Internat. J. Math. & Math. Sci.
VOL. 18 NO. 4 (1995) 735-744

REGULARIZATION AND ASYMPTOTIC EXPANSION OF CERTAIN DISTRIBUTIONS DEFINED BY DIVERGENT SERIES

RICARDO ESTRADA

Escuela de Matemática, Universidad de Costa Rica
San José, Costa Rica

(Received May 4, 1994 and in revised form September 6, 1994)

ABSTRACT. The regularization of the distribution $\sum_{n=-\infty}^{\infty}\delta(x-p^n)$, which gives a regularized value to the divergent series $\sum_{n=-\infty}^{\infty}\phi(p^n)$, is obtained in several spaces of test functions. The asymptotic expansion as $\varepsilon\to 0^+$ of series of the type $\sum_{n=0}^{\infty}\phi(\varepsilon p^n)$ is also obtained

KEY WORDS AND PHRASES Generalized functions, asymptotic expansions
1992 AMS SUBJECT CLASSIFICATION CODES. 41A60, 46F12

1. INTRODUCTION.

Divergent series of the type

$$\sum_{n=-\infty}^{\infty}\phi(p^n),\qquad \sum_{n=0}^{\infty}\phi(p^n), \tag{1.1}$$

where $p>1$ and where ϕ is a given function, as well as the asymptotic development as $\varepsilon\to 0^+$ of the related series

$$\sum_{n=-\infty}^{\infty}\phi(\varepsilon p^n),\qquad \sum_{n=0}^{\infty}\phi(\varepsilon p^n), \tag{1.2}$$

have been shown to be of importance in several areas, among which we could mention the counting algorithms used in database systems [8] and the use by Ramanujan of related series, with $p=2$, in his approach to the problem of the distribution of prime numbers [10, lecture 2].

The aim of the present article is to provide a regularization of the generalized function

$$g(x)=\sum_{n=-\infty}^{\infty}\delta(x-p^n), \tag{1.3}$$

where $\delta(x-\xi)$ is the Dirac delta function concentrated at the point ξ, so that the evaluation of $g(x)$ at the test function $\phi(x)$, $\langle g,\phi\rangle$, provides a regularized value to the generally divergent series

$$\sum_{n=-\infty}^{\infty}\phi(p^n). \tag{1.4}$$

We give regularizations in the standard spaces of distributions $\mathcal{D}$ or $\mathcal{S}$ as well as in the spaces $\mathcal{S}\{x^{\alpha_n}\}$ and $\mathcal{S}\{x^{\alpha_n}\ln x, x^{\alpha_n}\}$ introduced in [6] to study distributional asymptotic expansions. The regularization is achieved by using the Hadamard finite part ideas, employed in the regularization of distributions defined by divergent integrals [5].

Subsequently, we consider the asymptotic developments (1.2) by using the theory of asymptotic expansions of generalized functions. The close ties between generalized functions and asymptotic analysis have been studied by several authors [1,3,4,6,7,12,14,15], who have shown that generalized functions provide a very suitable framework for the asymptotic expansion of integrals and series.

In the present study we obtain the expansion of (1.2) as $\varepsilon \to 0^+$ for smooth functions with power or logarithmic behavior near the origin.

The plan of the article is as follows. In the second section we briefly review some ideas from the theory of asymptotic expansions of distributions. The third section studies related expansions of convergent series. The regularization in $\mathcal{S}'\{x^{\alpha_n}\}$ and in $\mathcal{S}'\{x^{\alpha_n} \ln x, x^{\alpha_n}\}$ is considered in the next two sections, respectively, where the asymptotic expansion is also given. The last section gives an ilustration of these results to the construction of counterexamples to some Hardy Littlewood type Tauberian theorems.

2. PRELIMINARIES.

In this section we provide a list of spaces of generalized functions needed in this paper. We also discuss the moment asymptotic expansion, a concept that plays a key role in our analysis.

The basic spaces of distributions that we are going to need are the spaces $\mathcal{D}'(\mathbf{R})$, $\mathcal{S}'(\mathbf{R})$ and $\mathcal{E}'(\mathbf{R})$. The space of test functions $\mathcal{D}$ consists of the smooth functions with compact support. The space of test functions $\mathcal{S}$ consist of those smooth functions ϕ for which

$$\phi^{(j)}(x) = O(|x|^{-n}), \qquad \text{as} \quad |x| \to \infty, \tag{2.1}$$

for each $j, n \geq 0$. The space $\mathcal{E}$ consists of all smooth functions with the topology of uniform convergence of all derivatives on compact sets. The dual spaces, $\mathcal{D}'$, $\mathcal{S}'$, and $\mathcal{E}'$ are, respectively, the spaces of standard, tempered and compactly supported distributions. For details, see [11,13].

