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NONSMOOTH ANAYLSIS APPROACH TO ISAAC'S EQUATION

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ABSTRACT. We study Isaacs' equation $(*)$ $w_t(t,x) + H(t,x,w_x(t,x)) = 0$ (H is a highly nonlinear function) whose "natural" solution is a value $W(t,x)$ of a suitable differential game. It has been felt that even though $W_x(t,x)$ may be a discontinuous function or it may not exist everywhere, $W(t,x)$ is a solution of $(*)$ in some generalized sense. Several attempts have been made to overcome this difficulty, including viscosity solution approaches, where the continuity of a prospective solution or even slightly less than that is required rather than the existence of the gradient $W_x(t,x)$. Using ideas from a very recent paper of Subbotin, we offer here an approach which, requiring literally no regularity assumptions from prospective solutions of $(*)$, provides existence results. To prove the uniqueness of solutions to $(*)$, we make some lower- and upper-semicontinuity assumptions on a terminal set Γ. We conclude with providing a close relationship of the results presented on Isaacs' equation with a differential games theory.

KEY WORDS AND PHRASES. Isaacs' equation, differential game, value function.
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1. INTRODUCTION.

There has been a constant interest for many years in solving Isaacs' equation, starting with the first serious treatment of Isaacs' equation by Fleming [6] (in the context of differential games), who approached this problem using several facts from stochastic processes and a theory of PDE; see also other treatments [1–5, 7–9, 13], including viscosity solution approaches, which basically assume (except for [13]) the continuity of a prospective solution. The present paper does not have much in common with the existing approaches, except for [10], which deals with fixed time problems.

Our starting point is a dynamical system (differential game) involving two control parameters $u \in U(t)$ and $v \in V(t)$, whose dynamics are described by the equation

$$\dot{x}(t) = f(t, x(t), u, v), \; x(t_0) = x_0, \; (t_0, x_0) \in \Omega_0 \subset R^{n+1}, \; t_0 \le t \le T, \tag{1.1}$$

where measurable control functions $u(t)$ and $v(t)$ are chosen with the aim of minimizing (resp. maximizing) a functional of the form

$$P = P[t_0, x_0, u(\cdot), v(\cdot)] = g(\tau, x(\tau)), \tag{1.2}$$

where $\tau = \tau[t_0, x_0, x(\cdot)]$ is the stopping time, the first time $t \in [t_0, T]$ for which $(t, x(t)) \in \Gamma \subset R^{n+1}$ (Γ is a fixed terminal set). This situation leads naturally (Section 5) to the lower Isaacs equation

$$\frac{\partial w}{\partial t}(t,x) + \max_{u \in U(t)} \min_{v \in V(t)} \langle \frac{\partial w}{\partial x}(t,x,), f(t,x,u,v) \rangle = 0 \tag{1.3}$$

with the boundary condition $w(t,x) = g(t,x)$ on Γ; since we assume Isaacs' condition:

$$\max_{u \in U(t)} \min_{v \in V(t)} \langle s, f(t,x,u,v) \rangle = H(t,x,s) = \min_{v \in v(t)} \max_{u \in U(t)} \langle s, f(t,x,u,v) \rangle, \tag{1.4}$$

the lower Isaacs equation, coincides with the upper Isaacs equation, and is commonly called Isaacs' equation: $\frac{\partial w}{\partial t}(t,x) + H(t,x,\frac{\partial w}{\partial x}(t,x)) = 0$. We study this highly nonlinear PDE on

$$\Omega = \{(\bar{t},\bar{x}) : \bar{x} = x(\bar{t}, t_0, x_0, u(\cdot), v(\cdot)), (t_0, x_0) \in \Omega_0, t_0 \le \bar{t} \le \tau \le T\}, \tag{1.5}$$

the set of all points attainable from Ω_0 (a given set of initial conditions). The interesting feature of Isaacs' equation is that H-J equations $\frac{\partial w}{\partial t}(t,x) + H(t,x,\frac{\partial w}{\partial x}(t,x)) = 0$, with $H(t,x,s)$ satisfying some regularity conditions, may be viewed as Isaacs' equations associated to certain differential games (see [5] and references in [10]). Throughout the paper, we assume conditions (1.4), (2.1)–(2.3), and (2.9)–(2.10); in Sections 4, 5 we additionally impose some continuity requirements on prospective solutions of Isaacs' equation.

Our goal is to adopt the notion of a minimax solution to the Cauchy problem for a H-J equation, introduced recently by Subbotin for fixed time problems ($\tau \equiv T$) [10], to the variable time setting. We define the notions of "upper" and "lower" solutions of Isaacs' equation, without imposing on them literally any continuity assumptions, in such a way that (Proposition 3.1) Isaacs' equation always admits lower and upper solutions. What is more, we prove $w_+(t,x)$ (resp.$w_-(t,x)$), being the infinum (resp. supremum) over all upper (lower) solutions of Isaacs' equation, is its solution (Theorem 3.5); it means, it is both an upper and lower solution.

When we compare solutions of Isaacs' equation and eventually prove uniqueness results, we impose some continuity assumptions. Typically, we require from a prospective solution to be lower- or upper-semicontinuous; sometimes we need continuity at each point $(t,x) \in \Gamma$. For example, any two solutions of Isaacs' equation must coincide if they are continuous at each point $(t,x) \in \Gamma$) (Corollary 4.3). Another result (Theorem 4.4) says that if $w_+(t,x)$ (resp. $w_-(t,x)$) is lsc (resp. usc) at each point $(t,x) \in \Gamma$, then Isaacs' equation admits exactly one solution $w^0(t,x) = w_+(t,x) = w_-(t,x)$, which happens to be continuous on Ω.

In the last section, we return to the differential game we started with to show, for example, that (i) if the upper value of the game $\overline{W}(t,x)$ is lsc (usc) on Ω, then it is a upper (lower) solution of Isaacs' equation; (ii) if $\underline{W}(t,x)$, the lower value of the game, is usc (lsc) on Ω then $\underline{W}(t,x)$ is a lower (upper) solution of Isaacs' equation; and (iii) if $\overline{W}(t,x)$ is usc on Ω and $\underline{W}(t,x)$ is lsc on Ω then the differential game has a value $W(t,x)$, which is a continuous solution of Isaacs' equation satisfying $w_-(t,x) \ge W(t,x) \ge w_+(t,x)$.

2. ASSUMPTIONS AND THE CONCEPT OF A SOLUTION OF ISAACS' EQUATION

Apart from (1.4), we make the following assumptions ((2.1)–(2.3), (2.9) and (2.10)).

$f(t,x,u,v)$ is a continuous function in all variables; $U(t)$ and $V(t)$ are compact sets varying continuously in time on the interval $[0,T]$. (2.1)

There is a constant λ such that $||f(t,x,u,v) - f(t,\bar{x},u,v)|| \le \lambda ||x - \bar{x}||, u \in U(t), v \in V(t), 0 \le t \le T$. (2.2)

Assumptions (2.1), (2.2) imply [11, Corollary 2,1] that $X(t_0, x_0)$, the set of all solutions of equation (1.1), is a precompact subset of $C^n[t_0,T]$, the space of continuous mappings from $[t_0,T]$ into R^n, equipped with the max norm. To simplify our presentation, we make two "convexity" assumptions, ensuring the compactness of certain sets of trajectories. For example, the condition below guarantees the compactness of $X(t_0,x_0)$ for all $(t_0,x_0) \in \Omega$.

