATION AND NONOSCILLATION THEOREMS FOR SOME MIXED DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we investigate the oscillatory and nonoscillatory behavior of solutions of certain mixed third and fourth order difference equations. Specific results are also obtained for the constant coefficient cases.

KEY WORDS AND PHRASES: Mixed difference equations, third and fourth order equations, oscillatory and nonoscillatory solutions.
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1. INTRODUCTION.

In this paper we study the oscillatory and nonoscillatory behavior of the solutions of certain third and fourth order difference equations. Until recently, excepting the studies by Cheng [1], Hooker and Patula [2], and [4, 5, 6, 7], there has not been much research devoted to the oscillation theory of difference equations of order greater than two.

For a sequence U_n, and a fixed real constant a, we define $\Delta_a U_n = U_{n+1} - aU_n$. When $a = 1$ we shall write ΔU_n instead of $\Delta_1 U_n$. We can define inductively $\Delta_a^k U_n = \Delta_a(\Delta_a^{k-1} U_n)$ for $k > 1$. The operator Δ_a was introduced by J. Popenda [3] in his study of certain nonlinear second order difference equations.

The objects of this study will be the mixed difference equations

$$\Delta^2(\Delta_a U_n) + (-1)^i P_n U_n = 0 \qquad i = 1,2, \tag{1.1}$$

and

$$\Delta^3(\Delta_a U_n) + (-1)^i P_n U_n = 0 \qquad i = 1,2, \tag{1.2}$$

where P_n is a sequence of positive numbers having a positive limit inferior, that is, there is a positive constant $c > 0$ such that $P_n > c$ for all n sufficiently large. We consider only nontrivial solutions. A solution is called nonoscillatory if it is eventually of constant sign (positive or negative) otherwise it is called oscillatory. The equations (1.1) and (1.2) are called <u>mixed</u> because of the two difference operators Δ and Δ_a appearing in the equations.

2. ASYMPTOTIC BEHAVIOR OF NONOSCILLATORY SOLUTIONS.

In this section we study the asymptotic behavior of the nonoscillatory solutions of (1.1) and (1.2).

THEOREM 1. Suppose U_n is a nonoscillatory solution of

$$\Delta^2(\Delta_a U_n) + P_n U_n = 0 \tag{2.1}$$

where $0 < a \le 1$. Then for all n sufficiently large we have

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta^2 U_n \neq \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^3 U_n \tag{2.2}$$

and

$$\lim_{n \to \infty} U_n = 0. \tag{2.3}$$

PROOF. For $a = 1$, Popenda and Schmeidel [4] have recently shown that (2.1) has a solution satisfying (2.2). A nonoscillatory solution may not exist if $0 < a < 1$, but if it does exist we show that it must satisfy (2.2) and (2.3). As the negative of a solution of equation (1.1) is also a solution of the same equation, it suffices to prove that an eventually positive solution of (2.1) satisfies (2.2). In this paper, we will assume that all inequalities about sequences hold for all n sufficiently large. Let $U_n > 0$ be a nonoscillatory solution of (2.1). Set

$$Z_n = \Delta_a U_n = U_{n+1} - aU_n, \tag{2.4}$$

then by (2.1)

$$\Delta^2 Z_n = -P_n U_n < 0 \tag{2.5}$$

so ΔZ_n is (eventually) strictly decreasing. From (2.5) it follows that if ΔZ_n is eventually negative we must have $Z_n \to -\infty$, however this is contradictory since $Z_n = U_{n+1} - aU_n = \Delta U_n + (1-a)U_n \to -\infty$ implies $\Delta U_n \to -\infty$, which forces U_n to be eventually negative. We must have

$$\Delta Z_n > 0 \tag{2.6}$$

for all large n. Indeed we will show that $\lim_{n \to \infty} U_n = 0$.

Writing (2.1) as $\Delta^2 Z_n = -P_n U_n$ and summing from N to $m-1$, where N is chosen large enough so that $\Delta Z_n > 0$ for all $n \ge N$, we get

$$\Delta Z_m - \Delta Z_N = -\sum_N^{m-1} P_n U_n.$$

The *lim inf* condition on P_n yields

$$0 < c \sum_N^{m-1} U_n \le \sum_N^{m-1} P_n U_n < \Delta Z_N.$$

Letting $m \to \infty$ we see that $\sum_N^{\infty} U_n < \infty$ and therefore $\lim_{n\to\infty} U_n = 0$. Because $U_n \to 0$ as $n \to \infty$ it follows that $Z_n \to 0$ as $n \to \infty$. From (2.6), Z_n is increasing, hence $Z_n < 0$ eventually. It then follows from the inequality $Z_n = \Delta U_n + (1-a)U_n < 0$ that $\Delta U_n < 0$ and from (2.6) $\Delta Z_n = \Delta^2 U_n + (1-a)\Delta U_n > 0$ and thus $\Delta^2 U_n > 0$. Finally from (2.5), $\Delta^2 Z_n = \Delta^3 U_n + (1-a)\Delta^2 U_n < 0$ and we get $\Delta^3 U_n < 0$ and the proof is complete.

Our next result though similar to the previous one requires $a > 1$.

