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REMARKS ON ORTHOGONAL POLYNOMIALS WITH RESPECT TO VARYING MEASURES AND RELATED PROBLEMS

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ABSTRACT. We point out the relation between the orthogonal polynomials with respect to (w.r.t.) varying measures and the so-called orthogonal rationals on the unit circle in the complex plane. This observation enables us to combine different techniques in the study of these polynomials and rationals. As an example, we present a simple and short proof for a known result on the weak-star convergence of orthogonal polynomials w.r.t. varying measures. Some related problems are also considered.

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1. INTRODUCTION.

Let σ be a finite positive measure on the unit circle $\Gamma := \{z \in C \mid |z| = 1\}$, and let $w_n(z) = z^n + \cdots$ be a sequence of polynomials whose zeros $\{z_{n,k}\}_{k=1}^n$ all lie in the closed unit disk $|z| \le 1$. Assume

$$\int_\Gamma \frac{d\sigma}{|w_n(z)|^2} < \infty,$$

then, for each n, we can consider the orthonormal polynomials $\varphi_{n,m}(z) = \alpha_{n,m} z^m + \cdots (\alpha_{n,m} > 0)$ w.r.t. $d\sigma / |w_n(z)|^2$, i.e., polynomials satisfying

$$\frac{1}{2\pi} \int_\Gamma z^{-j} \varphi_{n,m}(z) \frac{d\sigma}{|w_n(z)|^2} = 0, \quad j = 0, 1, \ldots, m-1,$$

and

$$\frac{1}{2\pi} \int_\Gamma |\varphi_{n,m}(z)|^2 \frac{d\sigma}{|w_n(z)|^2} = 1.$$

The sequences $\{\varphi_{n,m}(z)\}_{m=0}^\infty$ are called orthogonal polynomials w.r.t. varying measures $d\sigma / |w_n(z)|^2$ $(n = 1, 2, \ldots)$. They appear in the study of simultaneous Padé approximation and related problems. The following weak-star convergence result plays an important role in the study of convergence of $\{\varphi_{n,n}(z)\}_{n=1}^\infty$.

THEOREM 1. (Lopez, [8]) If the zeros of $w_n(z)$ satisfy the condition $lim_{n\to\infty}\sum_{i=1}^{n}(1-|z_{n,i}|)=\infty$, then

$$\left|\frac{w_n(z)}{\varphi_{n,n}(z)}\right|^2 |dz| \to d\sigma, \quad (n\to\infty),$$

in the weak-star topology.

One nice thing about this result is that it holds without any condition on the finite positive measure $d\sigma$. We will present a simple and short proof of Theorem 1. But before we get to the proof, we remark that the so-called orthogonal rationals (with prescribed poles) are also of current interest, especially in problems related to electrical engineering. (See, e.g., [2, 3, 4, 5]) These applications were more concerned with the algebraic properties of orthogonal rationals. It turns out that orthogonal rationals are just special case of orthogonal polynomials w.r.t. varying measures. It is the main purpose of this note to point out this relation. To see this relation, let us recall the definition of orthogonal rationals with respect to $d\sigma$.

Let $\{z_i\}_{i=1}^{\infty}$ be points in $|z|\le 1$, and denote $Q_n(z):=\prod_{i=1}^{n}(z-z_i)$.

DEFINITION 2. (cf. [2]) For $n=1,2,\ldots$, the rationals $\frac{P_n(z)}{Q_n(z)}$ with $P_n(z)=\alpha_n z^n+\cdots(\alpha_n>0)$ satisfying

$$\frac{1}{2\pi}\int_\Gamma \frac{P_n(z)}{Q_n(z)}\overline{r(z)}d\sigma = 0$$

for $r(z)\in\{\frac{q(z)}{Q_n(z)}: q\in\mathcal{P}_{n-1}\}$, and

$$\frac{1}{2\pi}\int_\Gamma \left|\frac{P_n(z)}{Q_n(z)}\right|^2 d\sigma = 1,$$

are called orthogonal rationals w.r.t. $d\sigma$.

Now, from Definition 2, it can be seen that with $Q_n(z)$ treated as $w_n(z)$, the corresponding $\varphi_{n,n}(z)$ is $P_n(z)$. So orthogonal rationals are indeed special case of orthogonal polynomials w.r.t. varying measures.

We go back to consider orthogonal polynomials w.r.t. varying measures. The following result, which is essentially proved in the orthogonal rational setting (cf. e.g., [2]), will help us to simplify the proof of Theorem 1 in [8].

LEMMA 3. If r and s belong to $\mathcal{R}(w_n):=\{p(z)/w_n(z): p\in\mathcal{P}_n\}$, then

$$\frac{1}{2\pi}\int_\Gamma r(z)\overline{s(z)}\left|\frac{w_n(z)}{\varphi_{n,n}(z)}\right|^2 |dz| = \frac{1}{2\pi}\int_\Gamma r(z)\overline{s(z)}d\sigma.$$

PROOF. By [6, formula (1.20)], we have

$$\frac{1}{2\pi}\int_\Gamma \frac{p(z)\overline{q(z)}}{|\varphi_{n,n}(z)|^2}|dz| = \frac{1}{2\pi}\int_\Gamma p(z)\overline{q(z)}\frac{d\sigma}{|w_n(z)|^2}$$

for all $p,q\in\mathcal{P}_n$. So,

$$\frac{1}{2\pi}\int_\Gamma \frac{p(z)}{w_n(z)}\overline{\left(\frac{q(z)}{w_n(z)}\right)}\left|\frac{w_n(z)}{\varphi_{n,n}(z)}\right|^2|dz| = \frac{1}{2\pi}\int_\Gamma \frac{p(z)}{w_n(z)}\overline{\left(\frac{q(z)}{w_n(z)}\right)}d\sigma,$$

which implies the lemma.