The sets $f(t,x,U(t),V(t))$ are convex in R^n for all $(t,x) \in \Omega$. (2.3)

To go ahead with our preliminary analysis, we use Isaacs' condition (1.4), which implies the existence of a saddle point pair (u_s^*, v_s^*), for any triplet (t, x, s), satisfying the saddle point inequality

$$\langle s, f(t,x,u,v_s^*)\rangle \leq s, f(t,x,u_s^*,v_s^*)\rangle = H(t,x,s) \leq \langle s, f(t,x,u_s^*,v)\rangle \tag{2.4}$$

holding for all $u \in U(t), v \in V(t)$. It is well known from elementary game theory that if (u_s^1, v_s^1) and (u_s^2, v_s^2) are saddle point pairs, then they are "exchangeable," i.e., (u_s^1, v_s^2) and (u_s^2, v_s^1) are also saddle point pairs. This observation leads us to set-valued maps $U(t,x,s) \subset U(t)$ and $V(t,x,s) \subset V(t)$, containing u_s^* and v_s^*, respectively, such that $\langle s, f(t,x,u,V(t,x,s))\rangle \leq \langle s, f(t,x,U(t,x,s), V(t,x,s))\rangle = H(t,x,s) \leq \langle s, f(t,x,U(t,x,s),v)\rangle$. These maps give rise to the two set-valued maps

$$F_+(t,x,s) = \{f(t,x,U(t,x,s),v) : v \in V(t)\}, \tag{2.5}$$
$$F_-(t,x,s) = \{f(t,x,u,V(t,x,s)) : u \in U(t)\}, \tag{2.6}$$

which are usc in (t,x) and compact-valued. In addition to (2.3), we make one more convexity assumption, (2.9), ensuring the compactness of $X_+(t_0,x_0,s), X_-(t_0,x_0,s)$, whose elements are solutions of the differential inclusions:

$$\dot{x}(t) \in F_+(t,x(t),s), \quad t \in [0,T], \quad x(t_0) = x_0, \tag{2.7}$$
$$\dot{x}(t) \in F_-(t,x(t),s), \quad t \in [0,T], \quad x(t_0) = x_0, \tag{2.8}$$

respectively.

The sets $F_+(t,x,s), F_-(t,x,s)$ are convex for all $(t,x) \in \Omega, s \in R^n$. (2.9)

The set Ω_0 is bounded, the terminal set $\Gamma \subset \Omega$ is closed and $\tau[x(\cdot)] \leq T$ for any trajectory $x(t) = x(t,t_0,x_0,u(\cdot),v(\cdot)), (t_0,x_0) \in \Omega_0$, of equation (1.1); in addition, $\inf\{t : (t,x) \in \Omega_0\} = 0$, $\sup\{t : (t,x) \in \Omega_0) \leq T$. (2.10)

REMARK 2.1. Our notion of a solution of Isaacs' equation will be applicable to a wide class of prospective solutions (we do not impose any continuity assumptions). If, however, a prospective solution $w(t,x)$ had a gradient at a point $(\bar{t},\bar{x})$, then our definition would imply the identity: $\frac{\partial w}{\partial t}(\bar{t},\bar{x}) + H(\bar{t},\bar{x},\frac{\partial w}{\partial x}(\bar{t},\bar{x})) = 0$.

DEFINITION 2.1. By an upper solution of Isaacs' equation (1.3) with the boundary condition $w(t,x) = g(t,x)$ on $\Gamma \subset \Omega$, we understand a function $w(t,x) : \Omega \to R$ satisfying $w(t,x) \geq g(t,x)$ on Γ and the inequality:

$$\forall(\bar{t},\bar{x}) \in \Omega,\ \forall t^* \in [\bar{t},T]\ \sup_{\|s\|=1}\ \inf_{x(\cdot)\in X_+(\bar{t},\bar{x},s)} w(\bar{t}^*, x(\bar{t}^*)) \leq w(\bar{t},\bar{x}),$$
$$\bar{t}^* = \min(t^*, \tau[x(\cdot)]). \tag{2.11}$$

When $w(t,x)$ is lsc on Ω, then (2.11) can be replaced by the statement that $w(t,x(t))$ decreases along some trajectory $x(\cdot) \in X_+(\bar{t},\bar{x},s)$ on the interval $[\bar{t},\tau[x(\cdot)]]$; clearly, the inf in (2.11) can then be replaced with min. Without the lower-semicontinuity assumption, one can always guarantee that, for each k,

$$w(\bar{t}+\frac{i}{k}, x(\bar{t}+\frac{i}{k})) \leq w(\bar{t}+\frac{r}{k}, x(\bar{t}+\frac{r}{k})) \leq w(\bar{t},\bar{x})$$
$$\text{for positive integers } i,r \text{ satisfying } r < i, \bar{t}+\frac{i}{k} \leq \bar{t}^*. \tag{2.12}$$

DEFINITION 2.2. By a lower solution of Isaacs' equation (1.3) one understands a function $w(t,x) : \Omega \to R$ satisfying $w(t,x) \leq g(t,x)$ on Γ and the inequality:

$$\forall(\bar{t},\bar{x}) \in \Omega, \forall t^* \in [\bar{t},T],\ \inf_{\|s\|=1}\ \sup_{x(\cdot)\in X_-(\bar{t},\bar{x},s)} w(\bar{t}^*, x(\bar{t}^*)) \geq w(\bar{t},\bar{x}), \tag{2.13}$$

where $\bar{t}^* = \min(t^*, \tau[x(\cdot)])$.

Similar remarks to those following (2.11) apply here, with obvious changes.

DEFINITION 2.3. A function $w : \Omega \to R$ is a solution of Isaacs' equation (1.3) if $w(t,x)$ is both a upper and lower solution of (1.3).

It is known [10] that (2.11) (resp. (2.13)) is equivalent to the inequality

$$\inf\{\partial^-_{(1,f)}w(t,x) : f \in F_+(\bar{t},\bar{x},s)\} \leq 0 \text{ (resp. } \sup\{\partial^+_{(1,f)}w(t,x) : f \in F_-(\bar{t},\bar{x},x)\} \geq 0),$$

for any $s, ||s|| = 1$, if $w(t,x)$ is lsc (resp. usc) on Ω; here $\partial^-_h w(t,x)$ and $\partial^+_h w(t,x)$ are the lower and upper Dini derivatives of $w(t,x)$ at the point $(\bar{t},\bar{x})$ in the direction of h. When $w(t,x)$ has a gradient at $(\bar{t},\bar{x})$ then (2.11) and (2.13) become equivalent to the inequalities: $\frac{\partial w}{\partial t}(\bar{t},\bar{x}) + \min_{v\in V(\bar{t})}\langle \frac{\partial w}{\partial x}(\bar{t},\bar{x}), f(\bar{t},\bar{x},U(\bar{t},\bar{x},s),v)\rangle \leq 0$ and $\frac{\partial w}{\partial t}(\bar{t},\bar{x}) + \max_{u\in U(\bar{t})}\langle \frac{\partial w(\bar{t},\bar{x})}{\partial x}, f(\bar{t},\bar{x},u,V(\bar{t},\bar{x},s))\rangle \geq 0$, respectively. By setting $s = \frac{\partial w}{\partial x}(\bar{t},\bar{x})$, we get (see (2.4)):

$$\frac{\partial w}{\partial t}(\bar{t},\bar{x}) + H(\bar{t},\bar{x}, \frac{\partial w}{\partial x}(\bar{t},\bar{x})) \leq 0, \quad \frac{\partial w}{\partial t}(\bar{t},\bar{x}) + H(\bar{t},\bar{x}, \frac{\partial w}{\partial x}(\bar{t},\bar{x})) \geq 0.$$

It follows from (2.4) that the latter two inequalities imply the former two, so that they are equivalent. It means our concept of a solution to Isaacs' equation is exactly what it should be when applied to regular functions (see Remark 3.1).

