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THE STRONG WCD PROPERTY FOR BANACH SPACES

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ABSTRACT. In this paper, we introduce a weakly compact version of the weakly countably determined (WCD) property, the strong WCD (SWCD) property. A Banach space X is said to be SWCD if there is a sequence (A_n) of weak* compact subsets of X^{**} such that if $K \subset X$ is weakly compact, there is an $(n_m) \subset N$ such that $K \subset \bigcap_{m=1}^{\infty} A_{n_m} \subset X$. In this case, (A_n) is called a strongly determining sequence for X. We show that SWCG $\Rightarrow$ SWCD and that the converse does not hold in general. In fact, X is a separable SWCD space if and only if $(X,$ weak$)$ is an $\aleph_0$-space. Using c_0 for an example, we show how weakly compact structure theorems may be used to construct strongly determining sequences.

KEY WORDS AND PHRASES. Banach spaces, WCG, WCD, SWCG.
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1. INTRODUCTION.

Let X be a Banach space with dual space X^* and second dual X^{**}. Let B and B^{**} denote the closed unit balls of X and X^{**} respectively.

X is said to be weakly compactly generated (WCG) if there is a weakly compact $K \subset X$ with the span of K dense in X [3]. The WCG property has been an active topic of research for several years (e.g., ([1], [3], [8]). Similarly, a generalization of this property, the WCD property, has been investigated, particularly since WCD spaces possess many of the same properties as WCG spaces (e.g., [6], [11], [12]). X is said to be WCD if there is a sequence (A_n) of weak* compact subsets of X^{**} such that for each $x \in X$ there is an $(n_m) \subset N$ with $x \in \bigcap_{m=1}^{\infty} A_{n_m} \subset X$ [12]. In this case, we say that (A_n) weakly determines X. We will see that each of these properties may be expressed as a property of the family of norm compact subsets of X. Our goal here is to introduce the weakly compact version of the WCD property, the SWCD property, and to examine its relationship to the strong WCG property of Schlüchtermann and Wheeler [9].

We first state some definitions and results.

X is strongly WCG (SWCG) if there is a sequence (K_n) of weakly compact subsets of X such that for each weakly compact subset H of X and each $\epsilon > 0$, there is an $n \in N$ such that $H \subset K_n + \epsilon B$ [9]. As noted in [9], restricting H to norm compact sets in the above definition gives a definition of WCG that is equivalent to the one above.

X is SWCD if there is a sequence (A_n) of weak* compact subsets of X^{**} such that for each weakly compact $K \subset X$ there is an $(n_m) \subset N$ with $K \subset \bigcap_{m=1}^{\infty} A_{n_m} \subset X$. In this case, we say (A_n) strongly determines X.

The following result affirms the claim that SWCD is the natural definition for the weakly compact version of WCD.

PROPOSITION 1. X is WCD if and only if there is a sequence (A_n) of weak* compact subsets of X^{**} such that for each norm compact $K \subset X$, there is an $(n_m) \subset N$ with $K \subset \bigcap_{m=1}^{\infty} A_{n_m} \subset X$.

PROOF. $(\Leftarrow)$ Clear.

$(\Rightarrow)$ Suppose (C_n) is a sequence of weak* compact subsets of X^{**} that weakly determines X. For each $i, j \in N$ let $F_{i,j} = C_j + \frac{1}{i} B^{**}$. Now let (A_n) be an enumeration of the finite unions of the $F_{i,j}$, and note that each A_n is weak* compact.

Suppose K is a norm compact subset of X. Choose a sequence $(n_m) \subset N$ so that $i \in (n_m) \Leftrightarrow K \subset A_i$. We certainly have $K \subset \bigcap_{m=1}^{\infty} A_{n_m}$, so we need only show that $\bigcap_{m=1}^{\infty} A_{n_m} \subset X$.

Let $x^{**} \in X^{**} \backslash X$. For each $x \in K$ there is an $i(x) \in N$ such that $x \in C_{i(x)}$ and $x^{**} \notin C_{i(x)}$. There is also a $j(x) \in N$ such that $x^{**} \notin C_{i(x)} + \frac{1}{j(x)} B^{**}$. Note that this last set has nonempty norm interior, so, in fact, we may find $x_1, \cdots, x_k \in K$ such that

$$K \subset \bigcup_{t=1}^{k} \left(C_{i(x_t)} + \frac{1}{j(x_t)} B^{**} \right).$$

and

$$x^{**} \notin \bigcup_{t=1}^{k} \left(C_{i(x_t)} + \frac{1}{j(x_t)} B^{**} \right).$$

The set on the right is one of the A_n containing K, hence it is one of the A_{n_m} so we have $x^{**} \notin \bigcap_{m=1}^{\infty} A_{n_m}$.□

We have WCG⇒WCD from [12], and the analogous result for the stronger properties from the following.

PROPOSITION 2. If X is SWCG then X is SWCD.

PROOF. Let the (A_n) be an enumeration of sets of the form $nK + \frac{1}{j} B^{**}$ with $j, n \in N$, where K is an SWCG generator for X.□

It is well-known that WCD spaces are Lindelöf in the weak topology [12]. Utilizing a natural strengthening of the Lindelöf property, we have a similar result for SWCD spaces.

