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OSCILLATION AND NON-OSCILLATION OF SOME NEUTRAL DIFFERENTIAL EQUATIONS OF ODD ORDER

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ABSTRACT. An existence criterion for nonoscillatory solution for an odd order neutral differential equation is provided. Some sufficient conditions are also given for the oscillation of solutions of some nth order equations with nonlinearity in the neutral term.

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1. INTRODUCTION.

In this paper we consider the first order neutral differential equations of the form

$$(x(t)-cx(t-\tau))' - p(t)x(g(t)) = 0\ , \quad t \geq t_0 \tag{1.1}$$

$$(x(t)+cx(t-\tau))' - p(t)x(g(t)) = F(t)\ , \tag{1.2}$$

and nonlinear neutral differential equation of nth order of the form

$$(x(t)-h(t)f_1(x(\tau(t))))^{(n)} + p(t)f_2(x(g(t))) = 0\ , \quad t \geq t_0\ . \tag{1.3}$$

Most of the works on oscillation theory for neutral equations deal with stable type equations. There are a few papers (see e.g. [2], [3], [5]) where nonoscillation of unstable type equations of order larger than one is discussed. In Section 2 we establish a result for the existence of an unbounded solution of Eq. (1.1) which tends to infinity exponentially. Some results on oscillation and nonoscillation for Eq. (1.3) are given in Section 3. To the best of our knowledge this is the first time that a differential equation with nonlinearity in the neutral term is being studied. As pointed out by Hale [1] it is useful to study neutral nonlinear differential equations of the form

$$(x(t)-G(t,x(t-\tau)))' = H(t,x(t-\tau))$$

As usual a solution $x(t)$ of Eq. (1.j), $j=1,2$, is said to be oscillatory on $[t_0,\infty)$ if the set of zeros of $x(t)$ is unbounded, otherwise it is called nonoscillatory. In Section 3 we need the following lemma:

LEMMA. (see [6]). Let X be a Banach space, Γ a bounded closed and convex subset of X, A, B be maps on Γ to X such that $Ax+By \in \Gamma$ for every pair $x,y \in \Gamma$. If A is a contraction and B is completely continuous then the equation

$$Ax + bx = x$$

has a solution in Γ.

2. RESULTS FOR EQUATIONS (1.1) AND (1.2).

We assume that

$$c \geq 0 , \quad \tau > 0 \quad g(t) \leq t , \quad \lim_{t \to \infty} g(t) = \infty \quad p(t) \geq 0 ,$$

and the functions p, g, F are continuous on $[t_0, \infty)$, $t_0 \geq 0$. In case

$$p(t) \equiv p , \quad g(t) = t - \sigma , \quad \sigma > 0 ,$$

from the analysis of the characteristic equation of Eq. (1.1) we know that Eq. (1.1) has always an unbounded solution

$$x(t) = Ae^{\alpha t} , \quad \alpha > 0 .$$

The question arises whether Eq.(1.1) has always an unbounded solution $x(t)$ which tends to infinity exponentially as t tends to infinity. We explore that possibility.

For $c \geq 0$, let $x(t)$ be a positive solution of Eq. (1.1). We put $z(t) = x(t) - cx(t-\tau)$. Then $z'(t) \geq 0$, and therefore two possibilities exist:

(i) $z(t) > 0$, eventually, or

(ii) $z(t) \leq 0$, eventually.

Consequently, the nonoscillatory solution $x(t)$ must satisfy one of the following type of asymptotic behavior:

(a) $\lim_{t \to \infty} x(t) = 0$;

(b) $\lim_{t \to \infty} x(t) = l \neq 0$;

(c) $\lim_{t \to \infty} x(t) = \infty$;

We prove the following:

THEOREM 2.1. Based on the value of c we have the conclusions:

(i) If $c \geq 0$, Eq. (1.1) has always a positive solution $x(t)$ satisfying (b) or (c);

(ii) If $c \geq 1$, Eq. (1.1) has always an unbounded solution $x(t)$ satisfying (c);

(iii) If $c > 1$, Eq. (1.1) has always an unbounded solution $x(t)$ which tends to infinity exponentially;

(iv) If $0 \leq c < 1$, and $\int_T^\infty p(t)dt = \infty$, $T \geq t_0$, Eq. (1.1) has always an unbounded positive solution, and every bounded solution of Eq. (1.1) either oscillates or tends to zero as t tends to infinity.