3. GENERAL CASE (WITHOUT CONTINUITY ASSUMPTIONS)

Let us start with the following three properties:

(F1) $\min_{f\in F_+(t,x,s)}\langle s,f\rangle = H(t,x,s) = \max_{f\in F_-(t,x,s)}\langle s,f\rangle, s \in R^n$;

(F2) $F_+(t,x,q) \cap F_-(t,x,p) = f(t,x,U(t,x,p),V(t,x,q)) \neq \emptyset$;

(F3) $|H(t,x,s) - H(t,y,s)| \leq \lambda||x-y|| \cdot ||s||$ for $(t,x) \in \Omega, (t,y) \in \Omega, s \in R^n$.

Note that properties F1, F2 are obvious, while F3 requires a short proof.

PROOF Without loss of generality, let $H(t,y,s) \geq H(t,x,s)$. With (u^x_s, v^x_s) meaning a saddle point pair for (t,x,s) and (u^y_s, v^y_s) meaning the same thing for (t,y,s), $|H(t,x,s) - H(t,y,x)|$ will be equal to (see (2.4))

$$\begin{aligned}
&\min_{v\in V(t)}[\max_{u\in U(t)}\langle s, f(t,y,u,v))\rangle] - \min_{v\in V(t)}[\max_{u\in U(t)}\langle s, f(t,x,u,v)\rangle] \\
= \;& \langle s, f(t,y,u^y_s,v^y_s) - f(t,y,u^y_s,v^x_s)\rangle + \langle s, f(t,y,u^y_s,v^x_s) - f(t,x,u^x_s,v^x_s)\rangle \leq 0+ \\
& \langle s, f(t,y,u^y_s,v^x_s) - f(t,x,u^y_s,v^x_s)\rangle + \langle s, f(t,x,u^y_s,v^x_s) - f(t,x,u^x_s,v^x_s)\rangle \\
& \leq \lambda||x-y|| \cdot ||s|| + 0.
\end{aligned}$$

It follows directly from F2 that $X_+(\bar{t},\bar{x},q) \cap X_-(\bar{t},\bar{x},p) \neq \emptyset$ for any $(\bar{t},\bar{x}) \in \Omega, q \in R^n, p \in R^n$; also, by virtue of (2.9), $X_+(\bar{t},\bar{x},q)$ and $X_-(\bar{t},\bar{x},p)$ are compact in $C^n[\bar{t},T]$.

PROPOSITION 3.1. The set of all upper solutions to Isaacs' equation (1.3) contains at least two (possible identical) elements, namely the functions $w^m(t,x) \equiv G = \max\{g(t,x) : (t,x) \in \Gamma\}$ and $w^{\max}(t,x)$, given by (3.1). Similarly, the set of all lower solutions to Isaacs' equation contains the constant function $W_w(t,x) \equiv g = \min\{g(t,x) : (t,x) \in \Gamma\}$ and $w^{\min}(t,x)$, given by (3.2). Both $w^{\max}(t,x)$ and $w^{\min}(t,x)$ coincide with $g(t,x)$ on Γ.

PROOF It is obvious that $w^m(t,x)$ and $w_m(t,x)$ are respectively an upper solution and a lower solution to Isaacs' equation. Setting

$$w^{\max}(t,x) = \max\{g(\tau, x(\tau)) : x(\cdot) \in \cup_{||s||=1}X_+(t,x,s)\}, \tag{3.1}$$

let us observe $w^{\max}(t,x) = g(t,x)$ on Γ. To show (2.11), select $(\bar{t},\bar{x}) \in \Omega$, $t^* \in [\bar{t},T]$, and $\bar{s} \in R^n$, $||\bar{s}|| = 1$, in order to demonstrate even something more than is required in (2.11), namely that $w^{\max}(t,x(t)) \leq w^{\max}(\bar{t},\bar{x})$ on the whole interval $[\bar{t},\tau[x(\cdot)]]$, including the point $\bar{t}^* = \min(t^*,\tau[x(\cdot)])$ for any $x(\cdot) \in X_+(\bar{t},\bar{x},\bar{s})$. Indeed, this fact follows from the very definition of $w^{\max}(t,x)$; as a matter of fact, we will have $w^{\max}(t, x(t)) < w^{\max}(\bar{t}, \bar{x})$ for all $x(\cdot) \in X_+(\bar{t}, \bar{x}, \bar{s})$ for which $g(\tau, x(\tau)) < w^{\max}(t, x)$. In a similar manner one shows the function

$$w^{\min}(t, x) = \min\{g(\tau, x(\tau)) : x(\cdot) \in \cup_{||x||=1} X_-(t, x, s)\} \tag{3.2}$$

is a lower solution of Isaacs' equation.

REMARK 3.1. Inequality (2.11) (a similar observation refers to (2.13)) cannot be replaced with the sharp inequality (without changing the meaning of the definition) because the constant upper solution $w^m(t, x) \equiv G$ would not satisfy it. We also cannot replace it with the equality because then $w^{\max}(t, x)$ would not be an upper solution.

LEMMA 3.2. The lsc envelope of an upper solution $w(t, x)$ to Isaacs' equation (1.3), given by

$$w_*(\bar{t}, \bar{x}) = \liminf_{(t,x)\to(\bar{t},\bar{x})} w(t, x), \quad (\bar{t}, \bar{x}) \in \Omega, \tag{3.3}$$

satisfies condition (2.11). It is an upper solution of (1.3) if $w(t, x)$ is lsc at each point $(t, x) \in \Gamma$.

PROOF Note that if $w(t, x)$ is lsc at each point $(t, x) \in \Gamma$ then $g(t, x) \leq w(t, x) = w_*(t, x)$ on Γ, as required in Definition 2.1. Let us choose a point $(\bar{t}, \bar{x}) \in \Omega, \bar{s} \in R^n$, and $t^* \in [\bar{t}, T]$. There is a sequence (t_k, x_k) convergent to $(\bar{t}, \bar{x})$ with $w(t_k, x_k)$ converging to $w_*(\bar{t}, \bar{x})$. Since $w(t, x)$ is an upper solution, there exist trajectories $x_k(\cdot) \in X_+(t_k, x_k, \bar{s})$ such that $w(\bar{t}_k^*, x_k(\bar{t}_k^*)) \leq w(t_k, x_k) + \frac{1}{k}, \bar{t}_k^* = \min(t^*, \tau[x_k(\cdot)])$. Since $X_+(t_k, x_k, \bar{s})$ are compact in the Banach space $C^n[t_k, T]$ and depend continuously on (t_k, x_k), we can assume $x_k(\cdot)$ converges to some $\bar{x}(\cdot) \in X_+(\bar{t}, \bar{x}, \bar{s})$ with $\tau[x(\cdot)] \leq \liminf_{k\to\infty} \tau[x_k(\cdot)]$. What is more, we can assume $x_k(\cdot)$ were chosen in such a way that

$$w(t_k, x_k) \geq w(t_k + \frac{i}{2^k}, x_k(t_k + \frac{i}{2^k})) \geq w(t_k + \frac{r}{2^k}, x_k + \frac{r}{2^k}))$$

whenever $1 \leq i \leq r$, with i, r being any natural numbers for which $t_k + \frac{r}{2^k} \leq \tau[x_k(\cdot)]$. We thus have $w_*(\bar{t} + \frac{r}{2^k}, x(\bar{t} + \frac{r}{2^k})) \leq \lim_{k\to\infty} w(t_k, x_k) = w_*(\bar{t}, \bar{x})$, for all natural numbers r, k satisfying $\bar{t} + \frac{r}{2^k} \leq \tau[\bar{x}(\cdot)]$. Since the set of points $\bar{t} + \frac{r}{2^k}$ is dense in $[\bar{t}, \infty)$, and $w_*(t, x)$ is a lsc function, we have $w_*(t, \bar{x}(t)) \leq w_*(\bar{t}, \bar{x})$ for any $t \in [\bar{t}, \tau[x(\cdot)]]$, $\bar{x}(\cdot) \in X_+(\bar{t}, \bar{x}, \bar{s})$.