The moment asymptotic expansion in the space $\mathcal{E}$ takes the following form

THEOREM 1. Let $f \in \mathcal{E}'$. Then

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}, \qquad \text{as} \quad \lambda \to \infty, \tag{2.2}$$

where

$$\mu_n = \langle f(x), x^n \rangle \tag{2.3}$$

are the moments.

The moment asymptotic expansion also hold in other spaces such as $\mathcal{O}'_C$, $\mathcal{O}'_M$, $\mathcal{K}'$ or $\mathcal{P}'$ of distributions of "rapid decay at infinity", but it does not hold in $\mathcal{S}'$ nor $\mathcal{D}'$ [6,7]. At present we shall have use for the result in $\mathcal{E}'$ only.

Another class of spaces of generalized functions, very suitable for the study of asymptotic expansions, is the following. Let $\{\phi_n(x)\}$ be a sequence of smooth functions defined in $(0, \infty)$

such that for each $k = 0, 1, 2, \ldots$ the sequence $\{\phi_n^{(k)}(x)\}$ is an asymptotic sequence as $x \to 0^+$. If $\mathcal{A}$ is any space of test functions, such as $\mathcal{E}$, $\mathcal{D}$, $\mathcal{S}$ or other, then the space $\mathcal{A}\{\phi_n(x)\}$ consists of those smooth functions $\psi(x)$ defined in $(0, \infty)$ which show the behavior of the type of the space $\mathcal{A}$ as $x \to \infty$ while for small x admit the asymptotic expansion for $k = 0, 1, 2, \ldots$

$$\psi^{(k)}(x) \sim a_1\phi_1^{(k)}(x) + a_2\phi_2^{(k)}(x) + a_3\phi_3^{(k)}(x) + \cdots \qquad \text{as} \quad x \to 0^+, \tag{2.4}$$

where $a_1, a_2, a_3, \ldots$ are certain constants.

The functionals $\delta_n \in \mathcal{A}'\{\phi_n\}$ are given as

$$\langle \delta_n(x), \psi(x)\rangle = a_n. \tag{2.5}$$

The generalized moment asymptotic expansion in the space $\mathcal{E}'\{x^{\alpha_n}\}$, where $\{\alpha_n\}$ is a sequence with $\Re e\, \alpha_n \nearrow \infty$, takes the form

$$f(\lambda x) \sim \sum_{n=1}^{\infty} \frac{\mu(\alpha_n)\delta_n(x)}{\lambda^{\alpha_n+1}}, \qquad \text{as} \quad \lambda \to \infty, \tag{2.6}$$

where $f \in \mathcal{E}'\{x^{\alpha_n}\}$ and where $\mu(\alpha_n) = \langle f(x), x^{\alpha_n}\rangle$ are the moments.

A moment expansion also holds in the space $\mathcal{E}'\{x^{\alpha_n}\ln x, x^{\alpha_n}\}$. If we use the notation $\delta_n(x)$ and $\delta_n'(x)$ for the functionals defined as

$$\langle \delta_n(x), \psi(x)\rangle = a_n, \tag{2.7a}$$

$$\langle \delta_n'(x), \psi(x)\rangle = a_n', \tag{2.7b}$$

for a function $\psi \in \mathcal{E}\{x^{\alpha_n}\ln x, x^{\alpha_n}\}$ with expansion

$$\psi(x) \sim \sum_{n=1}^{\infty}(a_n' \ln x + a_n)x^{\alpha_n}, \qquad \text{as} \quad x \to 0^+, \tag{2.8}$$

then the moment asymptotic expansion takes the form

$$f(\lambda x) \sim \sum_{n=1}^{\infty} \frac{-\mu'(\alpha_n)\delta_n'(x)\ln\lambda + [\mu'(\alpha_n)\delta_n'(x) + \mu(\alpha_n)\delta_n(x)]}{\lambda^{\alpha_n+1}}, \tag{2.9}$$

as $\lambda \to \infty$, where the $\mu'(\alpha_n) = \langle f(x), x^{\alpha_n}\ln x\rangle$ are the logarithmic moments.

