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CHARACTERIZATION OF FUZZY NEIGHBORHOOD COMMUTATIVE DIVISION RINGS II

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ABSTRACT. In [4] we produced a characterization of fuzzy neighborhood commutative division rings; here we present another characterization of it in a sense that we minimize the conditions so that a fuzzy neighborhood system is compatible with the commutative division ring structure. As an additional result, we show that Chadwick [5] relatively compact fuzzy set is bounded in a fuzzy neighborhood commutative division ring.

KEY WORDS AND PHRASES. Fuzzy neighborhood system, fuzzy neighborhood ring, fuzzy neighborhood commutative division ring, relatively compact fuzzy set, bounded fuzzy set.
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1. PRELIMINARIES.

Just like our previous work, we consider here the fuzzy neighborhood topology $t(\Sigma)$ on D, the one generated by the well-known fuzzy neighborhood system Σ of R. Lowen [8]. The pair $(D, t(\Sigma))$ is termed as a fuzzy neighborhood space. The triplet $(D, +, \cdot)$ (or D alone) is considered either a ring, division ring or commutative division ring (whichever we require). $D^*: = D\backslash\{0\}$ denotes the multiplicative group of nonzero elements of the commutative division ring D and D^+ is the additive group of D.

As usual $I_0: =]0,1], I: = [0,1]$ the unit interval, and $2^{(D)}$ denotes finite subsets of D. If μ is a fuzzy set of D then $\mu^\sim$ is given by

$$\mu^\sim(x): = \mu(x^{-1}) \forall x \epsilon D^*,$$

while for $x \epsilon D$,

$$x \oplus \mu(y): = 1_x \oplus \mu(y) = \mu(y - x)$$

$\forall y \epsilon D$, where 1_x denotes the characteristic function of the singleton set $\{x\}$.

For any $\mu, \nu, \theta \epsilon I^D$ and $x \epsilon D^*$,
$x \odot \mu$, $\nu \oplus \theta$ and $\nu \odot \theta$ are defined by:

$$x \odot \mu(y): = 1_x \odot \mu(y) = \mu(x^{-1}y)$$

$$\nu \oplus \theta(y): = \sup_{s+t=y} \nu(s) \wedge \theta(t)$$

and

$$\nu \odot \theta(y): = \sup_{st=y} \nu(s) \wedge \theta(t)$$

$\forall y \epsilon D$.

We define μ/ν as

$$\mu/\nu := \mu \odot \nu^{\sim}$$

and thus $1/(1 \oplus \nu)$ is written as:

$$1/(1 \oplus \nu)(x) = (1 \oplus \nu)^{\sim}(x) = (1 \oplus \nu)(x^{-1}) \Rightarrow$$

$$1/(1 \oplus \nu)(x) = \sup_{(l+s)^{-1} = x} \nu(s)$$

$\forall x \epsilon D^*$.

We call μ symmetric if and only if

$$\mu = \sim \mu, \text{ where } \sim \mu(x) := \mu(-x) \forall x \epsilon D.$$

The constant fuzzy set of D with value $\delta \in I$ is given by the symbol $\underline{\underline{\delta}}$ (ϵI^D). The saturation operation [8] is defined on a prefilter base $\mathfrak{F} \subset I^D$ by $\tilde{\mathfrak{F}} = \{v \epsilon I^D : \forall \delta \epsilon I_0 \exists v_\delta \epsilon \mathfrak{F} \ni v_\delta - \underline{\underline{\delta}} \le v\}$.

PROPOSITION 1.1 [8]. Let $(D, t(\Sigma))$ and $(D, t(\Sigma'))$ be fuzzy neighborhood spaces and $f: D \to D'$, then f is continuous at $x \epsilon D \Leftrightarrow \forall \nu' \epsilon \Sigma'(f(x)) \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(x_0) \ni \nu - \underline{\underline{\delta}} \le f^{-1}(\nu')$.

