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NUCLEAR JC-ALGEBRAS AND TENSOR PRODUCTS OF TYPES

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ABSTRACT. This article is a continuation of [1], to which the reader is referred for the definition and properties of the JC-tensor product of two JC-algebras. Our standard references for nuclear and postliminal C^*-algebras are [2, 3, 4, 5, 6, 7]. We extend the notion of nuclearity to JC-algebras and prove that postliminal JC-algebras are nuclear. In contrast with the situation which occurs for C^*-algebras, the JC-tensor product of two postliminal JC-algebras turns out, in general, to be non-postliminal and can even be anitliminal.

KEY WORDS AND PHRASES. C^*-algebras, Von Neumann algebra, nuclear C^*-algebra, Jordan algebra, JC-algebra, tensor products of operator algebras.
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0. PRELIMINARIES.

Let A be a JC-algebra and Φ_A the canonical involutory $*$-antiautomorphism of C^*-algebra of A. We may suppose that $A \subset C^*(A)$, so that Φ_A restricts to the identity on A. The real C^*-subalgebra of $C^*(A), R^*(A) = \{x \in C^*(A): \Phi_A(x) = x^*\}$ satisfies $R^*(A) \cap iR^*(A) = 0$ and $C^*(A) = R^*(A) \oplus iR^*(A)$. Let A be a JC-algebra contained in $\mathcal{C}_{s.a}$, where $\mathcal{C}$ is a C^*-algebra, then A is said to be **reversible** in $\mathcal{C}$ if $a_1 \cdots a_n + a_n \cdots a_1$ lies in A whenever $a_1, \cdots, a_n$ do. A is said to be **universally reversible** if it is reversible in $C^*(A)$ [8]. A JC-algebra A is said to be **postliminal (or of Type I)** if each JC-quotient of A contains a non-zero abelian projection. It is said to be **liminal** if for every Type I factor representation π of A, $\pi(A)$ contains a minimal projection. A JC-algebra is said to be **antiliminal** if it has no non-zero postliminal closed Jordan ideal. The reader is referred to [9, 10, 11, 12, 13] for a detailed account of the theory of JC-algebras.

Since our aim in this article is to extend some results on the tensor product of C^*-algebras to the tensor product of JC-algebras, we recall the following:

LEMMA 0.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $\mathcal{A} \otimes \mathcal{B}$ be their algebraic tensor product. A C^*-norm λ on $\mathcal{A} \otimes \mathcal{B}$ is a norm such that the completion $\mathcal{A} \otimes_\lambda \mathcal{B}$ of $\mathcal{A} \otimes \mathcal{B}$ is a C^*-algebra. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be C^*-algebras, and suppose that $\pi_1: \mathcal{A} \to \mathcal{C}$, $\pi_2: \mathcal{A} \to \mathcal{D}$ are $*$-homomorphisms. Then the natural map $\pi_1 \otimes \pi_2: \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} \otimes \mathcal{D}$ extends to a $*$-homomorphism $\pi_1 \otimes_{min} \pi_2: \mathcal{A} \otimes_{min} \mathcal{B} \to \mathcal{C} \otimes_{min} \mathcal{D}$, and if π_1, π_2 are injective then $\pi_1 \otimes_{min} \pi_2$ is injective. A C^*-algebra $\mathcal{A}$ is said to be **nuclear** if the maximal and the minimal C^*-norms on $\mathcal{A} \otimes \mathcal{B}$ coincide. Equivalently if the canonical $*$-homomorphism from $\mathcal{A} \otimes_{max} \mathcal{B}$ onto $\mathcal{A} \otimes_{min} \mathcal{B}$ is an isomorphism. The relevant background for the theory on tensor products of C^*-algebras can be found in [3, 5, 6, 7, 14, 15].

LEMMA 0.2. [2, Corollary 4], [4, Corollary 5]. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and I a norm closed ideal of $\mathcal{A}$. Then

(i) $\mathcal{A}$ is nuclear if and only if I and $\mathcal{A}/I$ are nuclear.

(ii) $\mathcal{A} \underset{min}{\otimes} \mathcal{B}$ is nuclear if and only if $\mathcal{A}$ and $\mathcal{B}$ are nuclear.