3. EXPANSION IN THE SPACE $\mathcal{S}$.

In this section we consider the asymptotic expansion as $\varepsilon \to 0^+$ of series of the type

$$\sum_{n=0}^{\infty} p^n \phi(\varepsilon p^n), \tag{3.1}$$

when the function ϕ belongs to the space $\mathcal{S}$. Here p is a constant, $p > 1$.

Notice that the values of $\phi(x)$ for negative x are irrelevant. What matters is that ϕ is smooth near $x = 0$ and thus the one-sided Taylor expansion

$$\phi(x) \sim \phi(0) + x\phi'(x) + \frac{x^2}{2!}\phi''(x) + \cdots, \qquad \text{as} \quad x \to 0^+, \tag{3.2}$$

holds.

In order to study (3.1) we introduce the three generalized functions $f(x)$, $f_+(x)$ and $f_-(x)$, defined as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} p^n \delta(x - p^n), \tag{3.3a}$$

$$f_+(x) = \sum_{n=0}^{\infty} p^n \delta(x - p^n), \tag{3.3b}$$

$$f_-(x) = \sum_{n=1}^{\infty} p^{-n} \delta(x - p^{-n}). \tag{3.3c}$$

Observe that f, f_+ and f_- are positive Radon measures and that

$$f(x) = f_+(x) + f_-(x). \tag{3.4}$$

Notice also that (3.1) can be written as

$$\sum_{n=0}^{\infty} p^n \phi(\varepsilon p^n) = \varepsilon^{-1}\langle f_+(x), \phi(\varepsilon x)\rangle = \langle f_+(\lambda x), \phi(x)\rangle, \tag{3.5}$$

where $\lambda = \varepsilon^{-1}$.

Let us start with the distribution $f(x)$. Clearly, $f \in \mathcal{S}'$. Moreover,

$$f(px) = f(x), \tag{3.6}$$

as follows directly from (3.3a). Thus

$$f(x) = F(\ln x), \tag{3.7}$$

where F is periodic of period $\ln p$. The mean of F, its average over an interval of length $\ln p$, is

$$\frac{1}{\ln p}\int_{\alpha}^{\alpha+\ln p} F(u)\, du = \frac{1}{\ln p}\int_{e^{\alpha}}^{pe^{\alpha}} f(x)\,\frac{dx}{x} = \frac{1}{\ln p}. \tag{3.8}$$

Therefore we can write

$$f(x) = \frac{H(x)}{\ln p} + g(x), \tag{3.9}$$

where $H(x)$ is the Heaviside function and where g is a periodic function of $\ln x$ of period $\ln p$ and zero mean. It follows that if $\phi \in \mathcal{S}$ then

$$\sum_{n=-\infty}^{\infty} p^n \phi(\varepsilon p^n) = \frac{1}{\varepsilon \ln p}\int_0^{\infty} \phi(x)\, dx + \frac{\psi(\varepsilon)}{\varepsilon}, \tag{3.10}$$

where the oscillatory component $\psi(\varepsilon) = \psi(\phi; \varepsilon)$, given by

$$\psi(\varepsilon) = \varepsilon\langle g(x), \phi(\varepsilon x)\rangle, \tag{3.11}$$

is a periodic function of $\ln \varepsilon$ of period $\ln p$ and zero mean.

Next, let us consider the generalized function $f_-(x)$. It has compact support and thus $f_-(\lambda x)$ admits the moment asymptotic expansion as $\lambda \to \infty$.

The moments are

$$\mu_k = \langle f_-(x), x^k \rangle = \sum_{n=1}^{\infty} p^{-n} p^{-nk} = \frac{1}{p^{k+1} - 1}, \tag{3.12}$$

hence

$$f_-(\lambda x) \sim \sum_{k=0}^{\infty} \frac{(-1)^k \delta^{(k)}(x)}{k!(p^{k+1} - 1)\lambda^{k+1}}, \qquad \text{as} \quad \lambda \to \infty. \tag{3.13}$$

Combining (3.4), (3.9) and (3.13) we obtain the development of $f_+(\lambda x)$ as

$$f_+(\lambda x) \sim \frac{H(x)}{\ln p} + g(\lambda x) - \sum_{k=0}^{\infty} \frac{(-1)^k \delta^{(k)}(x)}{k!(p^{k+1} - 1)\lambda^{k+1}}, \qquad \text{as} \quad \lambda \to \infty. \tag{3.14}$$