DEFINITION 1.2 [3]. Let $(D, +, \cdot)$ be a ring and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is said to be a fuzzy neighborhood ring if and only if the following are fulfilled:

(FNR1) $(D, +, t(\Sigma))$ is a fuzzy neighborhood group [1].

(FNR2) The mapping $m: (D \times D, t(\Sigma) \times t(\Sigma)) \to (D, t(\Sigma)), (x, y) \mapsto xy$ is continuous.

PROPOSITION 1.3 [3]. If $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood ring and $x \epsilon D$, then

(a) The left homothety $\mathcal{L}_x: (D, t(\Sigma)) \to (D, t(\Sigma))$, $y \mapsto xy$ (resp. right homothety $\mathcal{R}_x: (D, t(\Sigma)) \to (D, t(\Sigma)), y \mapsto yx$) is continuous. If x is a unit element of D then each homothety is a homeomorphism.

(b) The translation $T_x: (D, t(\Sigma)) \to (D, t(\Sigma))$, $y \mapsto y + x$ and the inversion $k: (D, t(\Sigma)) \to (D, t(\Sigma))$, $x \mapsto -x$ are homeomorphisms.

(c) $\nu \epsilon \Sigma(0) \Leftrightarrow x \oplus \nu \epsilon \Sigma(x)$, i.e., $T_x(\nu) \epsilon \Sigma(x)$.

(d) $\nu \epsilon \Sigma(x) \Leftrightarrow -x \oplus \nu \epsilon \Sigma(0)$, i.e., $T_{-x}(\nu) \epsilon \Sigma(0)$.

DEFINITION 1.4 [2]. Let $(D, +, \cdot)$ be a division ring, and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is said to be a fuzzy neighborhood division ring if and only if the following are true:

(FNDR1) $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood ring.

(FNDR2) The mapping $r: (D^*, t(\Sigma_{|D^*})) \to (D^*, t(\Sigma_{|D^*}))$, $x \mapsto x^{-1}$ is continuous where $\Sigma_{|D^*}$ is the fuzzy neighborhood system on D^* induced by D.

A commutative division ring structure and a fuzzy neighborhood Σ on D are said to be compatible if the conditions (FNDR1) and (FNDR2) are satisfied.

THEOREM 1.5 [3]. Let $(D, +, \cdot)$ be a ring and Σ a fuzzy neighborhood system on D. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood ring if and only if the following conditions are satisfied:

(1) $\forall x \epsilon D$: $\Sigma(x) = \{T_x(\nu): \nu \epsilon \Sigma(0)\}$.

(2) $\forall x_0 \epsilon D, \forall \mu \epsilon \Sigma(0)$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni x_0 \odot \nu \le \mu + \underline{\underline{\delta}}$, and $\nu \odot x_0 \le \mu + \underline{\underline{\delta}}$, i.e., the mapping $y \mapsto x_0 y$ and $y \mapsto y x_0$ are continuous at 0.

(3) $\forall \mu \epsilon \Sigma(0), \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu \oplus \nu \le \mu + \underline{\underline{\delta}}$, i.e., the mapping $(x, y) \mapsto x + y$ is continuous at $(0,0)$.

(4) $\forall \mu \epsilon \Sigma(0), \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu \le -\mu + \underline{\underline{\delta}}$, i.e., the mapping $x \mapsto -x$ is continuous at 0.

(5) $\forall \mu \epsilon \Sigma(0), \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu \odot \nu \le \mu + \underline{\underline{\delta}}$, i.e., the mapping $(x, y) \mapsto xy$ is continuous at

(0,0).

THEOREM 1.6 [4]. Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then the quadruple $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood commutative division ring if and only if the following are fulfilled:

(i) $\forall x \epsilon D: \Sigma(x) = \{T_x(\nu): \nu \epsilon \Sigma(0)\}$.

(ii) $\forall \mu \epsilon \Sigma(0)$, $\forall x \epsilon D$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni x \odot \nu \leq \mu + \underline{\underline{\delta}}$; i.e., $y \mapsto yx$ is continuous at 0.

