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ESTIMATES FOR THE CAUCHY MATRIX OF PERTURBED LINEAR IMPULSIVE EQUATION

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ABSTRACT. Estimates for the Cauchy matrix of a perturbed linear impulsive equation are obtained for given estimates for the Cauchy matrix of the corresponding unperturbed linear impulsive equation.

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1. INTRODUCTION.

Consider the linear impulsive equation

$$\begin{aligned} x' &= A(t)x, && t \neq \tau_k, \\ \Delta x &= A_k x, && t = \tau_k, \end{aligned} \tag{1.1}$$

where t belongs to the interval $J \subset \mathbf{R}: \tau_k < \tau_{k+1}$ $(k \in \mathbf{Z})$; the sequence $\{\tau_k\}$ has no finite accumulation point; $x \in \mathbf{R}^n\cdot$, $A_k \in \mathbf{R}^{n \times n}$. Suppose that $A(t)$ belongs to the space $PC(J, \mathbf{R}^{n \times n})$, i.e. $A(t)$ is an $n \times n$ matrix-valued function which is continuous for $t \in J$, $t \neq \tau_k$, and at the points $\tau_k \in J$ it has discontinuities of the first kind and is continuous from the left. We recall [1] that the solution $x(t)$ of (1.1) for $t \in J$, $t \neq \tau_k$ satisfies the equation

$$x' = A(t)x \text{ and for } t = \tau_k$$

the conditions

$$x(\tau_k^-) \overset{def}{=} \lim_{t \to \tau_k - 0} x(t) = x(\tau_k),\ x(\tau_k^+) \overset{def}{=} \lim_{t \to \tau_k + 0} x(t) = x(\tau_k) + \Delta x(\tau_k) = x(\tau_k) + A_k x(\tau_k).$$

Let $|x|$ be a norm of the vector $x \in \mathbf{R}^n$ and $|A| = sup\{|Ax| : |x| = 1\}$ be the corresponding norm of the matrix $A \in \mathbf{R}^{n \times n}$. Let the Cauchy matrix $W(t,s)$ of (1.1) satisfy an estimate of the form

$$|W(t,s)| \leq \varphi(t)\psi(s) \qquad (s, t \in J,\ s \leq t), \tag{1.2}$$

where the functions φ, $\psi: J \to \mathbf{R}_+$ continuous and positive.

Based on estimate (1.2), we shall seek for various estimates for the Cauchy matrix $Q(t,s)$ of the perturbed linear equation

$$\begin{aligned} y' &= [A(t) + B(t)]y, && t \neq \tau_k, \\ \Delta y &= [A_k + B_k]y, && t = \tau_k, \end{aligned} \tag{1.3}$$

where $B(t) \in PC(J, \mathbf{R}^{n \times n})$ and $B_k \in \mathbf{R}^{n \times n}$.

We shall use the following lemma:

LEMMA 1.1 [2]. Let the function $u \in PC(J, \mathbf{R}_+)$ satisfy the inequality

$$u(t) \le c + \int_s^t p(\tau)u(\tau)d\tau + \sum_{s \le \tau_k < t} p_k u(\tau_k) \qquad (s,\ t \in J,\ s \le t),$$

where $c \ge 0$ and $p_k \ge 0$ are constants and $p(\tau) \in PC(J, \mathbf{R}_+)$.

Then

$$u(t) \le c \prod_{s \le \tau_k < t} (1 + p_k) \mid exp\left[\int_s^t p(\tau)d\tau\right] \qquad (s, t \in J, s \le t).$$

2. MAIN RESULTS.

Recall [1] that if $U_k(t,s)$ is the Cauchy matrix for the equation

$$x' = A(t)x \qquad (\tau_{k-1} < t \le \tau_k),$$

then the Cauchy matrix for equation (1.1) is

$$W(t,s) = \begin{cases} U_k(t,s) & (s, t \in (\tau_{k-1}, \tau_k]), \\ U_{k+1}(t, \tau_k^+)(E + A_k)U_k(\tau_k, s) & (\tau_{k-1} < s \le \tau_k < t \le \tau_{k+1}), \\ U_{k+1}(t, \tau_k^+) \prod_{j=k}^{i+1} (E + A_j)U_j(\tau_j, \tau_{j-1}^+)(E + A_i)U_i(\tau_i, s) & (\tau_{i-1} < s \le \tau_i < \tau_k < t \le \tau_{k+1}). \end{cases}$$

