Internat. J. Math. & Math. Sci.
VOL. 18 NO. 2 (1995) 229-236

ON A GENERAL HYERS-ULAM STABILITY RESULT

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(Received June 18, 1993)

ABSTRACT. In this paper, we prove two general theorems about Hyers-Ulam stability of functional equations. As particular cases we obtain many of the results published in the last ten years on the stability of the Cauchy and quadratic equation.

KEY WORDS AND PHRASES. Functional equations, stability.
1991 AMS SUBJECT CLASSIFICATION CODES. 39B72, 47H15.

1. INTRODUCTION.

We consider the functional equation

$$g[F(x,y)] = H[g(x), g(y), g(R_1(x,y)), \cdots, g(R_p(x,y))] \tag{1.1}$$

where

$$F: S \times S \to S, \quad R_i: S \times S \to S \text{ and } H: X \times X \times X^p \to X$$

are given functions, S is a set, (X,d) is a complete metric space and $g: S \to X$ is the unknown function.

Together with equation (1.1) we consider the functional inequality

$$d\Big(f[F(x,y)], H[f(x), f(y), f(R_1(x,y)), \cdots, f(R_p(x,y))]\Big) \leq \rho(x,y) \tag{1.2}$$

where $\rho: S \times S \to \mathbb{R}^+$.

The aim of the present paper is to prove some general results of stability in the sense of Hyers-Ulam for the equation (1.1). This means to prove that, under suitable conditions on the functions involved, "near" any solution of the inequality (1.2) there exists a solution of the equation (1.1). The word "near" means that the distance of the solution of the equation from the solution of the inequality is explicitly evaluated through the function ρ. We point out that it is not necessary to know in advance either the form or the existence of any solution of (1.1).

The results we will prove contain as special cases many of the results of stability for the Cauchy and quadratic equation, published in the last ten years. The techniques we use are a generalization of those developed in [1].

2. MAIN RESULTS.

We introduce some notations and fix some assumptions on the functions involved throughout the paper.

Define $G(x) := F(x,x)$ and for each non-negative integer n let G^n denote the n-th iterate of G; if G is invertible, G^{-n} denotes the n-th iterate of G^{-1}.

We suppose that all functions R_i are constant on the diagonal of $S \times S$, i.e.,

$$R_i(x,x) = a_i \in S, \text{ for } x \in S \text{ and } i = 1, \cdots, p. \tag{2.1}$$

Moreover we assume that G and R_i commute:

$$R_i(G(x), G(y) = G(R_i(x,y)), \text{ for } i = 1, \cdots, p. \tag{2.2}$$

Note that conditions (2.1) and (2.2) imply that each a_i is a fixed point of $G: G(a_i) = a_i$, $i = 1, \cdots p$.

For every $\boldsymbol{z} = (z_1, \cdots, z_p)$, $z_i \in X$, we write

$$K_{\boldsymbol{z}}(u) := H(u, u, z_1, \cdots, z_p) = H(u, u, \boldsymbol{z})$$

and assume

$$K_{\boldsymbol{z}}(X) = X \text{ and } K_{\boldsymbol{z}} \text{ invertible.} \tag{2.3}$$

By $K^j_{\boldsymbol{z}}, j \in \mathbb{Z}$, we denote that j-th iterate (positive or negative) of $K_{\boldsymbol{z}}$. In the following, when it is clear from the context, we shall omit the subscript $\boldsymbol{z}$.

For a function $h: S \to X$, we denote by $\boldsymbol{h}(\boldsymbol{a})$ and by $\boldsymbol{h}(\boldsymbol{R}(x,y))$ the p-tuples $(h(a_1), \cdots, h(a_p))$ and $(h(R_1(x,y)), \cdots, h(R_p(x,y)))$ respectively.

For a function $f: S \to X$ we define for all $x, y \in S$

$$\delta(x,y) := d\left(f[F(x,y)], H[f(x), f(y), f(R_1(x,y)), \cdots, f(R_p(x,y))]\right)$$

and $\Delta(x) := \delta(x,x)$.