Arguing in an analogous manner, one can prove our next result.

LEMMA 3.3. The usc envelope $w^*(\bar{t}, \bar{x}) = \limsup_{(t,x)\to(\bar{t},\bar{x})} w(t, x)$ of a lower solution $w(t, x)$ of Isaacs' equation (1.3) satisfies condition (2.13). It is a lower solution if $w(t, x)$ is usc at each $(t, x) \in \Gamma$.

Let us denote by $w_+(t, x)$ the infimum over all upper solutions of Isaacs' equation, i.e.,

$$w_+(t, x) = \inf\{w(t, x) : w(\cdot) \text{ is an upper solution of (1.3) }\}, \ (t, x) \in \Omega. \tag{3.4}$$

Analogously,

$$w_-(t, x) = \sup\{w(t, x) : w(\cdot) \text{ is an lower solution of (1.3) }\}, \ (t, x) \in \Omega. \tag{3.5}$$

LEMMA 3.4. $w_+(t, x)$ is an upper solution of Isaacs' equation (1.3), and $w_-(t, x)$ is a lower solution of (1.3).

PROOF First observe $w_+(t, x) = g(t, x)$ and $w_-(t, x) = g(t, x)$ on Γ (see Proposition 3.1). Given a point $(\bar{t}, \bar{x}) \in \Omega, t^* \in [\bar{t}, T]$, and $\bar{s} \in R^n$, there exists a sequence $w_k(t, x)$ of upper solutions converging to $w_+(t, x)$, and a sequence of trajectories $x_k(\cdot) \in X_+(\bar{t}, \bar{x}, \bar{s})$ for which $w_+(t_k^*, x_k(t_k^*)) \leq w_k(t_k^*, x_k(t_k^*)) \leq w_k(\bar{t}, \bar{x}) + \frac{1}{k} \leq w_+(\bar{t}, \bar{x}) + \frac{2}{k}$, starting with a certain index $K (k > K)$, where $t_k^* = \min(t^*, \tau[x_k(\cdot)])$. It means (2.11) holds; arguing similarly, we demonstrate $w_-(t, x)$ satisfies condition (2.13).

THEOREM 3.5. $w_+(t, x)$ and $w_-(t, x)$ are solutions of Isaacs' equation (1.3).

PROOF In view of the previous lemma, it is sufficient to show $w_+(t, x)$ satisfies condition (2.13) and $w_-(t, x)$ satisfies condition (2.11). To show (2.13), we argue somewhat similarly as in the proof of Lemma 3.1 from [10]. We choose $t^* \in [0, T], p \in R^n, ||p|| = 1$, to define the auxiliary function (below $\bar{t}^* = \min(t^*, \tau[x(\cdot)])$):

$$w_p(t, x) = \begin{cases} w_+(t, x) \text{ if } t > t^* \\ \max\{w_+(\bar{t}^*, x(\bar{t}^*)) : x(\cdot) \in X_-(t, x, p)\} \text{ if } t \leq t^* \end{cases} \tag{3.6}$$

with the aim of showing $\max\{w_+(\overline{t}^*, x(\overline{t}^*)): x(\cdot) \in X_-(t,x,p)\} \geq w_+(t,x)$ if $t* \in [t,T]$, as required in (2.13). If we prove $w_p(t,x)$ is a upper solution of Isaacs' equation, then we will know $w_p(t,x) \geq w_+(t,x)$ on Ω.

First note $w_p(t,x) = w_+(t,x) = g(t,x)$ on Γ (see Lemma 3.4). To show $w_p(t,x)$ satisfies condition (2.11) at $(\overline{t},\overline{x}) \in \Omega$, consider two cases: $\overline{t} > t^*$ and $\overline{t} \leq t^*$. In the first case (2.11) holds because $w_p(t,x)$ coincides with $w_+(t,x)$ for $t > t^*$. In the second case, select $t_q^* \in [\overline{t},T], q \in R^n, ||q|| = 1$, to demonstrate

$$L = \min\{w_p(\overline{t}_q^*, x(\overline{t}_q^*)) : x(\cdot) \in X_+(\overline{t},\overline{x},q)\} \leq w_p(\overline{t},\overline{x}),\ \overline{t}_q^* = \min(t_q^*, \tau[x(\cdot)]). \tag{3.7}$$

With $x^0(\cdot) \in X_+(\overline{t},\overline{x},q) \cap X_-(\overline{t},\overline{x},p)(\dot{x}^0(t) \in F_+(t,x^0(t),q) \cap F_-(t,x^0(t),p))$, we study two subcases: (a) $t^0 = \min(t_q^*, \tau[x^0(\cdot)]) \leq t^*$, and (b) $t^0 > t^*$. In the first subcase, $L \leq w_p(t^0, x^0(t^0)) \leq w_p(\overline{t},\overline{x})$ (cf. (3.6)). In the second subcase (b), we are going to show the triplet inequality

$$L \leq w_p(t^0, x^0(t^0)) \leq w_p(t^*, x^0(t^*)) \leq w_p(\overline{t},\overline{x}), \tag{3.8}$$

the first part of which is obvious because $x^0(\cdot) \in X_+(\overline{t},\overline{x},q)$. The second part holds for $x^0(\cdot) \in X_-(t^*, x^0(t^*), p)$ (see (3.6)). The last (third) part of (3.8) is also clear because the set of all $x(\cdot) \in X_-(\overline{t},\overline{x},p)$ comprises all trajectories $x(\cdot) \in X_-(\overline{t},\overline{x},p)$ which coincide with $x^0(\cdot) \in X_-(\overline{t},\overline{x},p)$ on the interval $[\overline{t}, t^*]$.

Arguing similarly, one can demonstrate $w_-(t,x)$ satisfies condition (2.11).

4. SPECIAL CASE (UPPER AND LOWER SOLUTIONS SATISFY CONTINUITY ASSUMPTIONS)

REMARK 4.1. If $w_+(t,x)$ (given by (3.4)) and $w_-(t,x)$ (given by (3.5)) are respectively lsc and usc at each point $(t,x) \in \Gamma$, then they are respectively lsc and usc on Ω.

PROOF As $w_+(t,x)$ is lsc on Γ then, by Lemma 3.2, it coincides with its lsc envelope on Ω, which is of course lsc on Ω. Similarly, $w_-(t,x)$ is usc on Ω because it coincides with it usc envelope (Lemma 3.3).