Then an arbitrary solution $y(t)$ of (1.3) satisfies the integro-summary equation

$$y(t) = W(t,s)y(s) + \int_s^t W(t,\tau)B(\tau)y(\tau)d\tau + \sum_{s \le \tau_k < t} W(t, \tau_k^+)B_k y(\tau_k) . \tag{2.1}$$

From (2.1) and (1.2) it follows that

$$\mid y(t) \mid \le \varphi(t)\psi(s) \mid y(s) \mid + \int_s^t \varphi(t)\psi(\tau) \mid B(\tau) \mid \mid y(\tau) \mid d\tau + \sum_{s \le \tau_k < t} \varphi(t)\psi(\tau_k) \mid B_k \mid \mid y(\tau_k) \mid .$$

The the function $u(t) = \mid y(t) \mid / \varphi(t)$ satisfies the inequality

$$u(t) \le \psi(s) \mid y(s) \mid + \int_s^t \varphi(\tau)\psi(\tau) \mid B(\tau) \mid u(\tau)d\tau + \sum_{s \le \tau_k < t} \varphi(\tau_k)\psi(\tau_k) \mid B_k \mid u(\tau_k).$$

We apply Lemma 1.1 and obtain the estimate

$$\mid y(t) \mid \le \mid y(s) \mid M(t,s), \tag{2.2}$$

where

$$M(t,s) = \varphi(t)\psi(s) \prod_{s \le \tau_k < t} (1 + \varphi(\tau_k)\psi(\tau_k) \mid B_k \mid) exp\left(\int_s^t \varphi(\tau)\psi(\tau) \mid B(\tau) \mid d\tau \right). \tag{2.3}$$

From (2.2) and the equality $y(t) = Q(t,s)y(s)$ there follow immediately the subsequent assertions:

THEOREM 2.1. Let the Cauchy matrix $W(t,s)$ of equation (1.1) satisfy estimate (1.2).

Then the Cauchy matrix $Q(t,s)$ of equation (1.3) satisfies the estimate

$$|Q(t,s)| \leq M(t,s) \qquad (s,t \in J, s \leq t),$$

where $M(t,s)$ is given by (2.3)

COROLLARY 2.1. If

$$|W(t,s)| \leq Ke^{\alpha(t-s)} \qquad (s,t \in J, s \leq t), \tag{2.4}$$

where $K > 1$ and α are constants, then

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} \prod_{s \leq \tau_k < t} (1 + K|B_k|) exp\left[\int_s^t K|B(\tau)|d\tau\right] (s, t \in J, s \leq t). \tag{2.5}$$

COROLLARY 2.2. If in the interval $J = \mathbf{R}_+$ estimate (2.4) is valid and there exists a constant $\delta > 0$ such that

$$\sup_{\tau \in \mathbf{R}_+} |B(\tau)| \leq \delta, \qquad \sup_{\tau_k \in \mathbf{R}_+} |B_k| \leq \delta, \tag{2.6}$$

then

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} e^{K\delta(t-s) + \ell n(1+K\delta)i[s,t)}, \tag{2.7}$$

where $i[s,t)$ is the number of points τ_k lying in the interval $[s,t)$.

Moreover, if there exist constants $q \geq 0$ and $\varepsilon \geq 0$ such that

$$i[s,t) \leq q(t-s) + \varepsilon, \tag{2.8}$$

then

$$|Q(t,s)| \leq K(1+K\delta)^{\varepsilon} exp\{[\alpha + K\delta + q\ell n(1+K\delta)](t-s)\} \qquad (c \leq s \leq t). \tag{2.9}$$

Taking into account that $\prod_{s \leq \tau_k < t} (1 + K|B_k|) \leq exp \sum_{s \leq \tau_k < t} K|B_k|$, we obtain

COROLLARY 2.3. In the interval $J = \mathbf{R}_+$ let estimate (2.4) be valid and let a constant $M > 0$ exist such that

$$\int_0^\infty |B(\tau)|d\tau + \sum_{\tau_k \geq 0} |B_k| \leq M. \tag{2.10}$$

Then

$$|Q(t,s)| \leq Ke^{KM} \cdot e^{\alpha(t-s)} \qquad (0 \leq s \leq t). \tag{2.11}$$

REMARK 1. If equation (1.1) is uniformly asymptotically stable, i.e., estimate (2.4) is valid with $\alpha < 0$, then under perturbations for which (2.6) is satisfied with δ small enough equation (1.3) is also uniformly asymptotically stable.