(iii) $\forall \mu \epsilon \Sigma(0)$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu \oplus \nu \leq \mu + \underline{\underline{\delta}}$, i.e., $(x,y) \mapsto x+y$ is continuous at (0,0).

(iv) $\forall \mu \epsilon \Sigma(0)$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu \odot \nu \leq \mu + \underline{\underline{\delta}}$, i.e., $(x,y) \mapsto xy$ is continuous at (0,0).

(v) $\forall \mu \epsilon \Sigma(0)$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni (1 \oplus \nu)^{\sim} \leq (1 \oplus \mu) + \underline{\underline{\delta}}$, i.e., the inversion $x \mapsto x^{-1} (x \neq 0)$ is continuous at 1.

PROPOSITION 1.7 [4]. Let $(D, +, \cdot)$ be a fuzzy neighborhood commutative division ring. If conditions (i)-(v) of Theorem 1.6 are satisfied, then the following inequality holds good:

$$\forall \mu \epsilon \Sigma(0),\ \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu/(1 \oplus \nu) \leq \mu + \underline{\underline{\delta}} .$$

Recently the notion of relatively f-compact fuzzy sets was introduced by Chadwick and studied in relation to fuzzy neighborhood spaces. We quote the following results from [5]. For a detailed account of compact fuzzy neighborhood spaces we refer to [9].

THEOREM 1.8 [5]. Let $(D, t(\Sigma))$ be a fuzzy neighborhood spaces, $\mu \epsilon I^D$. Then the following are equivalent:

(a) μ is relatively f-compact;

(b) for each family $(\nu_x)_{x \epsilon D} \subset \Pi_{x \epsilon D} \Sigma(x)$ and each $\delta > 0$ there is $F \epsilon 2^{(D)}$ such that $sup_{x \epsilon F} v_x \geq \mu - \underline{\underline{\delta}}$.

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We shall now present an equivalent form of condition (v) of Theorem 1.6, namely

$$\text{(v}'\text{)} \qquad \forall \mu \epsilon \Sigma(0) \forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni \nu/(1 \oplus \nu) \leq \mu + \underline{\underline{\delta}} , \tag{2.1}$$

or

$$\nu \odot (1 \oplus \nu)^{\sim} \leq \mu + \underline{\underline{\delta}} .$$

PROPOSITION 2.1. Referring to Theorem 1.6, we have conditions (i)-(v), are equivalent to (i)-(iv) and (v′).

PROOF. We need prove only the converse part, i.e., $\forall \mu \epsilon \Sigma(0)$, $\forall \delta \epsilon I_0 \exists \nu \epsilon \Sigma(0) \ni (1 \oplus \nu)^{\sim} \leq (1 \oplus \mu) + \delta$. Suppose conditions (i)-(iv) and (v′) hold, and $\mu \epsilon \Sigma(0)$ and $\delta \epsilon I_0$; choose $\nu = \sim \nu$ symmetric $\ni \nu \odot (1 \oplus \nu)^{\sim} \leq \mu + \underline{\underline{\delta}}$. Let $z \epsilon D^*$, then

$$\begin{aligned}
(1 \oplus \nu)^{\sim}(z) &= (1 \oplus \nu)(z^{-1}) \\
&= \sup_{(1+x)^{-1} = z} \nu(x) \\
&= \sup_{1 - \frac{x}{1+x} = z} \nu(x) \\
&= \sup_{-\frac{x}{1+x} = z-1} \nu(x) \\
&= \sup_{-x(1+x)^{-1} = z-1} \nu(x) \\
&= \sup_{x(1+x)^{-1} = z-1} \nu(-x)
\end{aligned}$$

$$\leq \sup_{x(1+x)^{-1}=z-1} \sim \nu(x) \wedge (1 \oplus \nu)^{\sim}((1+x)^{-1})$$