LEMMA 4.1. If $u(t,x)$, a upper solution of Isaacs' equation, is lsc at each point $(t,x) \in \Gamma$, and $w(t,x)$, a lower solution of Isaacs' equation, is usc at each point $(t,x) \in \Gamma$, then $u(t,x) \geq w(t,x)$ on Ω. In particular, with $u(t,x) = w_+(t,x)$ and $w(t,x) = w_-(t,x)$ (see (3.4) and (3.5)), we have $w_+(t,x) \geq w_-(t,x)$ on Ω and consequently $u(t,x) \geq w(t,x)$ on Ω, for any upper solution $u(t,x)$ and any lower solution $w(t,x)$ of Isaacs' equation.

Note that the second part of Lemma 4.1 gives us the strongest application of the first part, while the weakest application is obtained when $u(t,x) = w^m(t,x) \equiv G$ and $w(t,x) = w_m(t,x) \equiv g$ (see Proposition 3.1).

PROOF Choose a point $(\overline{t},\overline{x}) \in \Omega \backslash \Gamma$ to show $u(\overline{t},\overline{x}) \geq w(\overline{t},\overline{x})$. We argue partly similarly as in the proof of Lemma 3.3 from [10], where the special case $\tau \equiv T$ was dealt with. Namely, we define the set $M_k(t), \overline{t} \leq t \leq T$, as the collection of all pairs $(y(\cdot), z(\cdot)) \in X_+(\overline{t},\overline{x}) \times X_-(\overline{t},\overline{x})$ for which

(i) $u(r, y(r))$ is decreasing on the set of points $\overline{t} + \frac{i}{k} \in [\overline{t}, t_y]$, $t_y = \min(t, \tau[y(\cdot)])$, including the end point t_y (see (2.12));

(ii) $w(r, z(r))$ is increasing on the set of points $\overline{t} + \frac{i}{k} \in [\overline{t}, t_z]$, $t_z = \min(t, \tau[z(\cdot)])$, including the end point t_z;

(iii) $\langle z(r) - y(r), \dot{z}(r) - \dot{y}(r) \rangle \leq \lambda ||z(r) - y(r)||^2 + \frac{1}{k}$ on $[\overline{t}, \min(t_y, t_z)]$.

There is no problem with finding pairs $(y(\cdot), z(\cdot))$ satisfying (i), (ii) for $t = T$ (see (2.12)). What we want to show now is that condition (iii) also holds, for any $k > 0$, i.e., $M_k(T) \neq \emptyset$. Suppose we have done it. Then, for each $\frac{1}{k}$, there is a pair $[y_k(\cdot), z_k(\cdot)] \in M_k(T)$ satisfying (i)–(iii). Let $\tau_k^y = t_y = \min(T, \tau[y_k(\cdot)]) = \tau[y_k(\cdot)]$ (by virtue of (2.10)), and $\tau_k^z = t_z = \min(T, \tau[z_k(\cdot)]) = \tau[z_k(\cdot)]$. With $\tau_k = \min(\tau_k^y, \tau_k^z)$ we have $\langle z_k(r) - y_k(r), \dot{z}_k(r) - \dot{y}_k(r) \rangle \leq \lambda ||z_k(r) - y_k(r)||^2 + \frac{1}{k}$ on $[\overline{t}, \tau_k]$. Since $z_k(r) - y_k(r)$ tend to zero with k tending to infinity (uniformly in r), and $X(\overline{t},\overline{x})$ is compact in $C^n[\overline{t},T]$, we may assume $y_k(\cdot), z_k(\cdot)$ tend to some $x(\cdot) \in X(\overline{t},\overline{x})$. Without loss of generality, let us assume $\tau_k = \tau_k^y$. We will then have that $(\tau_k, y_k(\tau_k))$ and $(\tau_k, z_k(\tau_k))$ converge to some point $(\tau, x(\tau)) \in \Gamma$, taking into account $(\tau_k, y_k(\tau_k)) \in \Gamma$ and Γ is a closed set. In addition, $u(\tau_k, y_k(\tau_k)) \leq u(\bar{t}, \bar{x})$ and $w(\bar{t} - \frac{i_k}{k}, z_k(\bar{t} + \frac{i_k}{k})) \geq w(\bar{t}, \bar{x})$, where $\bar{t} + \frac{i_k}{k}$ is within the distance of $\frac{1}{2k}$ to τ_k. Finally,

$$u(\bar{t}, \bar{x}) \geq \lim_{k \to \infty} u(\tau_k, y_k(\tau_k)) \geq \lim_{k \to \infty} g(\tau_k, y_k(\tau_k)) = g(\tau, x(\tau))$$

and (because $w(t,x)$ is usc at each point $(t,x) \in \Gamma$)

$$w(\bar{t}, \bar{x}) \leq \lim_{k \to \infty} w(\bar{t} + \frac{i_k}{k}, z_k(\bar{t} + \frac{i_k}{k})) \leq w(\tau, x(\tau)) \leq g(\tau, x(\tau)).$$

If $\tau_k = \tau_k^z$ then we need the lower-semicontinuity of $u(t,x)$ at each point $(t,x) \in \Gamma$.

What remains to do now is to prove condition (iii) holds for any positive integer k on the maximal interval $[\bar{t}, \min(\tau[y(\cdot)], \tau[z(\cdot)])]$, that is to say, $M_k(T) \neq \emptyset$. Suppose on the contrary that, for any pair $(y(\cdot), z(\cdot))$ satisfying conditions (i), (ii), the maximal interval on which (iii) holds is $[\bar{t}, t_k]$ with $t_k < \min(\tau[y(\cdot)], \tau[z(\cdot)])$. With $(y(\cdot), z(\cdot))$ being such a pair, set $y_k = y(t_k), z_k = z(t_k), s_k = z_k - y_k$, to observe, based on property F1, that

$$\min_{f \in F_+(t_k, y_k, s_k)} \langle s_k, f \rangle = H(t_k, y_k, s_k), \quad \max_{f \in F_-(t_k, z_k, s_k)} \langle s_k, f \rangle = H(t_k, z_k, s_k), \tag{4.1}$$

and consequently $\langle s_k, f(t_k, y_k, u, v_{s_k}) \rangle - \langle s_k, f(t_k, z_k, u_{s_k}, v) \rangle \leq H(t_k, z_k, s_k) - H(t_k, y_k, s_k) \leq \lambda \|z_k - y_k\| \cdot \|s_k\| \leq \lambda \|s_k\|^2$ (see (2.4) and property F3), for any $v \in V(t_k), u \in U(t_k)$, which implies $\langle z_k - y_k, \dot{z}_k - \dot{y}_k \rangle \leq \lambda \|s_k\|^2$ (see (iii)). Since the sets $F_+(t, x, s_k), F_-(t, x, s_k)$ depend upper-semicontinuously on (t,x), the function $s(r) = z(r) - y(r)$ is continuous and, based on (2.11), (2.13), $y(r)$ and $z(r)$ can be extended on $[t_k, t_k + \sigma]$ (for some σ) in such a way that $\dot{y}(r) \in F_+(r, y(r), s_k)$, $\dot{z}(r) \in F_-(r, z(r), s_k)$, with (i) and (ii) satisfied on $[\bar{t}, \min(t + \sigma, \tau[y(\cdot)])]$, we will have

$$\langle z(r) - y(r), \dot{z}(r) - \dot{y}(r) \rangle \leq \lambda \|z(r) - y(r)\|^2 + \frac{1}{k}, \; t_k \leq r \leq t_k + \sigma.$$

This contradiction shows $M_k(T) \neq \emptyset$.

COROLLARY 4.2. If $w_+(t,x)$ is lsc at each point $(t,x) \in \Gamma$ and $w_-(t,x)$ is usc at each point $(t,x) \in \Gamma$ then there is at most one solution $w^0(t,x)$ of Isaacs' equation (1.3). If it exists, it is continuous on Ω and satisfies $w^0(t,x) = w_+(t,x) = w_-(t,x)$ on Ω.

