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OUTER COMPOSITIONS OF HYPERBOLIC/LOXODROMIC LINEAR FRACTIONAL TRANSFORMATIONS

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ABSTRACT. It is shown, using classical means, that the outer composition of hyperbolic or loxodromic linear fractional transformations $\{f_n\}$, where $f_n \to f$, converges to α, the attracting fixed point of f, for all complex numbers z, with one possible exception, z_o. I.e.,

$$F_n(z) : = f_n o f_{n-1} o \dots o f_1(z) \to \alpha$$

When z_o exists, $F_n(z_o) \to \beta$, the repelling fixed point of f. Applications include the analytic theory of reverse continued fractions.

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1. INTRODUCTION.

The iterative behavior of the non-singular linear fractional transformation (LFT) $f(z) \colon = (az+b)/(cz+d)$, $a,b,c,d \in C$ and $ad-bc \neq 0$, is well documented. For example, Ford [1] describes the "multiplier" form of $f(z)$ for LFTs that are loxodromic, hyperbolic, or elliptic:

$$(f(z)-\alpha/(f(z)-\beta) = K\ (z-\alpha)/(z-\beta),\ K\colon = (a-c\alpha)/(a-c\beta). \tag{1.1}$$

Here α and β are the two distinct fixed points of f. A special equation analogous to (1.1) exists for the one remaining type of LFT - the parabolic transformation, having a single fixed point.

K is called the multiplier of f and provides an important means of classifying LFT's in terms of their fixed points. Briefly, (1.1) is either loxodromic or hyperbolic if $|K| < 1$, and it is this combined case - representing "most" LFTs - we shall consider exclusively in this article. It is then easily seen from the multiplier form that $f^n(z) \to \alpha$ for all $z \neq \beta$, and $f^n(\beta) \equiv \beta$.

If $f(z)\colon = a/(b+z)$, $f^n(0)$ is the normal nth approximant of a periodic continued fraction

$$\frac{a}{b} + \frac{a}{b} + \dots$$

Thus, this simple continued fraction converges to the attractive fixed point α of f. β is called the repulsive or repellent fixed point of f. The modified continued fraction generated by $f^n(z)$ instead of $f^n(0)$ also converges to α for all $z \neq \beta$. For this special $f(z)$, $|K| = |\alpha/\beta|$, so that $|\alpha| < |\beta|$.

Magnus and Mandell in 1971 [3] posed and answered the following question: If $\{f_n\}$ is a sequence of hyperbolic/loxodromic (H/L) LFTs that converge to a H/L LFT $f(z)$, then what may one predict of the convergence behavior of the "inner" composition $F_n(z)\colon = f_1 o f_2 o \dots o f_n(z)$ for $z \in C$? They obtained the following result which resembles the iterative case.

THEOREM 1. If $\{f_n\}$ and $f = Lim\ f_n$ are all H/L, then the inner compositional sequence $\{F_n(z)\}$ converges to a constant function for all values of $z \neq \beta$.

As a consequence, the modified limit periodic continued fraction generated in this way by setting $f_n(z) := a_n/(b_n + z)$ with $f_n(z) \rightarrow f(z) = a/(b+z)$, and where all the f_n's and f are H/L, converges to a constant for all $z \neq \beta$. This result has proven fruitful in both accelerating the convergence of such continued fractions and analytically continuing them beyond initial regions of convergence [4], [5].

In the current paper the question posed by Magnus and Mandell with regard to <u>inner</u> compositions is answered in the context of outer compositional structures:

$$F_n(z) := f_n o f_{n-1} o \ldots o f_1(z), \text{ where } \{f_n\} \text{ and } f = Lim\ f_n \text{ are all } H/L. \tag{1.2}$$

The proof of Theorem 1 is elementary, but complicated. Surprisingly, the proof of an analogous theorem for outer composition is, if anything, slightly more delicate. One would think that in (1.2) $F_n(z)$ becomes very like $f^n(z)$ for large values of n, and that this should simplify matters. However, the initial segment $F_j(z)$ for $j \ll n$ is not easy to control.

In analogy to Theorem 1 (and even closer to the simple iterative case) we shall prove

THEOREM 2. If $\{f_n\}$ and $f = Lim\ f_n$ are all H/L, then the outer compositional sequence $\{F_n(z)\}$ described in (1.2) converges to α, the attractive fixed point of f, for all values of z except possibly one, z_o. In this exceptional case $F_n(z_o) \rightarrow \beta$, the repulsive fixed point of f.