$$= \sup_{x(1+x)^{-1}=z-1} \nu(x) \wedge (1 \oplus \nu)^{\sim}((1+x)^{-1})$$

$$= \nu \odot (1 \oplus \nu)^{\sim}(z-1)$$

$$= 1 \oplus (\nu \odot (1 \oplus \nu)^{\sim})(z)$$

$$\leq (1 \oplus \mu)(z) + \delta \text{ (by (v'))}$$

i.e., $(1 \oplus \nu)^{\sim} \leq (1 \oplus \mu) + \underline{\delta}$. □

THEOREM 2.2. Let $(D, +, \cdot)$ be a commutative division ring and Σ a fuzzy neighborhood system on D. Then the following conditions are equivalent:

1° $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood commutative division ring.

2° $(D, +, t(\Sigma))$ is a fuzzy neighborhood group, and division

$$d: D \times D^{*} \to D, (x,y) \mapsto x/y$$

is continuous.

3° $(D, +, t(\Sigma))$ and $(D^{*}, \cdot, t(\Sigma_{|D^{*}}))$ are fuzzy neighborhood groups.

PROOF. 1°⇒2°. Let $(D, +, \cdot, t(\Sigma))$ be a fuzzy neighborhood commutative division ring. Then by definition it is an additive group and therefore a fuzzy neighborhood group. We show division

$$d: D \times D^{*} \to D, (x,y) \mapsto x/y$$

is continuous.

But from the following scheme we have:

$$D \times D^{*} \to D \times D \to D,$$

$$(x,y) \mapsto (x, y^{-1}) \mapsto xy^{-1},$$

i.e., $d: D \times D^{*} \to D, (x,y) \mapsto x/y$ is continuous.

2°⇒3°. If the division d is continuous on $D \times D^{*}$, then certainly the restriction to $D^{*} \times D^{*}$ is continuous, i.e., $(D^{*}, \cdot, t(\Sigma_{|D^{*}}))$ is a fuzzy neighborhood group.

3°⇒1°. We need to show that $m: D \times D \to D, (x,y) \mapsto xy$ is continuous. But this follows from Theorem 3.3 [4].

THEOREM 2.3. Let $(D, +, \cdot)$ be a commutative division ring with characteristic $Char(D) \neq 2$. Then a fuzzy neighborhood group on the commutative division ring D with respect to which the inversion is continuous is a fuzzy neighborhood commutative division ring.

PROOF. The continuity of multiplication follows from the equality:

$$xy = [(x+y-2)^{-1} - (x+y+2)^{-1}]^{-1} - [(x-y-2)^{-1} - (x-y+2)^{-1}]^{-1}. \quad \square$$

THEOREM 2.4. Let $(D, +, \cdot)$ be a commutative division ring and Σ a fuzzy neighborhood system on D such that

(i) multiplication, $m: D \times D \to D, (x,y) \mapsto xy$,

(ii) inversion, $r: D^{*} \to D^{*}$, $x \mapsto x^{-1}$,

(iii) addition of 1, $p: D \to D$, $x \mapsto 1 + x$, are continuous, then $(D, +, \cdot, t(\Sigma))$ is a fuzzy neighborhood commutative division ring.

PROOF. Negation $x \mapsto -x = (-1)x$ is continuous then $x \mapsto x-1$ is the composite:

$$x \mapsto -x \mapsto -x+1 \mapsto -(-x+1)$$

of continuous functions and therefore, continuous.
It remains to show that the addition is continuous. To this end, we show that the addition is continuous at $(0,0)$. In order to do so, we use the following identity:

$$x+y = [1+y(1+x)^{-1}](1+x)-1 \tag{2.1}$$

Let $\mu \epsilon \Sigma(0)$ and $\delta > 0$. Choose $\nu_i \epsilon \Sigma(0)$ and $\theta_i \epsilon \Sigma(1)$.
By continuity of $p: x \mapsto 1+x$, we get