PROPOSITION 1. Let $f: S \to X$ be a function and assume the following conditions are satisfied:

(i) defined $K(u) := H[u, u, \boldsymbol{f}(\boldsymbol{a})]$, there exists a strictly increasing superadditive function $k: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\forall u, v \in X \qquad d(K(u), K(v)) = k(d(u,v)) \tag{2.4}$$

(ii) the function H is continuous and for every $u \in X$

$$H[K^{-1}(u), K^{-1}(u), K^{-1}(\boldsymbol{f}(\boldsymbol{a}))] = K^{-1}[H(u, u, \boldsymbol{f}(\boldsymbol{a}))] \tag{2.5}$$

where

$$\boldsymbol{K}^{-1}(\boldsymbol{f}(\boldsymbol{a})) = (K^{-1}(f(a_n)), \cdots, K^{-1}(f(a_p)).$$

If the series $\sum_{n=1}^{\infty} k^{-n}[\Delta(G^{n-1}(x))]$ converges for every $x \in S$, then the sequence

$$q_n(x) := K^{-n}[f(G^n(x))] \tag{2.6}$$

converges for every $x \in S$ and defined $g(x) := \lim_{n \to \infty} q_n(x)$, we have

$$g[G(x)] = H[g(x), g(x), \boldsymbol{g}(\boldsymbol{a})]. \tag{2.7}$$

Moreover the following conditions hold

$$d(g(x), f(x)) \leq \sum_{n=1}^{\infty} k^{-n}[\Delta(G^{n-1}(x))], \qquad x \in S, \tag{2.8}$$

$$\lim_{n \to \infty} K^{-n}\left(K^n_{\boldsymbol{g}(\boldsymbol{a})}[g(x)]\right) = g(x), \qquad x \in S. \tag{2.9}$$

The function g is the only solution of (2.7) satisfying (2.8) and (2.9).

PROOF. As a first step we evaluate the distance between $f[G^n(x)]$ and $K^n[f(x)]$. We introduce the following notations:

$$k_1(t) := k(t), k_m(t_1, \cdots, t_m) := k[t_m + k_{m-1}(t_1, \cdots, t_{m-1})], t_i \in \mathbb{R}^+, \quad m > 1.$$

We prove by induction that the following inequality

$$d(f[G^n(x)], K^n[f(x)]) \leq \Delta(G^{n-1}(x)) + k_{n-1}\Big(\Delta(x), \cdots, \Delta(G^{n-2}(x))\Big) \tag{2.10}$$

holds. Take $n = 2$; we have

$$\begin{aligned} d\left(f[G^2(x)],\ K^2[f(x)]\right) &\leq d\left(f[G^2(x)],\ K[f(G(x))]\right) + d\left(K(f[G(x)]),\ K^2[f(x)]\right) \leq \\ &\leq d\left(f[G^2(x)],\ H[f(G(x)),\ f(G(x)),\ f(\boldsymbol{a})]\right) + k\left[d\left(f[G(x)],\ K[f(x)]\right)\right] = \\ &= \Delta(G(x)) + k\left[d\left(f(G(x)),\ H[f(x),\ f(x),\ f(\boldsymbol{a})]\right)\right] = \Delta(G(x)) + k_1[\Delta(x)]. \end{aligned}$$

Assume now (2.10) is true for $n - 1$; we obtain

$$\begin{aligned} d\left(f[G^n(x)],\ K^n[f(x)]\right) &\leq d\left(f[G^n(x)],\ K[f(G^{n-1}(x))]\right) + \\ &+ d\left(K[f(G^{n-1}(x))],\ K^n[f((x))]\right) = \Delta(G^{n-1}(x)) + k\left[d\Big(f[G^{n-1}(x)],\ K^{n-1}[f(x)]\Big)\right] \leq \\ &\leq \Delta(G^{n-1}(x)) + k\left\{\Delta(G^{n-2}(x)) + k_{n-2}\left[\Delta(x), \cdots, \Delta(G^{n-3}(x))\right]\right\} = \\ &= \Delta(G^{n-1}(x)) + k_{n-1}[\Delta(x), \cdots, \Delta(G^{n-2}(x))]. \end{aligned}$$