A family $\mathcal{U}$ of subsets of a topological space T is called a ***strong open cover*** of T, if $\mathcal{U}$ is an open cover of T and for each compact subset K of T there is a $U \in \mathcal{U}$ with $K \subset U$. If every strong open cover of T has a countable strong open subcover, T is said to be ***strongly Lindelöf*** (*SL*). This property was first studied in ([4], [5]) in relation to properties of the compact-open topology on spaces of continuous functions.

PROPOSITION 3. If X is SWCD then $(X, \textit{weak})$ is strongly Lindelöf.

PROOF. Suppose (A_n) is a sequence of weak* compact subsets of X^{**} strongly determining X, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a strong open cover of $(X, \textit{weak})$. For each $\alpha \in I$ there is a $V_\alpha \subset X^{**}$ such that V_α is $w*$ open and $V_\alpha \cap X = U_\alpha$.

Let $\mathcal{F}$ be the collection of all finite subsequences, π, of N such that $\bigcap_{n \in \pi} A_n \subset V_\alpha$ for some $\alpha \in I$. Infinitely many such π exist since there are infinitely many $(n_m) \subset N$ with $\bigcap_{m=1}^{\infty} A_{n_m} \subset X$. We may assume $\mathcal{F} = \{\pi_i\}_{i=1}^{\infty}$. For each $i \in N$ choose $\alpha_i \in I$ such that $\bigcap_{j \in \pi_i} A_j \subset V_{\alpha_i}$.

Let K be a weakly compact subset of X. By hypothesis there is an $(n_m) \subset N$ such that $K \subset \bigcap_{m=1}^{\infty} A_{n_m} \subset X$. There is also an $\alpha \in I$ such that $K \subset \bigcap_{m=1}^{\infty} A_{n_m} \subset U_\alpha \subset V_\alpha$, hence, since $V_\alpha \subset X^{**}$ is weak* open, there is a $t \in N$ such that $K \subset \bigcap_{i=1}^{i=t} A_{n_m} \subset V_\alpha$. Now $m_1, \cdots, m_t = \pi_i$ for some i, so $K \subset V_{\alpha_i}$. Since $K \subset X$, we have $K \subset U_{\alpha_i}$. Therefore, (U_{α_i}) is a countable strong open subcover of $\mathcal{U}$.□

The task of identifying SWCD spaces may be reduced by Theorem 1. In order to prove this result, we recall the following definition.

Let T be a completely regular topological space, and let $\mathcal{P}$ be a family of subsets of T. $\mathcal{P}$ is said to be a *pseudobase* for T if for each open $U \subset T$ and each compact $K \subset U$, there is a $P \in \mathcal{P}$ such that $K \subset P \subset U$. If T has a countable pseudobase, T is said to be an $\aleph_0$-space [7]. A recent study of $\aleph_0$-spaces in regard to Banach spaces is given in [10]. In [7] it is proved that if T is an $\aleph_0$-space then T is Lindelöf, and, in fact, an elementary modification of this proof reveals that T is strongly Lindelöf.

The following result indicates that separability provides a way to "isolate" a weakly compact convex set K from $X\backslash K$ using intersection of members of a countable family of weak* compact subsets of X^{**}. The SWCD property is the condition needed to isolate K from $X^{**}\backslash X$. $(X, weak)$ will be an $\aleph_0$-space precisely when K is isolated from $X^{**}\backslash K$ in this manner.

LEMMA 1. Let X be separable. Then there is a sequence (F_n) of $w*$ compact subsets of X^{**} such that if K is a weakly compact convex subset of X, there is an $(n_m) \subset \boldsymbol{N}$ with $K = \left(\bigcap_{m=1}^{\infty} F_{n_m}\right) \bigcap X$.

PROOF. Let the norm on X be denoted by $\|\cdot\|$. From [13] there is an equivalent norm, $|||\cdot|||$ on X, such that every weakly compact convex subset of X can be written as the intersection of closed $|||\cdot|||$-balls.

Now let (x_n) be a dense sequence of points in $(X, |||\cdot|||)$. Suppose that K is a weakly compact convex subset of X and $y \in X$. Let $x \in X$ and $\delta > 0$ be such that $K \subset B(x, \delta)$ and $y \notin B(x, \delta)$, where the ball is the closed ball with respect to $|||\cdot|||$. We clearly may enlarge this closed ball to radius r so that $r - \delta > 0$, r is rational, and $y \notin B(x, r)$. Set $p = min\{(r-\delta), |||y - x||| - r\}$, and find n so that $|||x_n - x||| < p$. Then $y \notin (x_n, p) \supset K$. This shows that for X separable, it is enough to consider those closed balls centered at some x_n and of rational radius in the previous paragraph. Let the (F_n) be an enumeration of the $w*$ closures in X^{**} of this collection of closed $|||\cdot|||$-balls.□

THEOREM 1. If X is separable, the following are equivalent.

(1) X is SWCD.

(2) $(X, weak)$ is an $\aleph_0$-space.