$$\theta_1 \ominus 1 \leq \mu + \underline{\delta/7} \tag{2.2}$$

Since multiplication $m: D \times D \rightarrow D, (x,y) \mapsto xy$ is continuous at $(1,1)$, we have a $\theta_2 \epsilon \Sigma(1)$ such that

$$\theta_2 \odot \theta_2 \leq \theta_1 + \underline{\delta/7} \tag{2.3}$$

Again applying the continuity of p, we can find $\nu_1 \epsilon \Sigma(0)$ such that

$$1 \oplus \nu_1 \leq \theta_2 + \underline{\delta/7} \tag{2.4}$$

Continuity of $m: (x,y) \mapsto xy$ at $(0,1)$ produces $\nu_2 \epsilon \Sigma(0)$ and $\theta_3 \epsilon \Sigma(1)$ such that

$$\nu_2 \odot \theta_3 \leq \nu_1 + \underline{\delta/7} \tag{2.5}$$

Since $r: x \mapsto x^{-1}$ is continuous at 1, we can find $\theta_4 \epsilon \Sigma(1)$ such that

$$\theta_4^{\sim} \leq \theta_3 + \underline{\delta/7} \tag{2.6}$$

Now again applying continuity of p at $x = 0$, we get for $\theta_4 \epsilon \Sigma(1)$, *a* $\nu_3 \epsilon \Sigma(0)$ such that

$$\begin{aligned} &\nu_3 - \underline{\delta/7} \leq -1 \oplus \theta_4 \\ &\Rightarrow 1 \oplus \nu_3 \leq \theta_4 + \underline{\delta/7} \\ &\Rightarrow 1 \oplus \nu_3 \leq (\theta_2 \wedge \theta_4) + \underline{\delta/7} \end{aligned} \tag{2.7}$$

Now if we can show that

$$\nu_2 \oplus \nu_3 \leq \mu + \underline{\underline{\delta}},$$

then we are done.
But, first we show the following inequality:

$$\nu_2 \oplus \nu_3 \leq [(1 \oplus \nu_2 \odot (1 \oplus \nu_3)^{\sim})] \odot (1 \oplus \nu_3) \ominus 1 \tag{2.8}$$

Let $z \epsilon D$, then

$$\begin{aligned} &\nu_3 \oplus \nu_2(z) \\ = \quad &\sup_{x+y=z} \nu_3(x) \wedge \nu_2(y) \\ = \quad &\sup_{x+y=z} \nu_2(y) \wedge \nu_3(x) \wedge \nu_3(x) \\ = \quad &\sup_{[1+y(1+x)^{-1}](1+x)=z+1} \nu_2(y) \wedge (1 \oplus \nu_3) \sim ((1+x)^{-1}) \wedge (1 \oplus \nu_3)(1+x) \end{aligned}$$

$$\leq \sup_{[1+y(1+x)^{-1}](1+x)=z+1} \nu_2 \odot (1 \oplus \nu_3) \sim (y(1 \oplus x)^{-1}) \wedge (1 \oplus \nu_3)(1+x)$$

$$= \sup_{[1+y(1+x)^{-1}](1+x)=z+1} [1 \oplus \nu_2 \odot (1 \oplus \nu_3) \sim][1 + y(1 \oplus x)^{-1}] \wedge (1 \oplus \nu_3)(1+x)$$

$$= [1 \oplus \nu_2 \odot (1 \oplus \nu_3) \sim] \odot (1 \oplus \nu_3)(z+1)$$

$$= [1 \oplus \nu_2 \odot (1 \oplus \nu_3) \sim] \odot (1 \oplus \nu_3) \ominus 1(z),$$

i.e.,

$$\nu_3 \oplus \nu_2 \leq [1 \oplus \nu_2 \odot (1 \oplus \nu_3) \sim] \odot (1 \oplus \nu_3) \ominus 1.$$

this proves the inequality (2.8).

Now we prove that the right side of (2.8) is less than or equal to $\mu + \delta$.