(To prove the last inequality, remember that k is increasing).
In the next step we use inequality (2.10) to show that $\{q_n(x)\}$ is a Cauchy sequence; thus by the completeness of X it is convergent.
Let $n > m$, then

$$\begin{aligned} d\left(q_n(x),\ q_m(x)\right) &= d\left(K^{-n}[f(G^n(x))],\ K^{-m}[f(G^m(x))]\right) = \\ &= k^{-n}\left[d\big(f[G^n(x)],\ K^{n-m}[f(G^m(x))]\big)\right]; \end{aligned}$$

since k is increasing and superadditive, k^{-n} is increasing and subadditive, so we get by (2.10)

$$\begin{aligned} d\left(q_n(x),\ q_m(x)\right) &\leq k^{-n}\left\{\Delta(G^{n-1}(x)) + k_{n-m-1}\left[\Delta(G^m(x)), \cdots, \Delta(G^{n-2}(x))\right]\right\} \leq \\ &\leq \sum_{j=m+1}^{n} k^{-j}[\Delta(G^{j-1}(x))]; \end{aligned}$$

the convergence of the series $\sum_{n=1}^{\infty} k^{-n}[\Delta(G^{n-1}(x))]$ implies that $\{q_n(x)\}$ is a Cauchy sequence for every $x \in S$.
Define $g(x) := \lim_{n\to\infty} q_n(x)$; we have

$$\begin{aligned} &d\left(K^{-n}[f(G^{n+1}(x))],\ H[K^{-n}(f(G^n(x))),\ K^{-n}(f(G^n(x))),\ K^{-n}(f(\boldsymbol{a}))]\right) \to \\ &\quad \to d\left(g[G(x)],\ H[g(x),\ g(x),\ g(\boldsymbol{a})]\right) \text{ for } n\to\infty \end{aligned}$$

(note that $G^n(a_i) = a_i$); on the other hand we get

$$\begin{aligned} &d\left(K^{-n}[f(G^{n+1}(x))],\ H[K^{-n}(f(G^n(x))),\ K^{-n}(f(G^n(x))),\ K^{-n}(f(\boldsymbol{a}))]\right) = \\ &\quad = d\left(K^{-n}[f(G^{n+1}(x))],\ K^{-n}[H(f(G^n(x)),\ f(G^n(x)),\ f(\boldsymbol{a}))]\right) = \\ &\quad = k^{-n}\left[d\left(f(G^{n+1}(x),\ H(f(G^n(x)),\ f(G^n(x)),\ f(\boldsymbol{a}))\right)\right] = k^{-n}\left[\Delta(G^n(x))\right] \to 0 \end{aligned}$$

for $n\to\infty$; thus we obtain (2.7).

$$\begin{aligned} d\left(q_n(x),\ f(x)\right) &= d\left(K^{-n}[f(G^n(x))],\ f(x)\right) = k^{-n}\left[d\left(f(G^n(x)),\ K^n[f(x)]\right)\right] \leq \\ &\leq k^{-n}\left\{\Delta(G^{n-1}(x)) + k_{n-1}[\Delta(x), \cdots, \Delta(G^{n-2}(x))]\right\} \leq \sum_{j=1}^{n} k^{-j}[\Delta(G^{j-1}(x))], \end{aligned}$$

taking the limit as n goes to infinity we have (2.8).

Assume h is a function satisfying (2.8), i.e.,

$$d\,(h(x),\ f(x)) \leq \sum_{n=1}^{\infty} k^{-n}\,[\Delta(G^{n-1}(x))], \qquad x \in S.$$

Then for every $s \in \mathbb{N}$ we have

$$d(h[G^s(x)], f[G^s(x)]) \leq \sum_{n=1}^{\infty} k^{-n}[\Delta(G^{n+s-1}(x))]$$

and, by (2.4),

$$k^s\,\{d\,(K^{-s}[h(G^s(x))],\ K^{-s}[f(G^s(x))])\} \leq \sum_{n=1}^{\infty} k^{-n}\,[\Delta(G^{n+s-1}(x))],$$

i.e.,

$$d\,(K^{-s}[h(G^s(x))],\ K^{-s}[f(G^s(x))]) \leq \sum_{m=s+1}^{\infty} k^{-m}\,[\Delta(G^{m-1}(x))].$$