REMARKS. (i) Taking $r=s=1$ in Lemma 3 gives us

$$\int_\Gamma \left|\frac{w_n(z)}{\varphi_{n,n}(z)}\right|^2|dz| = \int_\Gamma d\sigma. \tag{1}$$

(ii) Lemma 3 is equivalent to the following:

$$\frac{1}{2\pi}\int_\Gamma \frac{z^{-n}p_{2n}(z)}{|w_n(z)|^2}\left|\frac{w_n(z)}{\varphi_{n,n}(z)}\right|^2 |dz| = \frac{1}{2\pi}\int_\Gamma \frac{z^{-n}p_{2n}(z)}{|w_n(z)|^2}\,d\sigma, \tag{2}$$

for all $p_{2n}\in\mathcal{P}_{2n}$.

We also need the following result about weighted approximation with varying weights in L_∞-norm.

LEMMA 4. The space

$$T_n := \left\{\frac{z^{-n}p_{2n}(z)}{|w_n(z)|^2} : p_{2n}\in\mathcal{P}_{2n}\right\}$$

is dense in $C(\Gamma)$ in the L_∞-norm if and only if

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1-|z_{n,i}|)=\infty.$$

PROOF. This is a simple generalization of some known results (see, e.g., [1, p. 244] and [2, Theorem 8.1.2]). We only mention the following fact (cf. [1, p. 243])

$$\inf_{p\in\mathcal{P}_{2n+m-1}}\left\|\frac{z^{2n+m}+p(z)}{\prod_{z_{n,i}\neq 0}(z-z_{n,i})(z-1/\overline{z_{n,i}})}\right\| = \prod_{z_{n,i}\neq 0}|z_{n,i}|.$$

PROOF OF THEOREM 1. Now the proof of Theorem 1 follows from (1), (2), and Lemma 4 by using a standard argument (cf. e.g., [9, p. 248]).

Recently, in the study of frequency analysis (see, e.g., [7]), the following orthogonal polynomials w.r.t. varying measures appear naturally, i.e., polynomials $\varphi_{n,m}(z)=\beta_{n,m}z^m + \cdots (\beta_{n,m}>0)$ satisfying

$$\frac{1}{2\pi}\int_\Gamma z^{-j}\psi_{n,m}(z)|w_n(z)|^2 d\sigma = 0, j=0,1,\ldots,m-1$$

and

$$\frac{1}{2\pi}\int_\Gamma |\psi_{n,m}(z)w_n(z)|^2 d\sigma = 1.$$

For convenience, let us assume that the zeros of $w_n(z)$ all lie outside the unit circle Γ, i.e., $|z_{n,i}|>1$. The asymptotics of $\psi_{n,m}(z)w_n(z)$ are wanted in application ([7]). Here we will not discuss this problem but instead point out the relation of these polynomials $\{\psi_{n,m}\}$ with the following ones $\{\phi_{n,m}\}$: Assume $A_{n,i}>0 (1\le i\le n, n\ge 1)$ are given, then define polynomials $\phi_{n,m}(z)=\gamma_{n,m}z^{m+n} + \cdots (\gamma_{n,m}>0)$ satisfying

$$\frac{1}{2\pi}\int_\Gamma z^{-j}\,\phi_{n,m}(z)d\sigma + \sum_{i=1}^{n}A_{n,i}\,z_{n,i}^{-j}\,\phi_{n,i}(z_{n,i}) = 0$$

for $j=0,1,\ldots,m+n-1$, and

$$\frac{1}{2\pi}\int_\Gamma |\phi(z)|^2 d\sigma + \sum_{i=1}^{n}A_{n,i}|\phi_{n,m}(z_{n,i})|^2 = 1.$$

The polynomials $\{\phi_{n,m}\}_{m=0}^\infty$ are the orthonormal polynomials w.r.t. measure $d\sigma/(2\pi) + \sum_{i=1}^{n}A_{n,i}\delta_{z_{n,i}}$ having finitely many mass points off the unit circle Γ (where δ_z denotes the point unit measure with support at z). We have

THEOREM 5. For each n fixed, there hold

$$\lim_{\substack{A_{n,i}\to\infty \\ 1\le i\le n}} \phi_{n,m}(z) = \psi_{n,m}(z)w_n(z),$$

locally uniformly in C, where $m = 0, 1, 2, \cdots$.

PROOF. We use a normal family argument. For n and m chosen, since $\int_\Gamma |\phi_{n,m}(z)|^2 d\sigma \le 2\pi$, then the set $\Phi := \{\phi_{n,m}: A_{n,i} > 0, 1 \le i \le n\}$ as a subset of finite dimensional normed space is a uniformly bounded set. Thus Φ as a set of analytic functions (polynomials) is a normal family over C. Let $f(z) \in \mathcal{P}_{n+m}$ be a limit of a subsequence $\Phi' \subset \Phi$, then since $\sum_{i=1}^n A_{n,i} |\phi_{n,m}(z_{n,i})|^2 \le 1$, we have

$$f(z_{n,i}) = \lim_{\substack{\phi_{n,m} \in \Phi' \\ A_{n,i}\to\infty}} \phi_{n,m}(z_{n,i}) = 0$$

for $i = 1, 2, ..., n$. So $f(z) = w_n(z)g(z)$ with some $g(z) \in \mathcal{P}_m$. Now we just need to use the extremal property of (monic) orthogonal polynomials to assert that $g(z) = \psi_{n,m}(z)$. This completes the proof of Theorem 5.

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