As one example of Theorem 2, one easily obtains

COROLLARY 1. The modified reverse limit periodic continued fraction

$$\frac{a_n}{b_n} + \frac{a_{n-1}}{b_{n-1}} + \ldots + \frac{a_a}{b_a + z},$$

where $f_n(z) := a_n/(b_n + z)$ and $Lim\ f_n(z) = f(z) = a/(b+z)$ are all H/L, converges to α, the attractive fixed point of f for all $z \in C$, with one possible exception.

In order to prove Theorem 2, it is convenient to use a more general result from the analytic theory of contraction maps as applied to outer composition. The sufficiency part of the proof of the following theorem (all that is required in this exposition) is given in (1.2):

THEOREM 3. Let $\{g_n\}$ be a sequence of functions analytic on a simply connected region S and continuous on the closure of S. Suppose there exists a compact set Ω contained in S such that $\Omega \supset g_n(\Omega)$ for all n. Then, if $G_n(z) := g_n o \ldots o g_1(z)$, $G_n(z) \rightarrow \alpha$, a constant, uniformly for all $z \in S$ if, and only if, the sequence $\{\alpha_n\}$ of fixed points of $\{g_n\}$ in S converges to α.

2. PROOF OF THEOREM 2.

An explicit expression for $f_n(z)$ from the multiplier form (1.1) is

$$f_n(z) = \frac{(\alpha_n - K_n\beta_n)z + \alpha_n\beta_n(K_n - 1)}{(1 - K_n)z + K_n\alpha_n - \beta_n} \tag{2.1}$$

Let us begin with a lemma that will prepare the way for the use of Theorem 3 in the present context. In all that follows it will be assumed that $K := Lim\ K_n$, $\alpha := Lim\ \alpha_n$, and $\beta := Lim\ \beta_n$ exist.

LEMMA 1. Let $R := \rho\,|\alpha - \beta|$, $\rho := (r - |K|)/r\,|1 - K|$, $|K| < r < 1$. For n sufficiently large, there exists $t \in (r, 1)$ such that $|z - \alpha| \leq R \Rightarrow |f_n(z) - \alpha| < tR < R$.

PROOF. Writing

$$f_n(z) - \alpha = (f_n(z) - \alpha_n) + \varepsilon_n \qquad (\varepsilon_n := \alpha_n - \alpha)$$

$$= K_n(\alpha_n - \beta_n)\,[(z - \alpha) - \varepsilon_n]\,/\,[(1 - K_n)(z - \alpha) + \alpha - \beta_n + K_n\varepsilon_n] + \varepsilon_n$$

one gets

$$|f_n(z) - \alpha| \leq |K_n(\alpha_n - \beta_n)\,(R + |\varepsilon_n|)\,/\,(|\alpha - \beta_n| - |K_n\varepsilon_n| - R\,|1 - K_n|) + |\varepsilon_n|.$$

Letting $n \rightarrow \infty$ and replacing R by its defined value, this last expression becomes: $r\rho\,|\alpha - \beta| = rR\ (< R)$.

Thus, for all n sufficiently large and $|z - \alpha| \leq R$,

$$|f_n(z) - \alpha| < tR < R, \text{ for some } t \in (r, 1).$$

QED

We next decompose the final segment of $F_{n+m}(z)$ in order to show that as m increases applying f_{n+m} is similar to applying f.

From (1.2) one can write

$$f_p(z) = \lambda_p^{-1} o K_p o \lambda_p(z), \text{ where } \lambda_p(z) := (z-\alpha_p) / (z-\beta_p) \text{ and } K_p(z) := K_p z.$$

Here, for all p, $|K_p| \leq K_o < 1$.