If equation (1.1) is uniformly stable, i.e., $\alpha = 0$ in (2.4) and condition (2.10) is valid, then equation (1.3) is also uniformly stable.

The goal of the following considerations is to obtain estimates for $Q(t,s)$ in which instead of the integral and the sum of the norms of $B(\tau)$ and B_k the norm of the following function should enter

$$D(s) = \int_s^t B(\tau)d\tau + \sum_{s \leq \tau_k < t} B_k \qquad (s,t \in J, s \leq t).$$

We shall note that $D(s)$ is continuous for $s \neq \tau_k$, $D(t^-) = 0$ and $D(\tau_k^-) = D(\tau_k) = D(\tau_k^+) + B_k$.

Let $y(t)$ be an arbitrary solution of (1.3). From (2.1), taking into account that

$$W(t,t^-) - W(t,s) = \int_s^t \frac{\partial W}{\partial \tau}(t,\tau)D(\tau)y(\tau)d\tau + \int_s^t W(t,s)D'(\tau)y(\tau)d\tau + \int_s^t W(t,s)D(\tau)y'(\tau)d\tau$$

$$+ \sum_{s \le \tau_k < t} [W(t,\tau_k^+)D(\tau_k^+)y(\tau_k^+) - W(t,\tau_k^-)D(\tau_k^-)y(\tau_k^-)];$$

$$\frac{\partial W}{\partial t}(t,\tau) = -W(t,\tau)A(\tau), \qquad y'(\tau) = [A(\tau) + B(\tau)]y(\tau)$$

and

$$W(t,\tau_k^+)D(\tau_k^+)y(\tau_k^+) - W(t,\tau_k^-)D(\tau_k^-)y(\tau_k^-) + W(t,\tau_k^+)B_k y(\tau_k)$$
$$= W(t,\tau_k^+)[D(\tau_k^+)(E + A_k + B_k) - (E + A_k)D(\tau_k^-) + B_k]y(\tau_k)$$
$$= W(t,\tau_k^+)[D(\tau_k^+)(A_k + B_k) - A_k D(\tau_k^-)]y(\tau_k)$$

we obtain that

$$y(t) = W(t,s)[E + D(s)]y(s) + \int_s^t W(t,s)[D(\tau)(A(\tau) + B(\tau)) - A(\tau)D(\tau)]y(\tau)d\tau$$

$$+ \sum_{s \le \tau_k < t} W(t,\tau_k^+)[D(\tau_k^+)(A_k + B_k) - A_k D(\tau_k^-)]y(\tau_k). \tag{2.12}$$

If $W(t,s)$ satisfies estimate (1.2) and there exist constants $M \ge 0$, $m \ge 0$ and $\eta \ge 0$ such that

$$|A(t)| \le M, \ |B(t)| \le M, \ |A_k| \le m, \ |B_k| \le m \qquad (t, \tau_k \in J) \tag{2.13}$$

and

$$\left| \int_s^t B(\tau)d\tau + \sum_{s \le \tau_k < t} B_k \right| \le \eta \qquad (s \le t), \tag{2.14}$$

then from (2.12) we obtain that

$$|y(t)| \le \varphi(t)\psi(s)(1+\eta)|y(s)| + \int_s^t \varphi(t)\psi(\tau)\cdot 3M\eta\,|y(\tau)|\,d\tau + \sum_{s \le \tau_k < t} \varphi(t)\psi(\tau_k)\cdot 3m\eta\,|y(\tau_k)|$$

and by Lemma 1.1 we obtain that

$$|y(t)| \le |y(s)|\,N(t,s) \qquad (s,t \in J, s \le t), \tag{2.15}$$

where

$$N(t,s) = (1+\eta)\varphi(t)\psi(s) \prod_{s \le \tau_k < t} (1 + 3m\eta\varphi(\tau_k)\psi(\tau_k))exp\left(\int_s^t 3M\eta\varphi(\tau)\psi(\tau)d\tau\right). \tag{2.16}$$

From the estimate (2.15) obtained there follows immediately.