PROOF. For a given continuous function p there exists a continuous function $H(t) > 0$ such that

$$\int_{t_0}^{\infty} p(t)H(t)dt = \infty \quad \lim_{t \to \infty} \left[\frac{p(t)}{\exp\left(\int_{t_0}^{t} p(s)H(s)ds\right)} \right] = 0 , \tag{2.1}$$

Define

$$z(t) = \exp\left[\int_{t_0}^{t} \exp\left(\int_{t_0}^{s} p(u)H(u)du \right) ds \right] . \tag{2.2}$$

Let $BC([t_0, \infty), R)$ be a Banach space of bounded and continuous functions $y:[t_0,\infty) \to R$. Define a subset Ω of BC as follows:

$$\Omega = \{ y \in BC : 0 \leq y(t) \leq 1 , \quad t_0 \leq t < \infty \} .$$

Clearly Ω is a bounded, closed and a convex subset of BC. Now we define a mapping S on Ω as follows:

$$(Sy)(t) = \begin{cases} c\frac{z(t-\tau)}{z(t)} + \frac{1}{z(t)}\int_T^t p(s)z(g(s))y(g(s))ds + \frac{1}{2z(t)} & \text{if } t \geq T, \\ \frac{t}{T}(Sy)(T) + (1-\frac{t}{T}), & \text{if } t_0 \leq t \leq T, \end{cases} \tag{2.3}$$

where T is chosen sufficiently large so that $t-\tau \geq t_0$, $y(t) \geq t_0$, $z(t) \geq 1$,

and

$$c\frac{z(t-\tau)}{z(t)} + \frac{1}{z(t)}\int_T^t p(s)z(g(s))ds \leq \frac{1}{2}, \tag{2.4}$$

for $t \geq T$. Using (2.1) and (2.2) one finds that

$$\frac{z(t-\tau)}{z(t)} \to 0, \quad \text{and} \quad \frac{\int_{t_0}^t p(s)z(g(s))ds}{z(t)} \to \infty, \quad \text{as} \quad t\to\infty,$$

which shows that (2.4) is possible. Thus we have $S\Omega \subset \Omega$.
Let y_1 and y_2 be elements of Ω. Then

$$\begin{aligned} |(Sy_2)(t)-(Sy_1)(t)| &\leq c\frac{z(t-\tau)}{z(t)}|y_2(t-\tau)-y_1(t-\tau)| \\ &+ \frac{1}{z(t)}\int_T^t p(s)z(g(s))|y_2(g(s))-y_1(g(s))|\,ds \\ &\leq \frac{1}{2}\|y_2-y_1\|, \quad t \geq T. \end{aligned}$$

and

$$\begin{aligned} \|Sy_2-Sy_1\| &= \sup|(Sy_2)(t)-(Sy_1(t)| \\ &= \sup_{t\geq T}|(Sy_2)(t)-(Sy_1)(t)| \\ &\leq \frac{1}{2}\|y_2-y_1\|, \end{aligned}$$

which shows that S is a contraction on Ω. Hence, there is an element $y \in \Omega$ such that $Sy = y$. That is,

$$y(t) = \begin{cases} c\frac{z(t-\tau)}{z(t)} + \frac{1}{z(t)}\int_T^t p(s)z(g(s))y(g(s))ds + \frac{1}{2z(t)} & \text{if } t \geq T, \\ \frac{t}{T}y(T) + (1-\frac{t}{T}), & \text{if } t_0 \leq t \leq T, \end{cases} \tag{2.5}$$