Therefore

$$F_{n+m}(z) = f_{n+m} o \ldots o f_{n+1}(F_n(z)) =$$

$$\lambda_{n+m}^{-1} o K_{n+m} o \lambda_{n+m} o \lambda_{n+m-1}^{-1} o K_{n+m-1} o \ldots o \lambda_{n+1}^{-1} o K_{n+1} o \lambda_{n+1}(z_n),$$

where $z_n := F_n(z)$. Then

$$F_{n+m}(z) = \lambda_{n+m}^{-1} o K_{n+m} o T_m o T_{m-1} o \ldots o T_2(w_n),$$

where

$$T_p(z) := \lambda_{n+p} o \lambda_{n+p-1}^{-1} o K_{n+p-1}(z) \text{ and } w_n := \lambda_{n+1}(F_n(z)).$$

The idea behind the proof of the next lemma is that $T_p(z) \approx K_{n+p-1}(z)$ for large values of p. This is a device initiated by Magnus and Mandell [3].

LEMMA 2. Suppose that $|w_n| < M(=M(z))$ for all n sufficiently large. Then, for preassigned $\varepsilon > 0$, there exists $N(=N(z))$ and $P(=P(z))$ such that $n > N$ and $p > P$ implies $|T_p o \ldots o T_2(w_n)| < 2\varepsilon/(1-K_o)$.

PROOF. From

$$T_2(w_n) = \{K_{n+1} w_n(\beta_{n+1} - \alpha_{n+2}) + (\alpha_{n+2} - \alpha_{n+1})] / [K_{n+1} w_n(\beta_{n+1} - \beta_{n+2}) + (\beta_{n+2} - \alpha_{n+1})]$$

one obtains

$$|T_2(w_n)| \leq [K_o M |\beta_{n+1} - \alpha_{n+2}| + |\alpha_{n+2} - \alpha_{n+1}|] / [|\beta_{n+2} - \alpha_{n+1}| - K_o M |\beta_{n+1} - \beta_{n+2}|] < M$$

and

$$|T_2(w_n) - K_{n+1} w_n| < \varepsilon, \text{ for large values of } n.$$

Hence $|T_2(w_n)| < \varepsilon + |K_{n+1} w_n| < \varepsilon + K_o M$.

Similarly $|T_3 o T_2(w_n)| < M$ and $|T_3 o T_2(w_n)| < \varepsilon + K_o |T_2(w_n)| < \varepsilon(1+K_o) + K_o M^2$.

Continuing in this manner, one arrives at the general form $|T_p o \ldots o T_2(w_n)| < M$ and $|T_p o \ldots o T_2(w_n)| < \varepsilon(1 + K_o + K_o^2 + \ldots + K_o^{p-2}) + K_o^{p-1} M \leq \varepsilon/(1-K_o) + K_o^{p-1} M < 2\varepsilon/(1-K_o)$ for large p. QED

At several points later on we will refer back to this system of inequalities albeit under slightly different hypotheses.

In order to proceed, we need to know more about the exceptional point z_o described in Theorem 2. If $f_n \equiv f$, then $z_o = \beta$. That z_o can be any point in C is easily seen by setting $f_n \equiv f$ for $n > 1$ and allowing f_1 to be any LFT. Since f_1 is one to one in C, and $z_o = f_1^{-1}(\beta)$, clearly z_o could be any point we wish, including α.

It will become apparent later that the method of proof of Theorem 2 requires that $\{F_n(z)\}$ be uniformly bounded away from β for large values of n. Consequently, the possibility that $\{F_n(z)\}$ has a cluster point at β must be explored.

LEMMA 3. If $\{F_n(z)\}$ has a cluster point at β, then $F_n(z) \to \beta$.

PROOF. Suppose that $\{F_n(z)\}$ has a cluster point at β, but does not converge to β. Then there exists an additional cluster point $\beta^* \neq \beta$. Assum $\beta^* \neq \infty$ (a slight variation on the following argument works for this special case).

No matter how large N is, there exists $n > N$ such that $F_n(z) \in N_\varepsilon(\beta^*)$, where $\varepsilon > 0$ is chosen to exclude by a large margin β or any β_j from $N_\varepsilon(\beta^*)$. We will show that this has the effect of eliminating the possibility of β being a cluster point of the sequence, thus providing a contradiction.

Under these conditions $|\lambda_{n+1}(F_n(z))| < M$ for an infinite subsequence of n's. For such an n large enough the entire structure of the proof of Lemma 2 remains intact, thus giving, for a suitable choice of ε, $|T_p o \ldots T_2(w_n)| < 2\varepsilon/(1-K_o) = 1$ for all p sufficiently large.