Taking the limit as $s \to \infty$, we get

$$K^{-s}[h(G^s(x))] \to g(x), \qquad x \in S. \tag{2.11}$$

Since g is a solution of (2.7), we have

$$g[G^n(x)] = K^n_{g(a)}\,[g(x)] \text{ and } K^{-n}\{g[G^n(x)]\} = K^{-n}\{K^n_{g(a)}\,[g(x)]\};$$

by (2.11), since g satisfies (2.8), we get $K^{-n}\{K^n_{g(a)}[g(x)]\} \to g(x)$, i.e., (2.9).
If h is a solution of (2.7), then $K^{-n}\{h[G^n(x)]\} = K^{-n}\{K^n_{h(a)}\,[h(x)]\}$ and if h satisfies (2.9), we get $K^{-n}\{K^n_{h(a)}[h(x)]\} \to h(x)$. This and (2.11) imply $h = g$. □

A similar result can be obtained under different assumptions on the functions G and K.

PROPOSITION 2. Let $f: S \to X$ be a function and assume the following conditions are satisfied:

(i) defined $K(u) := H[u, u, \boldsymbol{f}(\boldsymbol{a})]$, there exists a strictly increasing subadditive function $k: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\forall u, v \in X \qquad d(K(u), K(v)) = k(d(u,v)) \tag{2.12}$$

(ii) the function H is continuous and for every $u \in X$

$$H[K(u),\ K(u),\ \boldsymbol{K}(\boldsymbol{f}(\boldsymbol{a}))] = K[H(u, u, \boldsymbol{f}(\boldsymbol{a}))] \tag{2.13}$$

where

$$\boldsymbol{K}(\boldsymbol{f}(\boldsymbol{a})) = (K(f(a_n)), \cdots, K(f(a_p))).$$

(iii) the function G is invertible.

If the series $\sum_{n=1}^{\infty} k^{n-1}[\Delta(G^{-n}(x))]$ converges for every $x \in S$, then the sequence

$$p_n(x) := K^n[f(G^{-n}(x))] \tag{2.14}$$

converges for every $x \in S$ and defined $g(x) := \lim_{n \to \infty} p_n(x)$, the function g is a solution of equation (2.7).
Moreover the following conditions hold

$$d\,(g(x),\ f(x)) \leq \sum_{n=1}^{\infty} k^{n-1}[\Delta(G^{-n}(x))], \qquad x \in S, \tag{2.15}$$

$$\lim_{n \to \infty} K^n\left(K^{-n}_{g(a)}\,[g(x)]\right) = g(x), \qquad x \in S. \tag{2.16}$$

The function g is the only solution of (2.7) satisfying (2.15) and (2.16).

PROOF. The proof follows step by step that of Proposition 1. After the evaluation of the distance between $f[G^{-n}(x)]$ and $K^{-n}[f(x)]$, one proves that $\{p_n(x)\}$ is a Cauchy sequence for

every $x \in S$, and thus is convergent. So we immediately obtain (2.7) and (2.15).
Also for the uniqueness we proceed as in Proposition 1.

REMARK 1. Note that in Propositions 1 and 2, only the function $\Delta(x)$ has been used, so they are stability theorems in the sense of Hyers-Ulam for the functional equation in a single variable (2.7). The special case of the equation $g(2x) = 2g(x)$ (where x belongs to a semigroup and the values of g are in a Banach space) with $\Delta(x) \leq const$ follows immediately from [2] and is presented in [3]. The stability of the equation $g(2x) = 4g(x)$ (x in a group and $g(x)$ in a Banach space) again with $\Delta(x) \leq const$, is contained in [4].

REMARK 2. In Proposition 1, in order to have the convergence of the series $\sum_{n=1}^{\infty} k^{-n}[\Delta(G^n(x))]$, it is enough to require that the function k satisfies the inequality $k(t) \geq ct$ for some $c > 1$ and that the series $\sum_{n=1}^{\infty} c^{-n}\Delta(G^{n-1}(x))$ is convergent. An analogous remark holds for Proposition 2.