PROOF By Lemma 4.1 we have $w_+(t,x) \geq w_-(t,x)$ on Ω. If a solution $w^0(t,x)$ of Isaacs' equation exists then, by virtue of (3.4) and (3.5), $w_-(t,x) \geq w^0(t,x) \geq w_+(t,x)$ on Ω, and $w^0(t,x)$ is continuous on Ω (by virtue of Remark 4.1). The proof is completed.

COROLLARY 4.3. Any two solutions $u(t,x)$ and $w(t,x)$ of Isaacs' equation must coincide if they are continuous at each point $(t,x) \in \Gamma$.

PROOF By Lemma 4.1, we have $u(t,x) \geq w(t,x)$ and $w(t,x) \geq u(t,x)$.

THEOREM 4.4. If $w_+(t,x)$ (resp. $(w_-(t,x))$, given by (3.4), (3.5), is lsc (resp. usc) at each point $(t,x) \in \Gamma$, then there is exactly one solution $w^0(t,x)$ of Isaacs' equation satisfying $w^0(t,x) = w_-(t,x) = w_-(t,x)$. This solution is continuous on Ω.

PROOF The conclusion follows from Corollary 4.2 and Theorem 3.5.

COROLLARY 4.5. If $\Gamma = \{(t,x) \in \Omega : t = T]$, i.e., the boundary condition for Isaacs' equation becomes $w(T,x) = g(T,x)$, then $w_+(t,x)$ is lsc at each point $(t,x) \in \Gamma$ and $w_-(t,x)$ is usc at each point $(t,x) \in \Gamma$. Consequently, the conclusions of Theorem 4.4 are valid.

REMARK 4.2. The content of Corollary 4.5 is less general, although close enough to that of Theorem 2.1 from [10].

PROOF Assume $w_+(T, \bar{x}) > \lim_{(t,x) \to (T,\bar{x})} w_+(t,x)$ for some $(T, \bar{x}) \in \Gamma$. There must then exist a sequence $(t_n, x_n) \in \Omega$ for which $w_+(t_n, x_n) < w_+(T, \bar{x}) - \varepsilon, (t_n, x_n) \to (T, \bar{x})$, and, by virtue of (2.11), $w_+(T, x_n(T)) \leq w_+(t_n, x_n)$ with $g(T, x_n(\tau)) \leq w_+(T, x_n(T))$ for some trajectories $x_n(\cdot) \in X(t_n, x_n)$. This implies

$$g(T, \bar{x}) = \lim_{n \to \infty} g(T, x_n(T)) \leq w_+(T, \bar{x}) - \varepsilon,$$

which is impossible because (see Proposition 3.1) $w_+(T, \bar{x}) \leq w^{\max}(T, \bar{x}) = g(T, \bar{x}) \leq w_+(T, \bar{x}) - \varepsilon$. Arguing similarly, one can prove $w_-(t,x)$ is usc at each point $(T, \bar{x}) \in \Omega$.

5. RELATIONSHIP TO DIFFERENTIAL GAMES, CONTINUITU OF THE VALUE FUNCTION

Let us return now to the differential game (1.1), (1.2) to find interpretations for $w_+(t,x)$ and $w_-(t,x)$. To define the game completely, we have to specify what we mean by strategies for player I (choosing a control function $(u(t))$ and player II (choosing a control function $v(t)$). Next, we have to define the upper value of the game, $\overline{W}(t,x)$, and the lower value of the game, $\underline{W}(t,x)$ (if they are equal, the differential game has a value).

We recall the definitions of upper and lower values of the game introduced in [13], as well as the optimality principles of dynamic programming in differential games obtained first in [12], and next extended to a more general case in [13], to prove Theorems 5.1, 5.2, 5.3 ensuring, under semicontinuity assumptions imposed on $\overline{W}(t,x)$ and $\underline{W}(t,x)$, that $\overline{W}(t,x), \underline{W}(t,x)$ are upper or/and lower solutions of Isaacs' equation. The existence of a value is proved in Theorem 5.3 where we assume $\overline{W}(t,x)$ is usc on Ω and $\underline{W}(t,x)$ is lsc on Ω. It is claimed the value function $W(t,x)$ is then a continuous solution of Isaacs' equation satisfying $w_-(t,x) \geq W(t,x) \geq w_+(t,x)$ on Ω.

Below we invoke some results from [13], which were proved under weaker assumptions (without convexity conditions (2.3), (2.9)), with one exception concerning regularity of $U(t), V(t)$ articulated in condition (1.4) from [13]. This condition is readily satisfied when $U(t) \equiv \overline{U}, V(t) \equiv \overline{V}$ for some compact sets $\overline{U}, \overline{V}$.

Let Π denote the set of all finite partitions π of $[\bar{t}, T]$. For any $\alpha : X(\bar{t}, \bar{x}) \to (\Pi, U)$ (U is the class of all control functions $(u(\cdot))$, we write down $\alpha[x(\cdot)]$ as $(\alpha_1[x(\cdot)], \alpha_2[x(\cdot)])$ with $\alpha_1[x(\cdot)] = \pi, \alpha_2[x(\cdot)] = u(\cdot)$.

DEFINITION 5.1. An operator $\alpha : X(\bar{t},\bar{x}) \to (\Pi, U)$ is said to be a strategy of player I if whenever $x_1(t) = x_2(t), \bar{t} \leq t \leq t^*$, then (i) $t_1^1 = t_1^2, \dots, t_{k+1}^1 = t_{k+1}^2$, where $t_1^1, t_2^1, \dots, t_{k+1}^1$ are the first $k+1$ points of the partition $\pi_1 = \alpha_1[x_1(\cdot)]$, while $t_1^2, t_2^2, \dots, t_{k+1}^2$ are the first $k+1$ partition points of $\pi^2 = \alpha_1[x_2(\cdot)]$, with k being the index for which $t_k^1 \leq t^* < t_{k+1}^1$, and (ii) $\alpha_2[x_1(\cdot)](t) = \alpha_2[x_1(\cdot)](t), \bar{t} \leq t \leq t_{k+1} = t_{k+1}^1 = t_{k+1}^2$.

In an analogous fashion, we understood the concept of a strategy for player II. It is easy to see how the players proceed when employing their strategies. Namely, at time $\bar{t}$ player I (a similar observation refers to player II) chooses his/her partition point t_1 (knowing $\bar{x}$ only) and a control $u(s), \bar{t} \leq x < t_1$. At time t_1, knowing $x(s)$ for $\bar{t} \leq s < t_1$, player I selects t_2 and $u(s)$ for $s \in [t_1, t_2)$, and so forth.

Denote by $A(\bar{t},\bar{x}), B(\bar{t},\bar{x})$ the space of strategies for player I and player II, respectively (associated with a point $(\bar{t},\bar{x})$). Given $\alpha \in A(\bar{t},\bar{x})$, we say that a trajectory $x(\cdot) \in X(\bar{t},\bar{x})$ is an outcome of $\alpha(x(\cdot) \in 0[\alpha]$, for short) if there are controls $u(\cdot) \in U, v(\cdot) \in V$ such that $\dot{x}(t) = f(t,x(t),u(t),v(t)), x(\bar{t}) = \bar{x}$, and $\alpha_2[x(\cdot)] = u(\cdot)$. In a similar fashion, we understand $0[\beta], \beta \in B(\bar{t},\bar{x})$.