Obviously $y(t) > 0$ for $t \geq t_0$. Set

$$x(t) = y(t)z(t). \tag{2.6}$$

that is,

$$x(t) - cx(t-\tau) = \int_T^t p(s)x(g(s))ds + \frac{1}{2}, \quad t \geq T, \tag{2.7}$$

which shows that $x(t)$ is a positive solution of Eq.(1.1) for $t \geq T$. This proves (i), (ii) and first part of (iv); In case $c > 1$ we have

$$x(t) \geq cx(t-\tau) \geq \cdots \geq c^n x(t-n\tau).$$

or

$$x(t) \geq x(t_0)e^{\mu(t-t_0)}, \quad \text{for} \quad t \geq t_0$$

where $\mu = \frac{lnc}{\tau} > 0$, which shows that (iii) is true. In order to prove the second part of (iv) we let $x(t)$ to be a bounded positive solution of Eq. (1.1). Put

$$u(t) = x(t) - cx(t-\tau) \, .$$

The $u'(t) > 0$ and $\lim_{t\to\infty} u(t)$ exists. Let $\lim_{t\to\infty} u(t) = l$. If $l > 0$ then $x(g(t)) \geq l$. Consequently,

$$u(t) - u(T_1) = \int_{T_1}^{t} p(s)x(g(s))ds \geq l \int_{T_1}^{t} p(s)ds \, ,$$

since $\int_{T_1}^{t} p(s)ds \to \infty$ as $t\to\infty$ we have a contradiction to the boundedness of $x(t)$. In view of (2.7) we can assume that $c \neq 0$. Now for $0 < c < 1$ we cannot have the case that $l < 0$. Thus $\lim_{t\to\infty} u(t) = 0$ and hence we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem.

EXAMPLE 2.1. Consider the equation

$$\left(x(t) - \tfrac{1}{2}x(t-1)\right)' = \frac{t-2}{2t^2}x(t-1) \, , \quad \text{for} \quad t \geq 2 \tag{2.8}$$

which satisfies the assumptions of theorem (2.1(i). In fact, (2.8) has a solution: $x(t) = 1 + \frac{1}{t}$.

EXAMPLE 2.2. The equation

$$(x(t) - x(t-1))' = (1 - e^{-1})x(t) \, , \quad \text{for} \quad t \geq 2 \tag{2.9}$$

satisfies the hypotheses of theorem 2.1(ii). We note that (2.9) has an unbounded solution $x(t) = e^t$.

EXAMPLE 2.3. The equation

$$(x(t) - 2x(t-1))' = \frac{(e-2)t + e}{t-1}x(t-1) \, , \quad \text{for} \quad t \geq 2 \tag{2.10}$$

satisfies the assumptions of theorem 2.1(iii). In fact, (2.10) has a solution $x(t) = te^t$.

EXAMPLE 2.4. Consider the equation

$$\left(x(t) - \tfrac{1}{2}x(t-2\pi)\right)' = \tfrac{1}{2}x(t - \tfrac{3}{2}\pi) \, . \tag{2.11}$$

One can easily check that $x(t) = \sin t$ is a bounded oscillatory solution of (2.11).

EXAMPLE 2.5. The equation

$$\left(x(t) - \tfrac{1}{2}x(t-1)\right)' = \left(\tfrac{e}{2} - 1\right)x(t) \, , \tag{2.12}$$

satisfies the hypotheses of theorem 2.1(v). In fact, (2.12) has a solution $x(t) = e^{-t}$.

OPEN PROBLEM. What is a criterion for the existence of oscillatory solutions for Eq.(1.2)?

THEOREM 2.2. Consider the Eq.(1.2) and assume that there exists a function f such that $F(t) = f'(t)$ and

$$\limsup_{t\to\infty} f(t) = +\infty \, ,$$

$$\liminf_{t\to\infty} f(t) = -\infty \, . \tag{2.13}$$

Then every bounded solution of Eq. (1.2) is oscillatory.

PROOF. Set

$$z(t) = x(t) + cx(t-\tau)$$

and let $x(t)$ be a bounded positive solution of (1.2). Then (1.3) reduces to

$$(z(t)-f(t))' = p(t)x(g(t)) \geq 0 .$$

If $z(t)-f(t) \leq 0$ eventually, then $0 \leq z(t) \leq f(t)$ eventually, a contradiction. Hence $z(t)-f(t)>0$ eventually, which is impossible, in view of that fact that $z(t)$ is bounded. This completes the proof.