REMARK 3. Assume S is a topological space, F and f are continuous functions. The function k in Proposition 1 is continuous at 0, so from $d(K^{-1}(u), K^{-1}(v)) = k^{-1}(d(u,v))$ we get the continuity of K^{-1}. It follows that $q_n: S \to X$ is a continuous function for every $n \in \mathbb{N}$. We can conclude that if for each point $x \in S$ there is a neighborhood $U \ni x$ such that the series $\sum_{n=1}^{\infty} k^{-n}[\Delta(G^{n-1}(x))]$ converges uniformly on U, then the function g is continuous on S. Note that this happens when the $sup\{\Delta(G^{n-1}(x)): n \geq 1, x \in S\} < +\infty$. An analogous remark holds for Proposition 2, where we need the continuity of the function G^{-1}.

The next two theorems give results of stability for the functional equation (1.1). Essential tools are Propositions 1 and 2 and now a fundamental role is assumed by the "algebraic" properties of the functions F and H.

THEOREM 1. Assume that all hypotheses of Proposition 1 are satisfied. Moreover suppose:

$$\forall x,\ y \in S, \qquad F[F(x,y),\ F(x,y)] = F[F(x,x),\ F(y,y)] \tag{2.17}$$

$\forall u, v \in X, z = (z_1, \cdots, z_p) \in X^p$

$$K^{-1}[H(u,v,z)] = H[K^{-1}(u), K^{-1}(v), K^{-1}(z)] \tag{2.18}$$

$$\forall x,y \in S, \qquad k^{-n}(\delta(G^n(x), G^n(y))) \to 0 \text{ as } n \to \infty. \tag{2.19}$$

Then the function g defined in Proposition 1 is a solution of the functional equation (1.1).

PROOF. We recall that $g(x) := \lim_{n \to \infty} K^{-n}[f(G^n(x))]$. For every $x, y \in S$, by (2.2), (2.17), (2.18) and the continuity of H we have

$$\begin{aligned}
&d\,(K^{-n}\{f[F(G^n(x),\ G^n(y))]\},\ K^{-n}\{H[f(G^n(x)),\ f(G^n(y)),\ f(R(G^n(x),\ G^n(y)))]\}) = \\
&\quad = d\,(K^{-n}\{f[G^n(F(x,y))]\},\ K^{-n}\{H[f(G^n(x)),\ f(G^n(y)),\ f(G^n(R(x,y)))]\}) = \\
&\quad = d\,(q_n(F(x,y)),\ H[q_n(x), q_n(y),\ q_n(R(x,y))]) \to \\
&\qquad\qquad \to d\,(g[F(x,y)],\ H[g(x),\ g(y),\ g(R(x,y))])
\end{aligned}$$

as $n \to \infty$.

On the other hand, we have also

$$\begin{aligned}
&d\,(K^{-n}\{f[F(G^n(x),\ G^n(y))]\},\ K^{-n}\{H[f(G^n(x)),\ f(G^n(y)),\ f(R(G^n(x),\ G^n(y)))]\}) = \\
&\quad = k^{-n}\{d\,(f[F(G^n(x),\ G^n(y))],\ H[f(G^n(x)),\ f(G^n(y)),\ f(R(G^n(x),\ G^n(y)))]\})\} = \\
&\qquad\qquad = k^{-n}(\delta(G^n(x),\ G^n(y))) \to 0 \text{ as } n \to \infty, \text{ by (2.19).}
\end{aligned}$$

Thus the theorem follows. □

THEOREM 2. Assume that all hypotheses of Proposition 2 are satisfied. Moreover suppose:

$$\forall x, y \in S, \qquad F[F(x,y),\ F(x,y)] = F[F(x,x),\ F(y,y)]$$

$\forall u,\ v \in X,\ z = (z_1, \cdots, z_p) \in X^p$

$$K[H(u,v,z)] = H[K(u), K(v), K(z)]$$

$$\forall x, y \in S, \quad k^n(\delta(G^{-n}(x),\ G^{-n}(y))) \to 0 \text{ as } n \to \infty.$$

Then the function g defined in Proposition 2 is a solution of the functional equation (1.1).