(3) There is a sequence (A_n) of $w*$ compact subsets of X^{**} such that if K is a weakly compact convex subset of X, then there is an $(n_m) \subset \boldsymbol{N}$ with $K = \bigcap_{m=1}^{\infty} A_{n_m}$.

PROOF. $(1 \Rightarrow 2)$. Suppose (A_n) is a sequence of $w*$ compact subsets of X^{**} that strongly determines X. Choose the sequence (F_n) according to Lemma 1 and let (C_n) denote an enumeration of the members of (A_n) and (F_n). Then let $\mathcal{P}' = (P'_n)$ be a sequence formed from all finite intersections of members of (C_n), and set $\mathcal{P} = (P_n)$ where $P_n = P'_n \cap X$ for each $n \in \boldsymbol{N}$.

By the sub-base theorem in [7], it is enough to show that for each weakly open convex set $U \subset X$ and weakly compact convex set $K \subset U$, there is an $n \in \boldsymbol{N}$ such that $K \subset P_n \subset U$. Assume U and K are given this way, where $U = V \cap X$ for some $w*$ open $V \subset X^{**}$. Then there is an $(n_m) \subset \boldsymbol{N}$ such that $K = \bigcap_{m=1}^{\infty} C_{n_m}$. Hence there is a $t \in \boldsymbol{N}$ such that $\bigcap_{m=1}^{t} C_{n_m} \subset V$, but $\bigcap_{m=1}^{t} C_{n_m} = P'_j$ for some j, so $K \subset P'_j \subset V$. Thus $K \subset P_j \subset U$.

$(2 \Rightarrow 3)$. Assume that $(X, weak)$ has a countable pseudobase, $\mathcal{P} = (P_n)$. Without loss of generality, assume that each member of $\mathcal{P}$ is bounded and weakly closed in X. For each $n \in \boldsymbol{N}$, let $A_n = \overline{P}_n^{w*} \subset X^{**}$ and note that each A_n is $w*$ compact in X^{**}.

Let $K \subset X$ be weakly compact, and choose $(n_m) \subset \boldsymbol{N}$ so that $i \in (n_m) \Leftrightarrow K \subset A_i$. Suppose $x^{**} \in X^{**}\backslash K$. Then there is a $w*$ open set $V \subset X^{**}$ such that $K \subset V$ and $x^{**} \notin \overline{V}^{w*}$. By hypothesis, there is an $n \in \boldsymbol{N}$ such that $K \subset P_n \subset V$, so $K \subset \overline{P}_n^{w*} \subset \overline{V}^{w*}$, and hence $K \subset A_n$ and $x^{**} \notin A_n$. Therefore $K = \bigcap_{m=1}^{\infty} A_{n_m}$.

$(3 \Rightarrow 1)$. Obvious.

It should be noted that SWCD does not imply separability, since all reflexive spaces are SWCD, and separability does not imply SWCD, because there exist separable Banach spaces,

$C([0,1])$ for instance, which are not $\aleph_0$-spaces in the weak topology [7]. A simple example of an SWCD space that is not SWCG is given in the following.

EXAMPLE. c_0.

Since c_0 is not weakly sequentially complete it cannot be SWCG [9]. However, $(c_0, weak)$ is an $\aleph_0$-space, since it has separable dual [7], so c_0 is SWCD. The following example demonstrates how strongly determining sequences may be produced by utilizing results about the structure of weakly compact sets.

From [2] we obtain the following result.

Let $M \subset c_0$. Then M is relatively weakly compact if and only if M is bounded and for every $(m_k) \subset N$ we have

$$\delta_{n, m_k} := \sup_{x \in M} \left(\inf_{k=1, \cdots, n} |x_{m_k}| \right) \to 0,$$

as $n \to \infty$.

Let $\mathfrak{C}$ be the collection of all finite subsequences of N. For $n, t \in N$ and $\pi \in \mathfrak{C}$, set

$$A_{\pi, n, t} = \left\{ x \in nB_{\ell^\infty}: \inf_{k \in \pi} |x_k| \le \frac{1}{t} \right\}.$$

Then each $A_{\pi,n,t}$ is bounded and $w*$ closed, hence $w*$ compact. The collection of all $A_{\pi,n,t}$ is countable, so let C_n be an enumeration of the $A_{\pi,n,t}$.

Suppose K is a weakly compact subset of c_0. Let $(m_k) \subset N$ be the collection of all $n \in N$ such that $K \subset C_n$, noting that there are indeed infinitely many such C_n by the above result.

Now suppose $x^{**} \in \ell_\infty \backslash c_0$. Then there is a $j \in N$ and a $(t_k) \subset N$ such that $|x^{**}_{t_k}| > 1/j$ for all $k \ge 1$. By the above result [2] again, $K \subset A_{\pi,s,j}$ for some $s \in N$ and π of the form $\pi = t_1, t_2, \cdots, t_p$, yet x^{**} is contained in no set of this form. Thus $x^{**} \notin \bigcap_{k=1}^{\infty} C_{n_k}$. Therefore $\bigcap_{k=1}^{\infty} C_{n_k} \subset X$, hence (C_n) is a strongly determining sequence for c_0.

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