THEOREM 2. Consider the following equation

$$\Delta^2(\Delta_a U_n) - P_n U_n = 0 \tag{2.7}$$

where $a > 1$. If U_n is a nonoscillatory solution of (2.7) then for all n sufficiently large

$$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n,$$

and

$$\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} |\Delta U_n| = \lim_{n\to\infty} |\Delta^2 U_n| = \infty.$$

PROOF. Assume without loss of generality that $U_n > 0$ for all n sufficiently large. Set

$$Z_n = \Delta_a U_n = U_{n+1} - aU_n,$$

then by (2.7)

$$\Delta^2 Z_n = P_n U_n > 0. \tag{2.8}$$

So ΔZ_n is increasing. If ΔZ_n is eventually positive then as $n \to \infty$, $Z_n \to \infty$ and since $Z_n = \Delta U_n + (1-a)U_n$ and $a > 1$ it follows that $\Delta U_n \to \infty$, which in turn implies $U_n \to \infty$. To see why $\Delta^2 U_n \to \infty$, note that $U_n \to \infty$ implies $\Delta^2 Z_n \to \infty$ and $\Delta Z_n \to \infty$ because of (2.8). But $\Delta Z_n = \Delta^2 U_n + (1-a)\Delta U_n$ and the result follows. Now, if ΔZ_n were eventually negative and increasing then ΔZ_n would have a limit as $n \to \infty$. However ΔZ_n having a limit implies that $\sum^{\infty} U_n < \infty$ and this implies $U_n \to 0$. But $U_n \to 0$ implies $Z_n \to 0$ also and therefore since Z_n is decreasing to zero, $Z_n > 0$. But $Z_n = \Delta U_n + (1-a)U_n > 0$ implies $\Delta U_n > 0$, a contradiction since $U_n > 0$ and $\Delta U_n > 0$ is inconsistent with $U_n \to 0$. Hence (2.7) cannot have a nonoscillatory solution with $\Delta Z_n \Delta^2 Z_n < 0$ for all n sufficiently large.

It should be noted that the condition $0 < a < 1$ was crucial in the proof of Theorem 1. Our next result requires $a > 1$ and is similar to one obtained in [2] for the equation

$$\Delta^4 U_n - P_{n+2} U_{n+2} = 0.$$

THEOREM 3. Consider the equation

$$\Delta^3(\Delta_a U_n) - P_n U_n = 0 \tag{2.9}$$

where $a \geq 1$. If U_n is a nonoscillatory solution of (2-) then for all n sufficiently large either

(I) $$\operatorname{sgn} U_n = \operatorname{sgn} \Delta U_n = \operatorname{sgn} \Delta^2 U_n = \operatorname{sgn} \Delta^3 U_n$$

or

(II) $$\operatorname{sgn} U_n = \operatorname{sgn} \Delta(\Delta_a U_n) \neq \operatorname{sgn} \Delta_a U_n = \operatorname{sgn} \Delta^2(\Delta_a U_n).$$

PROOF. We prove the case for $a > 1$. The proof for $a = 1$ is similar. There is no loss of generality in assuming U_n is an eventually positive solution of (2.9). Set $Z_n = \Delta_a U_n = U_{n+1} - aU_n$. Then by (2.9)

$$\Delta^3 Z_n = P_n U_n > 0. \tag{2.10}$$

Clearly $\Delta^2 Z_n$ is increasing. In case $\Delta^2 Z_n$ is eventually positive we will have $\lim_{n\to\infty} \Delta Z_n = \lim_{n\to\infty} Z = \infty$, and since $Z_n < U_{n+1}$ it follows that $U_n \to \infty$. Since $P_n > c$ for large n

$$\lim_{n\to\infty} \Delta^3 Z_n = \lim_{n\to\infty} \Delta^2 Z_n = \infty.$$

Since $Z_n = \Delta U_n + (1-a)U_n \to \infty$ and $a > 1$ it follows that $\Delta U_n \to \infty$. Examining $\Delta Z_n = \Delta^2 U_n + (1-a)\Delta U_n$ we see that $\Delta^2 U_n \to \infty$ as $n \to \infty$. Continuing in this manner we see that (I) holds eventually.

Next we consider the case where $\Delta^3 Z_n > 0$ and $\Delta^2 Z_n < 0$. Then $\lim_{n\to\infty} \Delta^2 Z_n$ exists and summing (2.10) from N to $m-1$ yields

$$-\Delta^2 Z_N > \Delta^2 Z_m - \Delta^2 Z_N = \sum_N^{m-1} P_n U_n \ge c \sum_N^{m-1} U_n.$$

Letting $m \to \infty$ it then follows that $\sum_N^\infty U_n < \infty$ and hence $\lim_{n\to\infty} U_n = 0$ which implies $Z_n \to 0$. Thus if $\Delta^3 Z_n > 0$ and $\Delta^2 Z_n < 0$ then eventually we must have

$$\Delta^3 Z_n > 0,\ \Delta^2 Z_n < 0,\ \Delta Z_n > 0, \tag{2.11}$$

because $\Delta^2 Z_n < 0$ and $\Delta Z_n < 0$ is inconsistent with $Z_n \to 0$, it then follows that either (i) $Z_n > 0$ or (ii) $Z_n < 0$ eventually. We will show that (i) is impossible. If (i) held then since $Z_n = \Delta U_n + (1-a)U_n > 0$ it follows that $\Delta U_n > 0$, in fact we have that $\Delta U_n > k + (a-1)U_n > k$ for some positive constant k and so $U_n \to \infty$ as $n \to \infty$. But this implies $\Delta^3 Z_n \to \infty$, so we must have $\Delta^2 Z_n > 0$ eventually, contradicting (2.11). So (i) cannot hold, resulting in (ii) holding eventually.