(iii) $I \underset{\lambda}{\otimes} \mathcal{B}$ is the norm-closure of $I \otimes \mathcal{B}$ in $\mathcal{A} \underset{\lambda}{\otimes} \mathcal{B}$, where $\lambda = min, max$, the minimal and the maximal C^*-norms on $\mathcal{A} \otimes \mathcal{B}$.

(iv) $I \underset{max}{\otimes} \mathcal{B}$ is the kernel of the natural map $\mathcal{A} \underset{max}{\otimes} \mathcal{B} \to \mathcal{A}/I \underset{max}{\otimes} \mathcal{B}$.

DEFINITION 0.3. Let A and B be any pair of JC-algebras. We may suppose that A and B are canonically embedded in their respective universal enveloping C^*-algebras $C^*(A), C^*(B)$. Let λ be any C^*-norm on $C^*(A) \otimes C^*(B)$. Then the JC-tensor product of A and B with respect to λ is the completion $JC(A \underset{\lambda}{\otimes} B)$ of the real Jordan algebra $J(A \otimes B)$ generated by $A \otimes B$ in $C^*(A) \underset{\lambda}{\otimes} C^*(B)$.

The reader is referred to [16] for the properties of the JC-tensor product of two JC-algebras.

THEOREM 0.4. Let A and B be JC-algebras. Then

$$C^*(JC(A \underset{\lambda}{\otimes} B)) = C^*(A) \underset{\lambda}{\otimes} C^*(B), \text{ where } \lambda = min,\ max.$$

LEMMA 0.5. Given JC-algebras A and B, and a C^*-norm λ on $C^*(A) \otimes C^*(B)$, $JC(A \underset{\lambda}{\otimes} B)$ is universally reversible unless one of A, B has a scalar representation, and the other has a representation onto a spin factor $V_n, n \geq 4$.

1. NUCLEAR JC-ALGEBRAS.

In this section we introduce the notion of nuclear JC-algebras. We examine the relationship between a nuclear JC-algebra and its universal enveloping C^*-algebra, and establish the Jordan analogues of some results on nuclear C^*-algebras.

DEFINITION 1.1. Let A be a JC-algebra. Then A is said to be **nuclear** if, for any JC-algebra B, all restrictions of C^*-norms on $C^*(A) \otimes C^*(B)$ coincide on $J(A \otimes B)$. Equivalently, the natural surjective map $JC(A \underset{max}{\otimes} B) \to JC(A \underset{min}{\otimes} B)$ is an isomorphism for any JC-algebra B.

The following theorem is the basic result of this section.

THEOREM 1.2. Let A be a JC-algebra. Then A is nuclear if and only if its universal enveloping C^*-algebra $C^*(A)$ is nuclear.

PROOF. Suppose that $C^*(A)$ is nuclear, and let B be any JC-algebra. Then the surjective map $C^*(A) \underset{max}{\otimes} C^*(B) \to C^*(A) \underset{min}{\otimes} C^*(B)$ is an isomorphism, from which it follows that the surjective Jordan homomorphism $JC(A \underset{max}{\otimes} B) \to JC(A \underset{min}{\otimes} B)$ is an isomorphism.

Conversely, assume that A is nuclear, and let $\mathcal{B}$ be any C^*-algebra. Let I be the commutator ideal $[\mathcal{B}, \mathcal{B}]$ of $\mathcal{B}$. Then $\mathcal{B}/I$ is abelian, and hence nuclear, by [15, Theorem 1]. Since I has no one-dimensional representations we have

$$\begin{aligned} C^*(A) \otimes C^*(I_{s.a}) &\simeq C^*(A) \otimes (I \oplus I^o) \\ &\simeq (C^*(A) \otimes I) \oplus (C^*(A) \otimes I^o), \end{aligned}$$

by [10, 7.4.15]. By assumption $max = min$ on $J(A \otimes I_{s.a})$ and hence, $max = min$ on $C^*(A) \otimes C^*(I_{s.a})$, by [16, Lemma 4.4. (iii)] and so,

$$C^*(A) \underset{max}{\otimes} I = C^*(A) \underset{min}{\otimes} I. \tag{1.1}$$

By [7, 4.4.7., 4.4.9. and 4.4.22] there are homomorphisms ϕ_i, π_i, $i = 1,2$, making the following diagram commutative:

$$\begin{array}{ccc} C^*(A) \underset{max}{\otimes} \mathfrak{B} & \xrightarrow{\phi_1} & C^*(A) \underset{min}{\otimes} \mathfrak{B} \\ \Big\downarrow \pi_2 & & \Big\downarrow \pi_1 \\ C^*(A) \underset{max}{\otimes} \mathfrak{B}/I & \underset{\phi_2}{\rightarrow} & C^*(A) \underset{min}{\otimes} \mathfrak{B}/I \end{array}$$