PROOF. The proof is completely similar to that of Theorem 1. □

REMARK 4. In algebraic language, the set S with the operation F is a groupoid and in the proofs of both Theorems 1 and 2 we do not need power associativity of S as, for instance, in [5]. Conditions (2.17) and (2.18) give the inequalities

$$F[G^n(x), G^n(y)] = G^n[F(x,y)],\ x, y \in S, \qquad n \in \mathbb{N}, \tag{2.20}$$

$$K^{-n}[H(u,v,z)] = H[K^{-n}(u),\ K^{-n}(v),\ K^{-n}(z)],\ \ u, v \in X,\ z \in X^p,\ n \in \mathbb{N}. \tag{2.21}$$

which are used in the proof of Theorem 1.

If we stipulate the following conditions, there exists $\nu \geq 2$ such that

$$F[G^\nu(x),\ G^\nu(y)] = G^\nu[F(x,y)],\ x, y \in S, \tag{2.22}$$

$$K^{-\nu}[H(u,v,z)] = H[K^{-\nu}(u), K^{-\nu}(v), K^{-\nu}(z)], u, v \in X, z \in X^p, \tag{2.23}$$

it is easy to prove by induction that relations (2.20) and (2.21) hold for all n of the form ν^s, $s \in \mathbb{N}$, and this is sufficient for proving Theorem 1 (see for related questions [5] and [6]). Note that (2.22) is strictly weaker than (2.17). To see this take as (S, F) the dihedral group D_4: (2.17) is not true while (2.22) holds for $\nu = 2$.

A similar remark holds for Theorem 2.

The special case when H depends only on its first two variables, i.e. when (1.1) becomes

$$g[F(x,y)] = H[g(x), g(y)], \tag{2.24}$$

has been treated in [1], where examples and counterexamples are presented.

When X is a Banach space, equation (2.24) generalizes the additive Cauchy equation

$$g[F(x,y)] = g(x) + g(y) \tag{2.25}$$

and if $\rho(x,y) = const$, Theorem 1 gives the well known Hyers result (see [2]).

Also, if S is a normed space and F is its addition, we obtain, for various forms of $\rho(x,y)$ and using either Theorem 1 or Theorem 2, many of the results on stability published in the last years. More precisely:

$\rho(x,y) = c \parallel x \parallel^a \parallel y \parallel^b$, $\quad 0 \leq a + b < 1$ $\quad$ (Th. 1): see [17] and [6];

$\rho(x,y) = c(\parallel x \parallel^a + \parallel y \parallel^a)$, $\quad a < 1$ $\quad$ (Th. 1): see [18],

$a > 1$ $\quad$ (Th. 2): see [10];

$\rho(x,y) = \Phi(\parallel x \parallel, \parallel y \parallel)$, $\quad$ where $\Phi: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is increasing,

symmetric and homogeneous of degree $p \in [0, +\infty) \backslash \{1\}$ (Th. 1, Th. 2): see [20];

$\rho(x,y) = c\ (\Psi(\parallel x \parallel) + \Psi(\parallel y \parallel))$, where $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ is such that $t^{-1}\Psi(t) \to 0$ as $t \to 0$, $\Psi(ts) \leq \Psi(t)\Psi(s)$, $\Psi(t) < t$ for $t > 1$ (Th. 1): see [7].

Note that also a particular case of a stability result about set-valued functions, due to Smajdor, is contained in Theorem 1 for equation (2.24) ([8], [9]).

Let $S = X = B$ be a Banach space on $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and let $a, b, \alpha, \beta, \gamma_1, \cdots, \gamma_p \in \mathbb{K}$ with $a + b \neq 1$. Let $r_i: B \to B, i = 1, \cdots, p$, be functions satisfying the functional equation $\phi((a+b)t) = (a+b)\phi(t)$ and define $R_i(x,y) = r_i(x-y)$.

Consider now the functional equation

$$g(ax+by) = \alpha g(x) + \beta g(y) + \sum_{i=1}^{p} \gamma_i g(R_i(x,y)) \tag{2.26}$$

and let $f: B \to B$ be a function with $f(0) = 0$ such that

$$\| f(ax+by) - \alpha f(x) - \beta f(y) - \sum_{i=1}^{p} \gamma_i f(R_i(x,y)) \| \leq \vartheta(\| x \|^s + \| y \|^s) \tag{2.27}$$

for all $x, y \in B$, where $\vartheta \geq 0$ and $s \in \mathbb{R}$.