3. SUFFICIENT CONDITIONS FOR OSCILLATION AND/OR NONOSCILLATION.

THEOREM 4. Every nontrivial bounded solution of

$$\Delta^3(\Delta_a U_n) + P_n U_n = 0 \tag{3.1}$$

where $a > 1$, is oscillatory.

PROOF. Suppose (3.1) has a bounded nonoscillatory solution U_n satisfying $U_n > 0$ for large n. Letting $Z_n = \Delta_a U_n = U_{n+1} - aU_n$, we see that $Z_n \ge -aU_n$. From (3.1), $\Delta^3 Z_n = -P_n U_n < 0$. Obviously $\Delta^2 Z_n$ is decreasing, and if $\Delta^2 Z_n$ is eventually negative, we see that $Z_n \to \infty$. This clearly contradicts the boundedness of U_n. Thus, we consider the case where $\Delta^2 Z_n > 0$. In this case $\lim_{n\to\infty} \Delta^2 Z_n = t \ge 0$. Using the fact P_n is bounded away from zero for large n, it follows that $\Delta Z_n < 0$ and $Z_n > 0$ for large n. Furthermore, $\lim_{n\to\infty} U_n = 0$, since $\Delta^2 Z_n \to t$ implies $\sum^\infty U_n < \infty$. Since $a > 1$ and

$$Z_n = \Delta U_n + (1-a)U_n > 0,$$

$\Delta U_n > 0$ for all n sufficiently large. But this is a contradiction, since $U_n \Delta U_n > 0$ is incongruent with $U_n \to 0$.

EXAMPLE. The equation

$$\Delta^3(\Delta_2 U_n) + \frac{1}{16} U_n = 0.$$

has the sequence $U_n = \left(\frac{3}{2}\right)^n$ as a solution. Hence equation (3.1) may have nonoscillatory solutions.

Before stating our final results consider the constant coefficient case $P_n = Q,\ Q > 0$.

Equation (1.1) with $P_n = Q$ is

$$\Delta^3 U_n + (1-a)\Delta^2 U_n + (-1)^i Q U_n = 0 \qquad i = 1,2. \tag{3.2}$$

So the characteristic polynomial is

$$(C_1) \qquad f(t) = (t-1)^3 + (1-a)(t-1)^2 + (-1)^i Q \qquad i = 1,2.$$

Similarly, equation (1.2) can be written as

$$\Delta^4 U_n + (1-a)\Delta^3 U_n + (-1)^i Q U_n = 0 \qquad i = 1,2 \tag{3.3}$$

with characteristic polynomial

$$(C_2) \qquad g(t) = (t-1)^4 + (1-a)(t-1)^3 + (-1)^i Q \qquad i = 1,2.$$

The proofs of our final results follow from a careful examination of the characteristic polynomials (C_1) and (C_2).

THEOREM 5. Consider (2.4) where $0 < a < 1$, $P_n = Q$ constant and $Q > a$, then all nontrivial solutions of (2.4) are oscillatory.

EXAMPLE. The general solution of

$$\Delta^2(\Delta_{1/2}U_n) + 6U_n = 0 \tag{3.4}$$

is

$$U_n = K_1(-1)^n + K_2(11/2)^{n/2}\sin n\varnothing + K_3(11/2)^{n/2}\cos n\varnothing$$

where $\varnothing = arc\tan(\sqrt{39}/7)$ it follows from Theorem 1 or from Theorem 5 that all solutions of (3.4) are oscillatory.

THEOREM 6. Consider (2.7) where $0 < a < 1$, $Q = \frac{R^3}{64}$ constant and $0 < a < 1 - \frac{3}{4}R$, then all solutions of (2.7) are nonoscillatory.

THEOREM 7. Consider (2.9) where $0 < a < 1$, $P_n = Q$ constant and $Q > 0$, then (2.9) has oscillatory and nonoscillatory solutions. Moreover, all nonoscillatory solutions are bounded and converge to zero.

Note that when $a = 1$, $P_n = Q$ constant, equation (2.7) becomes $\Delta^3U_n - QU_n = 0$. Clearly this equation has oscillatory solutions for any $Q > 0$. Thus, the result of Theorem 6 depends upon $0 < a < 1$. Furthermore, it should be noted that Theorem 7 is interesting because, when $a > 1$ and P_n is constant (2.7) must have an unbounded nonoscillatory solution. Clearly the boundedness of the nonoscillatory solutions can be attributed to the parameter a in the operator Δ_a.

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