By [4, Proposition 14] and (1.1) we have

$$Ker(\pi_2) = C^*(A) \underset{max}{\otimes} I = C^*(A) \underset{min}{\otimes} I,$$

and hence the restriction of ϕ_1, to $Ker(\pi_2)$ is an isomorphism. We shall complete the proof by showing that ϕ_1 is injective.

Let $x \in C^*(A) \underset{max}{\otimes} \mathfrak{B}$ such that $\phi_1(x) = 0$. Then

$$(\phi_2 o \pi_2)(x) = (\phi_1 o \phi_1)(x) = 0,$$

which implies that $x \in Ker(\pi_2)$, and so $x = 0$. Therefore, ϕ_1 is an isomorphism, and $C^*(A)$ is nuclear, completing the proof.

The Jordan analogue of parts (i) and (ii) of Lemma 0.2 is given in the following result.

COROLLARY 1.3. Let A be a JC-algebra, and I a norm-closed Jordan ideal of A. Then

(i) A is nuclear if and only if I and A/I are nuclear.

(ii) $JC(A \underset{min}{\otimes} B)$ is nuclear if and only if A and B are nuclear.

PROOF. (i) This follows by Theorem 1.2., Lemma 0.2. and the fact that $C^*(I)$ can be identified with a norm-closed ideal of $C^*(A)$.

(ii) Since $C^*(JC(A \underset{min}{\otimes} B)) = C^* \underset{min}{\otimes} C^*(B)$, (ii) follows by Lemma 0.2. and Theorem 1.2.

It was shown by Takesaki in [7, Theorem 3] that all Type I C^*-algebras are nuclear. We will extend this result to JC-algebras. In order to overcome the obstacle presented by the Type I_2 JW-algebras we need to exploit the deep C^*-algebras theorem which states that a C^*-algebra is nuclear if and only if its second dual is an injective Von Neumann algebra [3, Theorem 6.4].

Let X be a compact hypersonean space, and A a JC-algebra. Let $C(X,A)$ denote the set of all continuous functions on X with values in A. We shall denote by $\underset{\mathbf{C}}{C}(X)$ (resp. $\underset{\mathbf{R}}{C}(X)$) the algebra of all continuous complex valued (resp. real-valued) functions on X.

It is easy to see that $C(X,A)$ is the JC-algebra $\underset{\mathbf{R}}{C}(X) \underset{min}{\otimes} A$ generated by $\underset{\mathbf{R}}{C}(X) \otimes A$ in $\underset{\mathbf{C}}{C}(X) \underset{min}{\otimes} C^*(A)$. By Grothendieck's result [7, 4.4.14, 4.7.3] and [16, Corollary 3.5] $C^*(C(X,A)) = C(X, C^*(A))$.

REMARK. Note that if A is an associative JC-algebra then A is nuclear, because $C^*(A)$ is a commutative C^*-algebra and therefore nuclear [5, 11.3.13].

THEOREM 1.4. Postliminal JC-algebras are nuclear.

PROOF. Let A be a postliminal JC-algebra. By [9, Theorem 5.6] A^{**} is a JW-algebra of Type I. So, $A^{**} = M \oplus N$, where M is a Type I_2 JW-algebra and N is a universally reversible Type I JW-algebra. Therefore

$$C^*(A)^{**} = W^*(A^{**}) = W^*(M) \oplus W^*(N).$$

by [10, 7.1.11]. By a result of *Størmer* [12, Theorem 8.2], $W^*(N)$ is a Type I Von Neumann algebra. Hence $W^*(N)$ is injective. We have to show that $W^*(M)$ is injective.