When $\alpha_2[x(\cdot)] = u(\cdot)$ and $\beta_2[x(\cdot)] = v(\cdot)$ with $\dot{x}(t) = f(t,x(t),u(t),v(t)), x(\bar{t}) = \bar{x}$, we say $x(\cdot)$ is an outcome of the pair $(\alpha,\beta)(x(\cdot) \in 0[\alpha,\beta]$, for short). Clearly, when $\dot{x}(t) = f(t,x(t),u(t),v(t))$, $x(\bar{t}) = \bar{x}$, then by $P[\bar{t},\bar{x},x(\cdot)]$ we shall mean $P[\bar{t},\bar{x},u(\cdot),v(\cdot)]$ (given by (1.2)); analogously, since each pair $(\alpha,\beta) \in A(\bar{t},\bar{x}) \times B(\bar{t},\bar{x})$ gives rise to exactly one outcome $x(\cdot)$, by $P[\bar{t},\bar{x},\alpha,\beta]$ we shall mean $P[\bar{t},\bar{x},x(\cdot)]$. Following [13], we define the upper and lower values of the game as follows:

$$\underline{W}(\bar{t},\bar{x}) = \sup_{\alpha \in A(\bar{t},\bar{x})} \inf_{x(\cdot) \in 0[\alpha]} P(\bar{t},\bar{x},x(\cdot)], \ \overline{W}(t,\bar{x}) = \inf_{\beta \in B(\bar{t},\bar{x})} \sup_{x(\cdot) \in 0[\beta]} P[\bar{t},\bar{x},x(\cdot)]. \tag{5.1}$$

These formulas may be, clearly, replaced with the following ones:

$$\underline{W}(\bar{t},\bar{x}) = \sup_{\alpha \in A(\bar{t},\bar{x})} \inf_{\beta \in B(\bar{t},\bar{x})} P[\bar{t},\bar{x},\alpha,\beta], \ \overline{W}(\bar{t},\bar{x}) = \inf_{\beta \in B(\bar{t},\bar{x})} \sup_{\alpha \in A(\bar{t},\bar{x})} P[\bar{t},\bar{x},\alpha,\beta]. \tag{5.2}$$

DEFINITION 5.2. The differential game (defined above) has a value $W(\bar{t},\bar{x})$ at a point $(\bar{t},\bar{x}) \in \Omega$, if $W(\bar{t},\bar{x}) = \overline{W}(\bar{t},\bar{x}) = \underline{W}(\bar{t},\bar{x})$.

It was proved in [13] (Theorem 3.1) that, for each $(\bar{t},\bar{x}) \in \Omega - \Gamma$,

$$\sup_{\alpha \in A(\bar{t},\bar{x})} \inf_{x(\cdot) \in 0[\alpha]} \underline{W}(t,x(t)) = \underline{W}(\bar{t},\bar{x}), \quad \bar{t} \leq t \leq \tau[x(\cdot)], \tag{5.3}$$

and

$$\inf_{\beta\in B(\bar t,\bar x)}\sup_{x(\cdot)\in 0[\beta]} \overline{W}(t,x(t)) = \overline{W}(\bar t,\bar x), \quad \bar t \le t \le \tau[x(\cdot)]. \tag{5.4}$$

Note that in the relations above, called in [13] "optimality principles of dynamic programming for differential games," $x(\cdot)\in 0[\alpha]$ (resp. $x(\cdot)\in 0[\beta]$) can be replaced with $\beta\in B(\bar t,\bar x)$ (resp. $\alpha\in A(\bar t,\bar x)$).

The natural question arises how to find a relationship between strategies $\alpha\in A(t,x), \beta\in B(t,x)$ and set-valued maps $U(t,x,s), V(t,x,s)$. For each $U(t,x,s)\subset \overline{U}$, denote by $u^{le}(t,x,s)$ the selection from $U(t,x,s)$ which is the maximal element of $U(t,x,s)$ in the lexicographic order. Given $U(t,x,s)$ and the partition $\pi_n = \{\bar t, t_i\}$ of $[\bar t, T], t_i = \bar t + \frac{i}{n}(i=0,1,\ldots)$ one is naturally led to the differential inclusion (cf. (2.5), (2.7)): $\dot x_n(t)\in f(t,x_n(t),u^{le}(t_i,x_n(t_i),s),\overline{V}), t_i\le t< t_{i+1}, x(\bar t)=\bar x$, which (for a fixed control $v(\cdot)\in V$) uniquely determines a strategy $\alpha^n = (\alpha_1^n,\alpha_2^n)\in A(\bar t,\bar x)$. In fact, it is enough to set

$$\alpha_1^n[x(\cdot)] = \pi_n \quad \alpha_2^n[x(\cdot)] = u^{le}(t_i, x(t_i), s), \quad t_i\le t< t_{i+1}, \tag{5.5}$$

$t_i\in\pi_n(i=0,1,\ldots), u^{le}(t,x,s)\subset U(t,x,s)$.

THEOREM 5.1. If $\overline{W}(t,x)$ is lsc (resp. usc) on Ω then $\overline{W}(t,x)$ is a upper (resp. lower) solution of Isaacs' equation (1.3). Consequently, if $\overline{W}(t,x)$ is continuous on Ω then it is a solution of Isaacs' equation satisfying $w_-(t,x)\ge \overline{W}(t,w)\ge w_+(t,x)$ on Ω.

PROOF First notice $\overline{W}(t,x) = g(t,x)$ on Ω. Throughout the proof we fix a point $(\bar t,\bar x)\in\Omega$ and a vector $s, ||s||=1$. To demonstate $\overline{W}(t,x)$ is a upper solution, we select a natural number n to uniquely determine the strategy $\alpha^n = (\alpha_1^n,\alpha_2^n)$ given by (5.5). Next we observe the first part of

$$\sup_{\alpha\in A(\bar t,\bar x)}\inf_{\beta\in B(\bar t,\bar x)} \overline{W}(t,x,(t)) \le \inf_{\beta\in B(\bar t,\bar x)}\sup_{\alpha\in A(\bar t,\bar x)} \overline{W}(t,x(t)) = \overline{W}(t,\bar x), \tag{5.6}$$

$\bar t\le t<\tau[x(\cdot)]$, is a trivial fact from elementary game theory, while the second part is exactly identity (5.4). It follows from (5.6) and the remark immediately following (5.4), that there is a $x_n(\cdot)\in 0[\alpha^n]$ such that $\overline{W}(t,x_n(t))\le \overline{W}(\bar t,\bar x), \bar t\le t<\tau[x_n(\cdot)]$, with $\dot x_n(t)\in f(t,x_n(t),U(t_i,x_n(t_i),s),\overline{V}), x_n(\bar t) = \bar x, t_i\le t<t_{i+1}$. Using assumption (2.9) and the upper-semicontinuity of $U(t,x,s)$ in (t,x), we can assume $x_n(\cdot)$ is convergent to some $x(\cdot), \dot x(t)\in f(t,x(t),U(t,x(t),s),\overline{V})$, i.e., $x(\cdot)\in X_+(\bar t,\bar x)$. Since $\overline{W}(t,x)$ was assumed to be lsc on Ω, we conclude from $\overline{W}(t,x_n(t))\le \overline{W}(\bar t,\bar x)$ that $\overline{W}(t,x(t))\le \overline{W}(\bar t,\bar x), \bar t\le t\le\tau[x(\cdot)]\le \liminf_{n\to\infty}\tau[x_n(\cdot)]$, as required in (2.11).

