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RESEARCH NOTES

TWO INEQUALITIES FOR MEANS

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ABSTRACT. We prove two new inequalities for the identric mean and a mean related to the arithmetic and geometric mean of two numbers

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1. INTRODUCTION.

The logarithmic and identric means of two positive numbers a and b are defined by

$$L = L(a,b) := \frac{b-a}{\log b - \log a} \quad \text{for } a \neq b; \qquad L(a,a) = a$$

and

$$I = I(a,b) := \frac{1}{e}\left(b^b/a^a\right)^{1/(b-a)} \quad \text{for } a \neq b; \qquad I(a,a) = a,$$

respectively.

Let $A = A(a,b) := \frac{a+b}{2}$ and $G = G(a,b) := \sqrt{ab}$ denote the arithmetic and geometric means of a and b, respectively. Many interesting results have been proved for these means, see e.g. ([1] - [3], [5] - [10]). Let us introduce the mean U defined by

$$U = U(a,b) := \left(\frac{(2a+b)(a+2b)}{9}\right)^{1/2} = \left(\frac{8A^2+G^2}{9}\right)^{1/2}$$

The aim of this note is to prove the following:

THEOREM. For $a \neq b$ one has

$$(U^3G)^{1/4} < I < \frac{U^2}{A} \quad . \tag{1.1}$$

2. PROOF OF THE THEOREM.

For the first inequality we apply the Newton quadrature formula (see [4])

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^5}{648} f^{(4)}(\xi) \quad , \tag{2.1}$$

where $\xi \in (a,b)$ and $f: [a,b] \to \mathbb{R}$ has a continuous 4-th derivative on (a,b). Let $f(x) = -\log x (x > 0)$ in (2.1). Then $f^{(4)}(x) > 0$, and after certain transformations we get the left side of 1.1.

In order to prove the second inequality of (1 1) divide all terms by $a < b$ and denote $x. = \frac{b}{a} > 1$. Then the inequality to be proved becomes

$$(4x^2 + 10x + 4)/(x+1)g(x) > 9/e \tag{2 2}$$

where $g(x) = x^{x/(x-1)}$, $x > 1$.

Introduce the function $f. [1, \infty) \to \mathbb{R}$ defined by

$$f(x) = (4x^2 + 10x + 4)/(x+1)g(x),\ x > 1;\ f(1) = \lim_{x \to 1} f(x) = 9/e$$

We shall prove that f is strictly increasing, and this proves (2 2) We have

$$g'(x) = g(x)\left[\frac{1}{x-1} - \frac{\log x}{(x-1)^2}\right] \quad ,$$

and, after some elementary computations, we can deduce

$$(x^2-1)^2 g(x) f'(x) = (4x^2 + 10x + 4)(x+1)\log x - 10x^3 - 6x^2 + 6x + 10 \quad . \tag{2.3}$$

We now show that the right side of (2 3) is strictly positive, or equivalently

$$L < (8A^2 + G^2)A/(10A^2 + G^2) \quad , \tag{2.4}$$

where $L = L(x, 1)$ etc Since it is known that $L < (2G + A)/3$ (See [3]) we try to prove that $(2G + A)/3 < (8A^2 + G^2)A/(10A^2 - G^2)$. This holds true iff $14x^3 - 20x^2y + 4xy^2 + 2y^3 > 0$, with $x = A, y = G$, i e,

$$(x-y)(7x^2 - 3xy - y^2) > 0 \quad . \tag{2.5}$$

We have

$$7x^2 - 3xy - y^2 = \left[x + y\left(\frac{\sqrt{37}-3}{14}\right)\right]\left[x - y\left(\frac{\sqrt{37}+3}{14}\right)\right] > 0 \quad \text{by } \frac{\sqrt{37}-3}{14} > 0 \quad ,$$

and $0 < \frac{\sqrt{37}+3}{14} < 1$. Thus (2 5) is proved, concluding the proof of (2.2) and of the theorem.

3. REMARKS.

(1) Clearly, $G < U < A$ (for $a \neq b$). Relation (1.1) offers the improvement

$$G < (U^3G)^{1/4} < I < \frac{U^2}{A} < U < A \tag{2.6}$$

(2) It is well-known that (see e g. [7]) $A > I$, so from the right inequality in (1.1) we have

$$9I^2 < 8A^2 + G^2 \quad . \tag{2.7}$$

On the other hand, it is known that [8] $I > (2A + G)/3$, which according to $A > G$ and (2 7) yields the following double-inequality:

$$4A^2 + 5G^2 < 9I^2 < 8A^2 + G^2 \tag{2.8}$$

(3) The two sides of (1 1) imply

$$U^5 > A^4G \tag{2.9}$$

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