By using Theorems 1 and 2 we obtain the stability of equation (2.26) under the condition $|a+b|^s |\alpha+\beta|^{-1} \neq 1$ (In the case $|a+b|^s|\alpha+\beta|^{-1} = 1$ in general we have no stability; see [10] and [11]).

Note that the condition $f(0) = 0$ cannot be dispensed with, otherwise conditions (2.5), (2.13), (2.18) and the analogous of Theorem 2 are not satisfied. Nevertheless when $\alpha + \beta + \gamma_1 + \cdots + \gamma_p = 1$ the function $f^*(x) = f(x) - f(0)$ satisfies the inequality (2.27), so we can apply Theorems 1 or 2 to it. In the case $\alpha + \beta + \gamma_1 + \cdots + \gamma_p \neq 1$, if $s > 0$ the condition $f(0) = 0$ is forced by (2.27) if $s \leq 0$ the function f^* satisfies (2.27) with $|\alpha + \beta + \gamma_1 + \cdots + \gamma_p - 1| \, \| f(0) \| + \vartheta(\| x \|^s + \| y \|^s)$ as the right-hand side, so again we can apply Theorem 1.

Note that equation (2.26) when $a = b = 1, \alpha = \beta = 2, p = 1, \gamma_1 = -1$ and $r_1(t) = t$ becomes the quadratic equation and our results contain as particular cases those in [12], [13], [14], [4].

If in (2.27) instead of $\| x \|^s + \| y \|^s$ we have the quantity $\| x \|^s \| y \|^q$, we obtain the stability under the condition $|a+b|^{s+q}|\alpha+\beta|^{-1} \neq 1$. In the case of the quadratic equation we obtain as a particular case the result in [15].

3. A SEPARATION THEOREM.

In the last section we assume X is a closed subinterval of $\mathbb{R}$. By using the stability results we can prove a separation theorem (for analogous results see [16]).

Assume $f, g: S \to X$ are functions such that

$$\begin{cases} g(x) \leq f(x) \\ H[g(x), g(y), \boldsymbol{g}(\boldsymbol{R}(x,y))] \leq g[F(x,y)] \leq f[F(x,y)] \leq H[f(x), f(y), \boldsymbol{f}(\boldsymbol{R}(x,y))] \end{cases} \tag{3.1}$$

for all $x, y \in S$. We ask whether there exists a function $h: S \to X$ solution of equation (1.1) separating f and g, i.e., such that

$$g(x) \leq h(x) \leq f(x), \qquad x \in S. \tag{3.2}$$

We prove that under the hypotheses of Theorem 1 (or Theorem 2) with some additional conditions on the functions involved, the answer is affirmative.

THEOREM 3. Let $f, g: S \to X$ be functions satisfying (3.1) and such that $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{g}(\boldsymbol{a})$. Define

$$\sigma(x,y) = H[f(x),\ f(y),\ \boldsymbol{f}(\boldsymbol{R}(x,y))] - H[g(x),\ g(y),\ \boldsymbol{g}(\boldsymbol{R}(x,y))],\ x, y \in S.$$

Under the hypotheses of Theorem 1 (Theorem 2) where instead of δ we write σ, and if K is strictly increasing on X, there exists a function $h: S \to X$ solution of (1.1) such that $g(x) \leq h(x) \leq f(x)$ for all $x \in S$.

PROOF. We prove the theorem under the assumptions of Thecrem 1. By the hypothesis on σ we can conclude that the sequence $\{K^{-n}[f(G^n(x))]\}$ converges and its limit, say $h(x)$, is a solution of equation (1.1).

By hypothesis, the function $K(u) = H[u, u, f(a)] = H[u, u, g(a)]$ is increasing. This and (3.1) imply $f(G^n(x)) \leq K^n(f(x))$, so we get $K^{-n}[f(G^n(x))] \leq f(x)$ and $h(x) \leq f(x)$. Moreover we have $g(x) \leq K^{-n}[g(G^n(x))] \leq K^{-n}[f(G^n(x))]$ and so $g(x) \leq h(x)$.

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