To prove (2.13), we assume $\overline{W}(t,x)$ is usc on Ω and argue similarly, with (5.5) replaced by: $\beta^n = (\beta_1^n,\beta_2^n), \beta_1^n[x(\cdot)] = \pi_n, \beta_2^n[x(\cdot)] = v^{le}(t_i,x(t_i),s), t_i\le t<t_{i+1}$, where $v^{le}(t,x,s)$ is the selection from $V(t,x,s)$ which is the maximal element of $V(t,x,s)$ in the lexicographic order. We obtain from (5.4) that there are trajectories $x_n(\cdot)\in 0[\beta^n]$ for which $\overline{W}(t,x_n(t))\ge \overline{W}(\bar t,\bar x)$, which gives us $\overline{W}(t,x(t))\ge \overline{W}(\bar t,\bar x), \bar t\le t\le\tau[x(\cdot)], x(\cdot)\in X_-(\bar t,\bar x)$, using the upper-semicontinuity of $\overline{W}(t,x)$ as required in (2.13).

Finally, if $\overline{W}(t,x)$ is continuous on Ω then $\overline{W}(t,x)$ must be a solution of Isaacs' equation satisfying $w_-(t,x)\ge \overline{W}(t,x)\ge w_+(t,x)$, by virtue of (3.4), (3.5).

THEOREM 5.2. If $\underline{W}(t,x)$ is usc (lsc) on Ω then $\underline{W}(t,x)$ is a lower (upper) solution of Isaacs' equation (1.3). Consequently, if $\underline{W}(t,x)$ is continuous on Ω, then $\underline{W}(t,x)$ is a solution of Isaacs' equation with $w_-(t,x)\ge \underline{W}(t,x)\ge w_+(t,x), (t,x)\in\Omega$.

PROOF The second part of the theorem is an obvious consequence of the first part. We start with observing $\underline{W}(t,x) = g(t,x)$ on Γ, choosing a point $(\bar t,\bar x)\in\Omega$ and $s\in R^n, ||s||=1$. As in the proof of Theorem 5.1, we "approximate" the multifunction $V(t,x,s)$ with the sequence $\beta^n = [\beta_1^n,\beta_2^n]$ defined therein to obtain trajectories $x_n(\cdot)\in 0[\beta^n]$ for which $\overline{W}(t,x_n(t))\ge \underline{W}(\bar t,\bar x), \bar t\le t\le\tau[x_n(\cdot)]$, with the inequality

$$\inf_{\beta\in B(\bar t,\bar x)}\sup_{\alpha\in A(\bar t,\bar x)} \underline{W}(t,x(t)) \ge \sup_{\alpha\in A(\bar t,\bar x)}\inf_{\beta\in B(\bar t,\bar x)} \underline{W}(t,x(t)) = \underline{W}(\bar t,\bar x), \tag{5.7}$$

playing the role of (5.6); as previously, $\alpha\in A(\bar t,\bar x)$ can be replaced with $x(\cdot)\in 0[\beta]$. The final step to make is to conclude $\underline{W}(t,x(t))\ge \underline{W}(\bar t,\bar x)$ for some $x(\cdot)\in X_-(\bar t,\bar x)$, as required in (2.13), using the upper-semicontinuity of $\underline{W}(t,x)$ on Ω.

If $\underline{W}(t,x)$ is lsc on Ω then, based on (5.3), we choose, for $\alpha^n = (\alpha_1^n,\alpha_2^n)$ given by (5.5), a trajectory $x_n(\cdot)\in 0[\alpha]$ such that $\underline{W}(t,x_n(t))\le \underline{W}(\bar t,\bar x), \bar t\le t<\tau[x_n(\cdot)]$. As in the first part of the proof of Theorem 5.1, $x_n(\cdot)$ are convergent to some $x(\cdot) \in X_+(\bar{t},\bar{x})$ for which $\underline{W}(t,x(t)) \leq \underline{W}(\bar{t},\bar{x}), \bar{t} \leq t \leq \tau[x(\cdot)]$. The proof is completed.

THEOREM 5.3. If $\overline{W}(t,x)$ is usc on Ω and $\underline{W}(t,x)$ is lsc on Ω then the game has a value $W(t,x) = \overline{W}(t,x) = \underline{W}(t,x)$, which is a continuous solution of Isaacs' equation satisfying $w_-(t,x) \geq W(t,x) \geq w_+(t,x), (t,x) \in \Omega$. Any solution of Isaacs' equation continuous at each point $(t,x) \in \Gamma$ coincides with $W(t,x)$ on Ω.

PROOF By virtue of Theorem 5.1 (resp. Theorem 5.2) $\overline{W}(t,x)$ (resp. $\underline{W}(t,x)$) is a lower (resp. upper) solution of Isaacs' equation. Using Lemma 4.1, we conclude $\underline{W}(t,x) \geq \overline{W}(t,x)$. Since the inverse inequality always holds true (a trivial fact in elementary game theory), we infer $\underline{W}(t,x) = \overline{W}(t,x) = W(t,x)$ on Ω, and (by (3.4), (3.5)) $w_-(t,x) \geq W(t,x) \geq w_+(t,x)$. The last part of the theorem follows immediately from Corollary 4.3.

COROLLARY 5.4. If assumptions of Theorem 5.3 are satisfied, $W_+(t,x)$ is lsc at each point $(t,x) \in \Gamma$ and $w_-(t,x)$ is usc at each point $(t,x) \in \Gamma$, then $W(t,x)$ is the only solution of Isaacs' equation. $W(t,x)$ is continuous on Ω and satisfies $W(t,x) = w_-(t,x) = w_+(t,x)$.

PROOF The conclusions follow immediately from Corollary 4.2 and Theorem 5.3.

REFERENCES

1. E. Barron, L. C. Evans & R. Jensen, Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls, J. Differential Eqs., 53 (1984), 213–233.

2. L. D. Berkovitz, Characterizations of the values of differential games, Appl. Math. Optim. 17 (1988), 177–183.

3. R. J. Elliott & N. J. Kalton, Cauchy problems for certain Isaac-Bellman equations and games of survival, Trans. Amer. Math. Soc. 198 (1974), 45–72.

4. L. C. Evans & H. Ishii, Differential games and nonlinear first order PDE on bounded domains, Manuscripta Math. 49 (1984), 109–139.

5. L. C. Evans & P. E. Souganidis, Differential games and representation formulas for solutions of *H-J-I* equations, Indiana Univ. Math. J. 33 (1984), 773–797.

6. W. H. Fleming, The Cauchy problem for degenerate parabolic equations, J. Math. Mech. 13 (1964), 987–1008.

7. W. H. Fleming & P. E. Souganidis, PDE-viscosity solution approach to some problems of large deviations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 13 (1986), 2, 171–192.

8. W. H. Fleming & P. E. Souganidis, Asymptotic series and the method of vanishing viscosity, Indiana Univ. Math. J. 35 (1986), 2, 425–447.

9. A. Subbotin, A generalization of the fundamental equation of the theory of differential games, Dokl. Akad. Nauk. SSSR 254 (1980), 292–297.

10. A. Subbotin, Existence and uniqueness results for Hamilton-Jacobi equations, Nonlinear Analysis, to appear.

11. L. S. Zaremba, Existence of value in generalized pursuit-evasion games, SIAM J. Control 22 (1984), 894–904.

12. L. S. Zaremba, Optimality principles of dynamic programming in differential games, J. Math. Anal. Appl. 138 (1989), 43–51.

13. L. S. Zaremba, Viscosity solutions of the Isaacs equation on an attainable set, Problems Control Inf. Theory, to appear.