EXAMPLE 2.6. The equation

$$(x(t)+x(t-\pi))' - tx(t-2\pi) = -t\sin t , \tag{2.14}$$

satisfies the assumptions of Theorem 2.2. Hence every bounded solution of (2.14) is oscillatory. In fact, $x(t)=\sin t$ is such a solution.

3. NONLINEAR NEUTRAL EQUATION (1.3)

(i) $h \in C(R_+, R)$;

(ii) $f_i \in C(R,R)$, $xf_i(x)>0$, $i=1,2$, as $x \neq 0$

$$|f_1(x)-f_1(y)| \leq L|x-y| , \quad x,y \in [0,1]$$

f_2 is a nondecreasing function, L is a positive constant;

(iii) $\tau, g \in C(R_+,R)$, $0 \leq t-\tau(t) \leq M$, M is a constant

$$\lim_{t\to\infty} \tau(t) = \infty , \quad \lim_{t\to\infty} g(t) = \infty$$

(iv) there exists $\alpha > 0$ such that

$$Lh(t)e^{\alpha(t-\tau))} \leq c < 1 ,$$

and

$$Lh(t)e^{\alpha(t-\tau))} + \frac{e^{\alpha t}}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)f_2(e^{-\alpha g(s)})ds \leq 1 .$$

Then Eq.(1.3) has an eventually positive solution $x(t)$ which tends to zero exponentially as $t\to\infty$.

PROOF. Let t_0 be sufficiently large so that

$$T = \min\{\inf_{t\geq t_0} \tau(t) , \inf_{t\geq T_0} g(t)\}$$

As before, $BC([T,\infty))$ denotes the Banach space of all bounded and continuous real valued functions defined on $[T,\infty)$. Let Ω be a subset of BC as defined in Se. 2. Define operators S_1 and S_2 on Ω as follows:

$$(S_1y)(t) = \begin{cases} h(t)e^{\alpha t}f_1\Big(y(\tau(t)e^{-\alpha\tau(t)}\Big), & t \geq t_0 \\ \frac{t}{t_0}(S_1y)(t_0) + (1-\frac{t}{t_0}) , & \text{for } T \leq t \leq t_0 , \end{cases}$$

$$(S_2y)(t) = \begin{cases} \frac{e^{\alpha t}}{(n-1)!}\int_t^\infty (s-t)^{(n-1)}p(s)f_2\Big(y(g(s))e^{-\alpha g(s)}\Big)\,ds, & \text{if } t \geq t_0 \\ \frac{t}{t_0}(S_2y)(t_0) + (1-\frac{t}{t_0}) , & \text{for } T \leq t \leq t_0 . \end{cases}$$

By (iv), for every $x,y \in \Omega$ we have $S_1x+S_2y \in \Omega$. Condition (iv) implies that S_1 is a contraction on Ω. It is easy to see that

$$\Big|\frac{d}{dt}(S_2y)(t)\Big| \leq M_1 \quad \text{for} \in \Omega$$

where M_1 is a positive constant. From this it follows that S_2 is completely continuous. By Lemma there exists a $y \in \Omega$ such that

$$(S_1 + S_2)y = y\ .$$

That is,

$$y(t) = \begin{cases} h(t)e^{\alpha t} f_1\Big(y(\tau)(t))e^{-\alpha\tau(t)}\Big) + \dfrac{e^{\alpha t}}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{(n-1)} p(s) f_2\Big(y(g(s))e^{-\alpha g(s)}\Big)\, ds, & \text{if } t \geq t_0 \\ \dfrac{t}{t_0}\, y(t_0) + (1 - \dfrac{t}{t_0})\ , & \text{for } T \leq t \leq t_0\ . \end{cases}$$

It is easy to see that $y(t) > 0$ for $t \geq T$. Set $x(t) = y(t)e^{-\alpha t}$.
Then

$$x(t) = h(t) f_1(x(\tau(t))) + \frac{1}{(n-1)} \int_t^\infty (s-t)^{n-1} [(s) f_2(x(g(s)))\, ds\ , \quad t \geq t_0$$

or

$$\Big(x(t) - h(t) f_1(x(\tau(t)))\Big)^n + p(t) f_2(x(g(t))) = 0\ , \quad t \geq t_0\ .$$

This completes the proof.