By virtue of Stacey's results [17] we may write

$$M = \sum_{k \in K}^{\oplus} M_k.$$

where K is a set of cardinal numbers and where, for each $k \in K$, M_k is a JW-algebra of Type $I_{2,k}$. Moreover, as is also proved in [17], there is for each $k \in K$ a compact hyperstonean space X_k and a surjective normal homomorphism

$$\pi_k: C(X_k, V_k)^{**} \to M_k,$$

which extends to a normal homomorphism

$$\hat{\pi}_k: W^*(C(X_k, V_k)^{**}) \to W^*(M_k).$$

However, using [10, 7.1.11] we see that

$$W^*(C(X_k, V_k)^{**}) = C^*(C(X_k, V_k))^{**} = C(X_k, C^*(V_k))^{**}.$$

Since (see [10, 6.2.1] or [18, pp. 75, 263]) $C^*(V_k)$ can be realized as an inductive limit of finite dimensional C^*-algebras, $C^*(V_k)$ is nuclear, by [5, 11.3.12]. Consequently $C(X_k, C^*(V_k)) = \mathbb{C}(X_k) \otimes_{min} C^*(V_k)$ is nuclear, by [2, Corollary 4] and Grothendieck's theorem mentioned above. This means that $C(X_k, C^*(V_k))^{**}$ is injective. Hence, being isomorphic to a W^*-closed ideal of this algebra, $W^*(M_k)$ must itself be injective by [3, Proposition 3.1]. Therefore,

$$W^*(M) = \sum_{k \in K}^{\oplus} W^*(M_k)$$

is injective, so that $C^*(A)$ is nuclear. Therefore A is a nuclear JC-algebra, by Theorem 1.2., and the proof is complete.

2. TENSOR PRODUCTS OF TYPES OF JC-ALGEBRAS.

In this section we investigate the result of tensoring types of postliminal JC-algebras. We also consider tensor products of antiliminal JC-algebras. For C^*-algebras we have the following theorem:

THEOREM 2.1. (Guichardet, [4, Theorems 7, 8]. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let λ be a C^*-norm on $\mathcal{A} \otimes \mathcal{B}$. Then

(i) $\mathcal{A}$ and $\mathcal{B}$ are postliminal if and only if $\mathcal{A} \otimes_{min} \mathcal{B}$ is postliminal.

(ii) $\mathcal{A}$ and $\mathcal{B}$ are liminal if and only if $\mathcal{A} \otimes_{min} \mathcal{B}$ is liminal.

(iii) $\mathcal{A}$ or $\mathcal{B}$ is antiliminal if and only if $\mathcal{A} \otimes_{min} \mathcal{B}$ is antiliminal.

Moreover, if $\mathcal{A} \otimes_{\lambda} \mathcal{B}$ is antiliminal for any C^*-norm λ, then $\mathcal{A}$ and $\mathcal{B}$ are antiliminal.

To begin with we recall the following result on universal enveloping algebras.

LEMMA 2.2 [9, Proposition 4.5], [19, Theorem 2.6 and Corollary 2.7]. Let A be a JC-algebra. Then

(i) $C^*(A)$ is postliminal (resp. liminal) if and only if A is postliminal (resp. liminal) with no infinite dimensional spin factor representations.

(ii) If $C^*(A)$ is antiliminal, and A has no infinite dimensional spin factor representations, then A is antiliminal.

It turns out that neither of the equivalences (i), (ii), (iii) of Theorem 2.1 are true in the context of JC-algebra. In fact, all can be dismissed by the same counter-example.

PROPOSITION 2.3. Let V be an infinite dimensional spin factor and let A be any JC-algebra without one dimensional representations. Then $JC(V \underset{min}{\otimes} A)$ is antiliminal.

PROOF. Put $B = JC(V \underset{min}{\otimes} A)$. Then we have $C^*(B) = C^*(V) \underset{min}{\otimes} C^*(A)$. The Clifford C^*-algebra $C^*(V)$ is antiliminal (it is simple, unital and infinite dimensional). Consequently, $C^*(B)$ is antiliminal by Theorem 2.1. But B is universally reversible. Hence B is antiliminal by Lemma 2.2. (ii).

This result shows that the next two theorems cannot be improved.

THEOREM 2.4. Let A and B be JC-algebras.

(i) If A and B are postliminal and neither has infinite dimensional spin factor representations, then $JC(A \underset{min}{\otimes} B)$ is postliminal.

(ii) If $JC(A \underset{min}{\otimes} B)$ is postliminal then A and B are postliminal.