In fact,

$$\begin{aligned}
\nu_3 \oplus \nu_2 &\leq [(1 \oplus \nu_2 \odot (1 \oplus \nu_3)^{\sim})] \odot (1 \oplus \nu_3) \ominus 1 \\
&\leq [1 \oplus \nu_2 \odot (\theta_4^{\sim} + \underline{\delta/7}))] \odot (1 \oplus \nu_3) \ominus 1 \text{ (by (2.7))} \\
&\leq [1 \oplus (\nu_2 \odot \theta_3) + (2\underline{\delta/7}))] \odot (1 \oplus \nu_3) \ominus 1 \text{ (by (2.6))} \\
&\leq [(1 \oplus \nu_1 + 3\underline{\delta/7})] \odot (1 \oplus \nu_3) \ominus 1 \text{ (by (2.5))} \\
&\leq [(\theta_2 + 4\underline{\delta/7}) \odot (1 \oplus \nu_3) \ominus 1 \text{ (by (2.4))} \\
&\leq [\theta_2 \odot (1 \oplus \nu_3) + 4\underline{\delta/7}] \ominus 1 \\
&\leq [(\theta_2 \odot \theta_2) + 5\underline{\delta/7}] \ominus 1 \\
&\leq [(\theta_1 + 6\underline{\delta/7}] \ominus 1 \text{ (by (2.3))} \\
&\leq (\theta_1 \ominus 1) + 6\underline{\delta/7} \\
&\leq \mu + \underline{\delta/7} + 6\underline{\delta/7} = \mu + \underline{\underline{\delta}}.
\end{aligned}$$

$\Rightarrow \nu_3 \oplus \nu_2 \leq \mu + \underline{\underline{\delta}}$, which proves the continuity of addition. Continuity of addition at (x,y), where $x \neq 0$ follows similarly from the identity $x + y = x(1 + x^{-1}y)$. □

THEOREM 2.5. Let $(D, +, \cdot)$ be a commutative division ring.

(i) If $(D, +, \cdot, t(\Sigma_j))_{j \epsilon J}$ is a family of fuzzy neighborhood rings, then $(D, +, \cdot, t(\Sigma) = sup_{j \epsilon J} t(\Sigma_j))$ is a fuzzy neighborhood ring, where

$$\Sigma(0) = \{\inf_{i \epsilon J_0} \nu_{j_i} : \nu_{j_i} \epsilon \Sigma_j(0);\ i \epsilon J_0,\ J_0 \epsilon 2^{(J)}\}^{\sim}$$

(ii) If $(D, +, \cdot, t(\Sigma_j))_{j \epsilon J}$ is a family of fuzzy neighborhood commutative division rings, then $(D, +, \cdot, t(\Sigma) = sup_{j \epsilon J} t(\Sigma_j))$ is a fuzzy neighborhood commutative division ring, where

$$\Sigma(0) = \{\inf_{i \epsilon J_0} \nu_{j_i} : \nu_{j_i} \epsilon \Sigma_j(0);\ i \epsilon J_0\ ;\ J_0 \epsilon 2^{(J)}\}^{\sim}$$

PROOF. We only prove a part of (ii). Verification of the conditions (i)-(iv) of Theorem 1.6 are straightforward. We check condition (v′) (inequality (2.1)).

Let $\mu = inf_{i \epsilon J_0} \mu_{j_i}$ and $\delta \epsilon I_0$. Choose ν_{j_i} satisfying the condition (v′) in (2.1) for all $i \epsilon J_0$; and let $\nu = inf_{i \epsilon J_0} \nu_{j_i}$. Now for any $z \epsilon D$:

$$\begin{aligned}
\nu/(1 \oplus \nu)(z) &= \nu \odot (1 \oplus \nu)^{\sim}(z) \\
&= \sup_{ab=z} \nu(a) \wedge (1 \oplus \nu)^{\sim}(b)
\end{aligned}$$

