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RESEARCH NOTES

NOTE ON POINTWISE CONTRACTIVE PROJECTIONS

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ABSTRACT: Let $C(X)$ be the space of real-valued continuous functions on a Hausdorff completely regular topological space X, endowed with the compact-open topology. In this paper necessary and sufficient conditions are given for a subspace of $C(X)$ to be the range of a pointwise contractive projection in $C(X)$

KEY WORDS AND PHRASES: Contractive projection, extreme point.
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Several authors have considered the problem of characterizing the subspaces of $C(K)$ admitting contractive projections, K being a compact Hausdorff space (cf. Lindestrauss [1], Lindestrauss-Wulbert [2] and Lindberg [3]). And when X is a Hausdorff completely regular topological space with a fundamental sequence of compact sets, we have discussed in [4] conditions for a subspace E of $C(X)$ to be the range of a compact contractive projection in $C(X)$. In this note we want to study the problem that arises when the projection p of $C(X)$ onto E is pointwise contractive, i.e. when for each $x \in X$ it is $|pf(x)| \le |f(x)|$ for every $f \in C(X)$.

Hereafter X will stand for any Hausdorff completely regular topological space and $C(X)$ for the space of the continuous real valued functions on X endowed with the compact-open topology. Given a linear subspace E of $C(X)$ and $x \in X$, we set $E_x = \{f \in E : |f(x)| \le 1\}$ and $C_x = \{f \in C(X) : |f(x)| \le 1\}$, holding E_x° and C_x° for their polar sets in the topological dual spaces of E and $C(X)$, respectively. E is called separating if for each $x, y \in X$, $x \neq y$, there is some $f \in E$ such that $f(x) \neq f(y)$. For each $x \in X$, δ_x will denote the linear form of $(C(X))'$ (E') such that $\delta_x(f) = f(x)\ \forall f \in C(X)$ (E). If A is a subset of $(C(X))'$, z is called an extreme point of A if $z = \lambda x + (1-\lambda)y$, with $0 < \lambda < 1$, $x, y \in A$, implies that $z = x = y$. Given $x \in X$, x is a double point of X if there is some $y_x \in X$ such that $f(x) + f(y_x) = 0$ for every $f \in E$. We shall say x is an autodouble point if $f(x) = 0$ for every $f \in E$, i.e. if x is a double point and $y_x = x$. If x is not a double point, x is called a single point. If E is separating and x is a double point then y_x is unique, there being at most only one autodouble point. Clearly, there are no double points if E contains the constant functions.

LEMMA. Let E be a separating linear subspace of $C(X)$. For each $x \in X$, $\pm\delta_x$ are the only extreme points of E_x°.

PROOF. Clearly the $\sigma(E',E)$-closed convex cover of $F = \{\delta_x, -\delta_x\}$ is contained in E_x° and if $\varphi \notin \overline{CF}^{\sigma(E',E)}$ there must be some some $f \in E$ such that $\varphi(f) > 1$ and $|f(x)| \le 1$, i.e. $\varphi \notin E_x^\circ$. On the other hand E_x° has some extreme point, since it is weakly compact, and it will be contained in F, [5, §25.1 (6)]. Hence $\pm\delta_x$ are the only extreme points of E_x°.

PROPOSITION. Let E be a separating subspace of $C(X)$ and p a pointwise contractive projection of $C(X)$ onto E. Then for each $x \in X$, the transpose linear mapping p^* of p satisfies

i) $p^*(\delta_x) = 0$ if x is an autodouble point

ii) $p^*(\delta_x) = \delta_x$ if x is not an autodouble point

PROOF. i) For each $f \in C(X)$, $p^*\delta_x(f) = \delta_x(pf) = (pf)(x) = 0$ since $pf \in E$. Hence $p^*(\delta_x) = 0$

ii) Let $E(x) = \{\varphi \in C_X^\circ, \varphi|_E = \delta_x\}$ which, by Krein-Millman's theorem, coincides with the closed convex cover of its extreme points. Now $p^*\delta_x \in E(x)$ since for each $g \in C_X$, $|p^*\delta_x(g)| = |\delta_x(pg)| = |pg(x)| \leq |g(x)| \leq 1$, and for each $f \in E$, $p^*\delta_x(f) = \delta_x(pf) = \delta_x(f)$. Besides, each extreme point of $E(x)$ is an extreme point of C_X° since if φ is an extreme point of $E(x)$ and $\varphi = \alpha u^* + (1-\alpha)v^*$, $u^*, v^* \in C_X^\circ$, then $\delta_x = \varphi|_E = \alpha u^*|_E + (1-\alpha)v^*|_E$, where $u^*|_E, v^*|_E \in E_X^\circ$. So, $\delta_x = u^*|_E = v^*|_E$, and u^*, v^* belong to $E(x)$ and coincide with φ. Therefore, by Proposition 1, $\varphi = \alpha\delta_x$ with $|\alpha| = 1$. As x is not autodouble, there is some $f \in E$ such that $0 \neq f(x) = \delta_x(f)$ which, taking into account that $\varphi \in E(x)$, takes the same value as $\varphi(f) = \alpha\delta_x(f) = \alpha f(x)$. Hence $\alpha = 1$ and $p^*\delta_x = \delta_x$.

From this Proposition it follows that if z is an autodouble point, then z is an isolated point since if 1_X is the function identically one on X, $(p1_X)(x) = \delta_x(p1_X) = (p^*\delta_x)(1_X) = \delta_x(1_X) = 1$ for each $x \neq z$, and $(p1_X)(z) = 0$. Moreover, for each $x \neq z$ the equation $f(x) + f(y) = 0$ $\forall f \in E$ has no solution since $p1_X \in E$ and no point $x \neq z$ is a double point. Consequently,

THEOREM 1. Let E be a separating subspace of $C(X)$ and p a pointwise contractive projection of $C(X)$ onto E. Then

i) If for each $x \in X$ there is some $f \in E$ such that $f(x) \neq 0$, every point of X is single.

ii) If there is some $x \in X$ such that $f(x) = 0$ for each $f \in E$, x is the only double point of X. Moreover x is isolated and, clearly, autodouble.

THEOREM 2. A locally convex topological vector space E is isomorphic to the range of a pointwise contractive projection in $C(X)$ if and only if E is isomorphic to either $C(X)$ or some $C_z(X) = \{f \in C(X) : f(z) = 0\}$, $z \in X$.

PROOF. Assume E is separating. If each point $x \in X$ is single, $pf(x) = \delta_x(pf) = p^*\delta_x(f) = \delta_x(f) = f(x)$ for every $f \in C(X)$. So $f = pf \in E$. On the other hand, if there exists some double point z, E is contained in $C_z(X)$. But for each $f \in C_z(X)$, $pf(z) = 0$ and $pf(x) = f(x)$ for $x \neq z$. If E is not separating, we are able to form the quotient by identifying those points which are not separated by E and the same conclusion yields.

Conversely, if E is isomorphic to some $C_z(X)$, then the mapping $p : C(X) \to C_z(X)$ defined by $p(f) = f_z$, where $f_z(x) = f(x)$ for $x \neq z$ and $f_z(z) = 0$, is pointwise contractive and $pf = f$ for each $f \in C_z(X)$

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