PROOF. (i) Suppose that A and B satisfy the stated conditions. Then, $C^*(A)$ and $C^*(B)$ are postliminal. Therefore,

$$C^*(JC(A \underset{min}{\otimes} B)) = C^*(A) \underset{min}{\otimes} C^*(B)$$

is postliminal. Also, it follows that because neither A nor B has infinite dimensional spin factor representations, $JC(A \underset{min}{\otimes} B)$ does not have any either. So, $JC(A \underset{min}{\otimes} B)$ must be postliminal.

(ii) Suppose now that $JC(A \underset{min}{\otimes} B)$ is postliminal. We will prove that A (and so, by implication, B) is postliminal.

Let $\pi_1 : A \to \mathfrak{B}(H_1)$ be an irreducible representation. We may suppose that $\pi_1(A)$ has neither one-dimensional nor spin factor representations. By [9, Proposition 5.5], it will be enough to show that $\pi_1(A) \cap C(H_1) \neq 0$, where $C(H_1)$ is the set of all compact operators on H_1.

Let $\pi_2 : B \to \mathfrak{B}(H_2)$ be irreducible, and let

$$\hat{\pi}_1 : C^*(A) \to \mathfrak{B}(H_1), \qquad \hat{\pi}_2 : C^*(B) \to \mathfrak{B}(H_2),$$

be the canonical extensions. Then $\hat{\pi}_1, \hat{\pi}_2$ are also irreducible, so that,

$$\hat{\pi} : C^*(A) \underset{min}{\otimes} C^*(B) \to \mathfrak{B}(H_1) \underset{min}{\otimes} \mathfrak{B}(H_2) \subset \mathfrak{B}(H_1 \otimes H_2)$$

is irreducible, by [5, 11.3.2] and [20, 2.11.3]. Consequently, since $C^*(JC(A \underset{min}{\otimes} B)) = C^*(A) \underset{min}{\otimes} C^*(B)$,

$$\hat{\pi} : JC(A \underset{min}{\otimes} B) \to \mathfrak{B}(H_1 \otimes H_2)$$

is irreducible, by [9, Proposition 5.5].

Note that the conditions imposed upon $\pi_1(A)$ imply that $\hat{\pi}$ cannot be a spin factor representation. Hence, since $JC(A \underset{min}{\otimes} B)$ is postliminal, we have

$$\hat{\pi}(JC(A \underset{min}{\otimes} B)) \cap C(H_1 \otimes H_2) \neq 0,$$

by [9, Proposition 5.5]. Thus

$$\hat{\pi}(C^*(A) \underset{min}{\otimes} C^*(B)) \supset C(H_1 \otimes H_2) = C(H_1) \underset{min}{\otimes} C(H_2).$$

By [4, Lemma 7], this implies that $C(H_1) \subset \hat{\pi}_1(C^*(A))$, in particular. Hence, since $\pi_1(A)$ is reversible in $\mathfrak{B}(H_1)$, this implies that $\pi_1(A) \cap C(H_1) \neq 0$, by [13, Lemma 3.7]. This completes the proof.

THEOREM 2.5. Let A, B be JC-algebras.

(i) If A and B are liminal JC-algebras without infinite dimensional spin factor representations, then $JC(A \underset{min}{\otimes} B)$ is liminal.

(ii) If $JC(A \otimes_{min} B)$ is liminal, then A and B are liminal.

PROOF. The proof of the first part is the same as Theorem 2.4 (i) transparently modified.

In order to prove (ii), suppose that $JC(A \otimes_{min} B)$ is liminal. Retaining the notation used in the proof of Theorem 2.4. (ii) we then see that

$$\hat{\pi}(JC(A \otimes_{min} B)) \subset C(H_1 \otimes H_2),$$

so that,

$$\hat{\pi}_1(C^*(A)) \otimes_{min} \hat{\pi}_2(C^*(B)) \subseteq C(H_1) \otimes C(H_2),$$

and hence,

$$\hat{\pi}_1(C^*(A)) \subset C(H_1), \text{ by [4, Lemma 7].}$$

Consequently, $\pi_1(A) \subset C(H_1)$, and the arguments used in Theorem 2.4 imply that A is therefore liminal.

The Jordan analogue of part (iii) of Theorem 2.1 is given in the following two results.

PROPOSITION 2.6. Let A and B be JC-algebras having no infinite dimensional spin factor representations, and λ a C^*-norm on $C^*(A) \otimes C^*(B)$. If $JC(A \otimes_{\lambda} B)$ is antiliminal, then either A or B is antiliminal.