Now $|F_{n+p}(z) - \beta_{n+p}| = |\lambda_{n+p}{}^{-1} o K_{n+p}(T_p o \ldots o T_2(w_n)) - \beta_{n+p}|$

$\geq |\beta_{n+p} - \alpha_{n+p}|/(1 + |K_{n+p}|\,|T_p o \ldots o T_2(w_n)|)$

$> |\beta - \alpha|/(2(1 + |K|))$ for <u>all</u> p sufficiently large.

Therefore, it is not possible that $\{F_j(z)\}$ has a cluster point at β. $(\rightarrow\leftarrow)$ QED

Next, we see that z_o, if it exists, is unique.

LEMMA 4. There exists no more than one value z_o such that $F_n(z_o) \to \beta$.

PROOF. Suppose there are two such values, z_1 and z_2. Set $V_n := F_n(z_1)$ and $W_n := F_n(z_2)$. Observe that $V_n \neq W_n$ since F_n is one to one. For large n (using a local uniform convergence argument) $|f_{n+1}(V_n) - f_{n+1}(W_n)|/|V_n - W_n| \approx |f_{n+1}'(V_n)| \approx f'(\beta)| > 1$.

Therefore, $|V_{n+1} - W_{n+1}|/|V_n - W_n|$ for all n sufficiently large. Hence, one of $\{V_n\}$ or $\{W_n\}$ does not converge to β. $(\rightarrow\leftarrow)$ QED

It is now possible to complete the proof of Theorem 2.

If $z \neq z_o$, then Lemmas 3 and 4 tell us that there exists $D > 0$ such that $|F_n(z) - \beta_{n+1}| > D$ for all n sufficiently large. We use this to insure the boundedness of $\{F_n(z)\}$ for large n. Then it is possible to show that $|F_n(z) - \alpha| < \rho|\alpha - \beta|$ for large n, thus allowing the use of Lemma 1 and Theorem 3.

Suppose that $\{F_n(z)\}$ has a cluster point at ∞. Choose n large enough to guarantee that $|\lambda_{n+1}(F_n(z))| \approx 1$ <u>and</u> that the inequalities of Lemma 2 are valid. Then, for suitable $\varepsilon > 0$, $|T_p o \ldots o T_2(w_n)| < \rho/4K_o < 1/2K_o$.

It then follows that

$|F_{n+p}(z)| = |\lambda_{n+p}{}^{-1} o K_{n+p}(T_p o \ldots o T_2(w_n))|$

$\leq (|\beta_{n+p} K_{n+p}(T_p o \ldots o T_2(w_n))| + |\alpha_{n+p}|)/(1 - |K_{n+p}(T_p o \ldots o T_2(w_n))|)$

$\leq |\beta_{n+p}| + 2|\alpha_{n+p}| < B$ for all p sufficiently large. $(\rightarrow\leftarrow)$

Therefore, for <u>all</u> sufficiently large n, $|\lambda_{n+1}(F_n(z))| < M$.

Lemma 2 then insures that $|T_p o \ldots o T_2(w_n)| < \rho/4K_o < 1/2K_o$, for all sufficiently large values of p. Next,

$|F_{n+p}(z) - \alpha| \leq |F_{n+p}(z) - \alpha_{n+p}| + |\varepsilon_{n+p}|$ (recall: $\varepsilon_n := \alpha_n - \alpha$)

$= |\lambda_{n+p}{}^{-1} o K_{n+p}(T_p o \ldots o T_2(w_n)) - \alpha_{n+p}| + \varepsilon_{n+p}|$

$\leq K_o|T_p o \ldots o T_2(w_n)|\,|\beta_{n+p} - \alpha_{n+p}|/(1 - K_o|T_p o \ldots o T_2(w_n)|) + |\varepsilon_{n+p}|$

$< (\rho/2)|\beta_{n+p} - \alpha_{n+p}| + |\varepsilon_{n+p}| < \rho|\beta - \alpha|$ for all p sufficiently large.

Therefore $z_p := F_{n+p}(z)$ lies in the disk $(|z - \alpha| \leq R)$ of Lemma 1.

Theorem 3 then implies $Lim_{m\to\infty} f_{n+m} o f_{n+m-1} o \ldots o f_{n+p+1}(z_p) = \alpha$.

Hence $Lim_{n\to\infty} F_n(z) = \alpha$. QED

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