EXAMPLE 3.1. Consider a nonlinear neutral equation of the form

$$\Big(x(t) - \tfrac{1}{4}\, x^3(t-1)\Big)' + p(t) x^{\frac{1}{3}}(t) = 0\ , \tag{3.1}$$

where

$$p(t) = e^{-\frac{2}{3}t} - \tfrac{3}{4}\, e^3 e^{-\frac{8}{3}t} > 0$$

for all large values of t. In our notation

$$h(t) = \tfrac{1}{4}\ , \quad f_1(x) = x^3\ , \quad L = 3\ , \quad f_2(x) = x^{1/3}\ .$$

Obviously the hypotheses of theorem 3.1 are satisfied. Therefore Eq.(3.1) has a solution $x(t)$ which tends to zero exponentially as $t \to \infty$. In fact, $x(t) = e^{-t}$ is such a solution of (3.1).

Now we establish an oscillation criterion for Eq.(1.3) for the case $n = 1$.

THEOREM 3.2. Assume that

(i) τ, g, f_1 and f_2 are continuous, f_2 is nondecreasing

$$x f_i(x) > 0\ , \quad \text{for } x \neq 0\ , \quad i = 1.2\ , \quad |f_1(x)| \leq L|x|\ ,$$

(ii) $$0 \leq h(t) \leq c \ \text{ and } \ (cL) < 1$$

(iii) $$\lim_{y\to 0} \frac{f_2(y)}{y} = q\ ,$$

(iv) $$\tau(t) < t\ , \quad \lim_{t\to\infty} \tau(t) = \infty\ , \quad g(t) < t\ , \quad \lim_{t\to\infty} g(t) = \infty\ ,$$

(v) $$\liminf_{t\to\infty} \int_{g(t)}^t p(s)\, ds = p\ , \quad pq > \tfrac{1}{e}\ .$$

Then every solution of Eq. (1.3) is oscillatory.

REMARK. We first prove a lemma which we need in the proof of the theorem.

LEMMA 3.2. Let $x(t)$ be an eventually positive solution of Eq. (1.3). Set

$$z(t) = x(t) - h(t) f_1(x(\tau(t)))\ . \tag{3.2}$$

Then $z(t) > 0$ eventually.

PROOF. For convenience we put $\sigma = \tau$, $\sigma^0 = I = \text{identity}, \sigma^n = \sigma o \sigma^{n-1}, n = 0, 1, \cdots$. From

(1.3) it follows that z is nonincreasing. In case $z(t) < 0$ eventually, then

$$z(t) \leq cf_1(x(\tau(t))) \leq cLx(\sigma(t)) \leq \cdots \leq (cL)^n x(\sigma^n(t)),$$

which implies that $\lim_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$, which is a contradiction.

PROOF OF THEOREM. Suppose the contrary, and let $x(t)$ be an eventually positive solution of Eq. (1.3). Then $z(t) > 0$, eventually. Since $x(t) \geq z(t)$ we have

$$f_2(x(t)) \geq f_2(z(t)) \tag{3.3}$$

Then from (1.3) and (3.3) we have

$$z'(t) + p(t)f_2(z(g(t))) \leq 0, \tag{3.4}$$

which implies that (3.4) has an eventually positive solution. However (v) implies that (3.4) cannot have a positive solution, by a known result [4]. This completes the proof.

EXAMPLE. Consider

$$(x(t) - cx(t-2\pi)\sin^2 x(t-2\pi))' + p(t)x(t-\tfrac{\pi}{2}) = 0, \quad \text{for } t \geq 2\pi \tag{3.5}$$

where

$$p(t) = 1 - c\sin^2(\sin t) - c\cos t\sin(2\sin t) \geq 0, \text{ and } 0 < c < \tfrac{1}{2}(1 - \tfrac{2}{e\pi}).$$

It is obvious that

$$\int_{t-\frac{\pi}{2}}^{t} p(s)ds \geq (1-2c)\frac{\pi}{2} > \frac{1}{e}.$$

Therefore the hypotheses of Theorem 3.2 are satisfied. Hence every solution of (3.5) is oscillatory. In fact, $x(t) = \sin t$ is such a solution.

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