PROOF. Let I, J be the largest liminal ideals of A, B, respectively. Then $C^*(I), C^*(J)$ are liminal (and hence nuclear) ideals of $C^*(A)$, $C^*(B)$, respectively. Thus the closure $\overline{C^*(I) \otimes C^*(J)}$ of $C^*(I) \otimes C^*(J)$ in $C^*(A) \otimes_{\lambda} C^*(J)$ is liminal, since it is isomorphic to $C^*(I) \otimes_{min} C^*(J)$, by Theorem 2.1 (ii). It follows that $JC(A \otimes_{\lambda} B) \cap \overline{C^*(I) \otimes C^*(J)} = 0$, which implies that $I \otimes J = 0$, and so, either I or J is zero, proving the proposition.

THEOREM 2.7. Let A be a universally reversible JC-algebra with no one-dimensional representations. If A is antiliminal, then $JC(A \otimes_{min} B)$ is antiliminal for any JC-algebra B.

PROOF. Let I be the largest postliminal ideal of $C^*(A)$ such that $C^*(A)/I$ is antiliminal. Then $A \cap I = 0$. Indeed, since the C^*-algebra $[A \cap I]$ generated by $A \cap I$ in I, being a C^*-subalgebra of I is again postliminal [22, Proposition 6.2.9], and therefore $A \cap I$ is a postliminal Jordan ideal of A. By [9, Lemma 3.1 (iii)], $A \cap I = 0$. Now, note that $\Phi_A(I) = I$, and hence $C^*(A \cap I) = I$, by [8, Lemma 4.3]. Therefore, $I = 0$, and so, $C^*(A)$ is antiliminal, which implies $C^*(JC(A \otimes_{min} B))$ is antiliminal. The proof is completed by Lemma 2.3 (ii), since $JC(A \otimes_{min} B)$ has no infinite dimensional spin factor representations.

Recall that [20, 4.7.20] a C^*-algebra $\mathcal{A}$ is said to be **dual** if and only if $\mathcal{A} \subset C(H)$, for some Hilbert space H. Then if $\mathcal{A}$ and $\mathcal{B}$ are dual C^*-algebras, since $\mathcal{A} \subset C(H_1), \mathcal{B} \subset C(H_2)$, H_1, H_2 are Hilbert spaces, then

$$\mathcal{A} \otimes_{min} \mathcal{B} \subset C(H_1) \otimes_{min} C(H_2) = C(H_1 \otimes H_2).$$

So, $\mathcal{A} \otimes_{min} \mathcal{B}$ is dual.

The following result shows that the converse is also true.

LEMMA 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\mathcal{A} \otimes_{min} \mathcal{B}$ is dual, then $\mathcal{A}$ and $\mathcal{B}$ are dual.

PROOF. Suppose that $C_o(X), C_o(Y)$ are maximal commutative C^*-subalgebras of $\mathcal{A}, \mathcal{B}$, respectively, where X, Y are locally compact Hausdorff spaces. Then $C_o(X \times Y) = C_o(X) \otimes_{min} C_o(Y)$ [14, Lemma 1.22.4] is a commutative subalgebra of $\mathcal{A} \otimes_{min} \mathcal{B}$, and hence dual. Thus $X \times Y$ is discrete, which implies that X and Y are discrete, and $\mathcal{A}$ and $\mathcal{B}$ are dual, by [20, 4.7.20].

Bearing in mind the counter-example given in Proposition 2.3., and the fact that spin factors are dual JC-algebras, we give the Jordan analogue of these results.

THEOREM 2.9. Let A, B be JC-algebras.

(i) If A and B are dual without infinite dimensional spin factor representations, then $JC(A \underset{min}{\otimes} B)$ is dual.

(ii) If $JC(A \underset{min}{\otimes} B)$ is dual, then A and B are dual.

PROOF. Suppose (i) hold, then $C^*(A), C^*(B)$ are dual, by [1, 3.3, 4.2, 4.4] and hence $C^*(JC(A \underset{min}{\otimes} B)) = C^*(A) \underset{min}{\otimes} C^*(B)$ is dual. By Lemma 0.5, $JC(A \underset{min}{\otimes} B)$ does not have infinite dimensional spin factor representations. Hence $JC(A \underset{min}{\otimes} B)$ is dual, by [1, 3.3, 4.2, 4.4].

(ii) This is identical to the argument given in the proof of Lemma 2.8.

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