$$= \sup_{ab=z} \nu(a) \wedge \sup_{(1+x)^{-1}=b} \nu(x)$$

$$= \sup_{ab=z} \inf_{\iota\in J_0} \nu_{j_\iota} \wedge \sup_{(1+x)^{-1}=b} \inf_{\iota\in J_0} \nu_{j_\iota}(x)$$

$$\leq \inf_{\iota\in J_0} \sup_{ab=z} \nu_{j_\iota}(a) \wedge (1 \oplus \nu_{j_\iota})^{\sim}(b)$$

$$= \inf_{\iota\in J_0} \nu_{j_\iota} \odot (1 \oplus \nu_{j_\iota})^{\sim}(z)$$

$$\leq \inf_{\iota\in J_0} \mu_{j_\iota}(z) + \delta$$

$$= \mu(z) + \delta,$$

$$\nu/1 \oplus \nu \leq \mu + \underline{\delta} \text{ or } \nu \odot (1 \oplus \nu)^{\sim} \leq \mu + \underline{\delta}.$$ □

DEFINITION 2.6. ([3], [4]). Let $(D, +, \cdot)$ be a commutative division ring and $(D, +, \cdot, t(\Sigma))$ a fuzzy neighborhood ring. Then a fuzzy set $\mu \epsilon I^D$ is called bounded in $(D, +, \cdot, t(\Sigma))$ if and only if for all $\nu\epsilon\Sigma(0)\forall\delta\epsilon I_0$ there exists $\theta\epsilon\Sigma(0)$ such that $\mu \odot \theta \leq \nu + \underline{\delta}$.

THEOREM 2.7. In a fuzzy neighborhood commutative division ring $(D, +, \cdot, t(\Sigma))$, every relatively $f-$compact fuzzy set is bounded.

PROOF. Let $\mu\epsilon I^D$, $\delta > 0$ and $\nu\epsilon\Sigma(0)$. Since multiplication is continuous, for each x, we can find a $\theta_x\epsilon\Sigma(0)$ and $\nu_x\epsilon\Sigma(x)$ such that

$$\nu_x \odot \theta_x \leq \nu + \underline{\delta/2} \tag{2.9}$$

Since μ is relatively $f-$ compact, by Theorem 1.8, there is $x_1, x_2, \cdots, x_n \epsilon D$ such that

$$\nu_{x_1} \vee \nu_{x_2} \vee \cdots \vee \nu_{x_n} + \underline{\delta/2} \geq \mu.$$

Let $\theta = \theta_{x_1} \wedge \cdots \wedge \theta_{x_n}$, then $\theta\epsilon\Sigma(0)$. Then for any $z\epsilon D$:

$$\mu \odot \theta(z) = \sup_{ab=z} \mu(a) \wedge \theta(b)$$

$$\leq \sup_{ab=z} ((\nu_{x_1} \vee \nu_{x_2} \vee \cdots \vee \nu_{x_n}(a) \wedge (\theta_{x_1} \wedge \theta_{x_2} \wedge \cdots \wedge \theta_{x_n})(b)) + \frac{\delta}{2}$$

$$\leq \sup_{ab=z} ((\nu_{x_1}(a) \wedge \theta_{x_1}(b)) \vee (\nu_{x_2}(a) \wedge \theta_{x_2}(b)) \vee \cdots \vee (\nu_{x_n}(a) \wedge \theta_{x_n}(b))) + \frac{\delta}{2}$$

$$= (\nu_{x_1} \odot \theta_{x_1})(z) \vee (\nu_{x_2} \odot \theta_{x_2}(z) \vee \cdots \vee (\nu_{x_n} \odot \theta_{x_n})(z) + \frac{\delta}{2}$$

$$\leq (\nu(z) + \delta/2) \vee \cdots \vee (\nu(z) + \delta/2) + \frac{\delta}{2}$$

$$= \nu(z) + \frac{\delta}{2} + \frac{\delta}{2}$$

$$= \nu(z) + \delta$$

$\Rightarrow \mu \odot \theta \leq \nu + \underline{\delta}$, proving that μ is bounded. □

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