THEOREM 2.2. Let the Cauchy matrix $W(t,s)$ of equation (1.1) satisfy estimate (1.2) and let conditions (2.13) and (2.14) hold.

Then the Cauchy matrix $Q(t,s)$ of equation (1.3) satisfies the estimate

$$|Q(t,s)| \le N(t,s) \qquad (s,t \in J, s \le t),$$

where $N(t,s)$ is given by (2.16).

COROLLARY 2.4. If $|W(t,s)| \le Ke^{\alpha(t-s)}$ $\quad (s,t \in J, s \le t),$ then

$$|Q(t,s)| \le (1+\eta)Ke^{\alpha(t-s)} \cdot e^{3KM\eta(t-s) + \ell n(1+3Km\eta)i[s,t)} \tag{2.17}$$

for $s,t \in J, s \le t$.

Moreover, if condition (2.8) holds, then

$$|Q(t,s)| \leq (1+\eta)(1+3Km\eta)^{\varepsilon} K e^{[\alpha + 3KM\eta + q\ell n(1+3Km\eta)](t-s)} \tag{2.18}$$

for $s,t \in J, s \leq t$.

COROLLARY 2.5. In the assumptions of Theorem 2.2 let condition (2.14) be replaced by the more general condition

$$\left| \int_s^t B(\tau)d\tau + \sum_{s \leq \tau_k < t} B_k \right| \leq \eta \qquad (s,t \in J, s \leq t \leq s+h), \tag{2.19}$$

where $h > 0$ is a constant. Then $Q(t,s)$ satisfies the estimate

$$|Q(t,s)| \leq K(1+\eta)\, exp\{[\alpha + 3KM\eta + \tfrac{1}{h}\ell n(K+K\eta)](t-s) + \ell n(1+3Km\eta)i[s,t)\} \tag{2.20}$$

for $s,t \in J, s \leq t$.

Indeed, estimate (2.20) follows immediately from (2.17) and the fact that the estimate

$$|y(t)| \leq |y(s)| L^{exp[\gamma(t-s) + ri[s,t)]} \qquad (s \leq t \leq s+h)$$

implies

$$|y(t)| \leq |y(s)| L\, exp[\gamma + \tfrac{1}{h}\ell nL)(t-s) + ri[s,t] \qquad (s \leq t).$$

REMARK 2. In some cases estimate (2.17) is better than estimate (2.7).

EXAMPLE 1. Let equations (1.1) and (1.3) be scalar and $A(t) = -1$,

$$B(t) = sin\omega t,\ A_k = 1,\ B_k = (-1)^k b, \qquad 0 \leq b \leq 1,\ \tau_k = \kappa = 0,1,2,\cdots, t \in \mathbf{R}_+.$$

Then

$$|W(t,s)| = e^{-(t-s) + \ell n2i[s,t)} \leq Ke^{\alpha(t-s)} \qquad (0 \leq s \leq t),$$

where $K = 2$, $\alpha = -1 + \ell n2$. In the notation introduced

$$\delta = 1,\ M = 1,\ m = 1, \qquad \left| \int_s^t B(\tau)d\tau + \sum_{s \leq \tau_k < t} B_k \right| \leq \frac{2}{\omega} + b = \eta.$$

Then $Q(t,s)$ is estimated:

(i) by estimate (2.7)

$$|Q(t,s)| \leq Ke^{\alpha(t-s)} \cdot e^{2(t-s) + \ell n(1+2)i[s,t)} \tag{2.21}$$

(ii) by estimate (2.17)

$$|Q(t,s)| \leq (1+\eta)Ke^{\alpha(t-s)} \cdot e^{6\eta(t-s) + \ell n(1+6\eta)i[s,t)} \tag{2.22}$$

Estimate (2.22) is better than estimate (2.21) if $6\eta < 2$, i.e., if $\frac{2}{\omega} + b < \frac{1}{3}$ which is fulfilled for large ω and small b.

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