Evaluating (3.14) at $\phi \in \mathcal{S}$ and setting $\lambda = \varepsilon^{-1}$ yields the expansion

$$\sum_{n=0}^{\infty} p^n \phi(\varepsilon p^n) \sim \frac{1}{\varepsilon \ln p} \int_0^{\infty} \phi(x)\, dx + \frac{\psi(\varepsilon)}{\varepsilon} - \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)\varepsilon^k}{k!(p^{k+1} - 1)}. \tag{3.15}$$

4. REGULARIZATION IN THE SPACE $\mathcal{S}'\{x^{\alpha_n}\}$.

Notice that (3.15) does not give the asymptotic development of the series $\sum_{n=0}^{\infty} \phi(\varepsilon p^n)$, for $\phi \in \mathcal{S}$, unless $\phi(0) = 0$. We now attend to this problem, in the more general framework of the spaces $\mathcal{S}\{x^{\alpha_n}\}$. Indeed, the development of $\sum_{n=0}^{\infty} p^n \psi(\varepsilon p^n)$ for $\psi \in \mathcal{S}\{x^{-1}, 1, x, \ldots\}$ would give the expansion of $\sum_{n=0}^{\infty} \phi(\varepsilon p^n)$ for $\phi \in \mathcal{S}$ by setting $\psi(x) = x^{-1}\phi(x)$.

Even though the series $\sum_{n=0}^{\infty} p^n \phi(p^n)$ converges for $\phi \in \mathcal{S}\{x^{\alpha_n}\}$ for any sequence $\{\alpha_n\}$ with $\Re e\alpha_n \nearrow \infty$, the method of the previous section requires the consideration of the two generally divergent series

$$\langle f(x), \phi(x) \rangle = \sum_{n=-\infty}^{\infty} p^n \phi(p^n), \tag{4.1a}$$

$$\langle f_-(x), \phi(x) \rangle = \sum_{n=1}^{\infty} p^{-n} \phi(p^{-n}). \tag{4.1b}$$

Therefore, we study the regularization of $f(x)$ and $f_-(x)$ in the spaces $\mathcal{S}\{x^{\alpha_n}\}$. Naturally the two regularization problems are equivalent in view of (3.5).

Let $\phi \in \mathcal{S}\{x^{\alpha_n}\}$, with expansion

$$\phi(x) \sim a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots, \qquad \text{as} \quad x \to 0^+. \tag{4.2}$$

Suppose $\alpha_m = -1$. Then the series giving the value of $\langle f_-(x), \phi(x) - \sum_{j=1}^{m} a_j x^{\alpha_j} \rangle$ converges. Thus it is enough to give the finite part of the sum $\sum_{n=1}^{\infty} p^{-n}\phi(p^{-n})$ if $\phi(x) = x^{\alpha}$, $\alpha \leq -1$. If $\alpha < -1$ we have

$$\sum_{n=1}^{N} p^{-n} p^{-n\alpha} = \frac{p^{-(1+\alpha)} - p^{N(1+\alpha)}}{1 - p^{-1+\alpha}}.$$

thus the partial sums consist of two parts, namely, the infinite part $\dfrac{p^{-N(1+\alpha)}}{1 - p^{-(1+\alpha)}}$, and the finite part

$$\langle f_-(x), x^{\alpha}\rangle = \mathrm{Fp}\sum_{n=1}^{\infty} p^{-n(1+\alpha)} = \frac{1}{p^{1+\alpha} - 1}. \tag{4.3}$$

When $\alpha = -1$, we have

$$\langle f_-(x), x^{-1}\rangle = \mathrm{Fp}\sum_{n=1}^{\infty} 1 = \mathrm{Fp}\lim_{N\to\infty} N = 0. \tag{4.4}$$

Therefore, the regularization of $f_-(x)$ is given by

$$\langle f_-(x), \phi(x)\rangle = \sum_{j=1}^{m-1} \frac{a_j}{p^{1+\alpha_j} - 1} + \sum_{n=1}^{\infty} p^{-n}\left(\phi(p^{-n}) - \sum_{j=1}^{m} a_j p^{-n\alpha_j}\right), \tag{4.5}$$

if $\phi \in \mathcal{S}\{x^{\alpha_n}\}$ has the expansion (4.2).

The regularization of $f(x)$ is given by the formula $f = f_+ + f_-$ since f_+ is a well-defined element of $\mathcal{S}'\{x^{\alpha_n}\}$. Observe that if $\phi \in \mathcal{S}\{x^{\alpha_n}\}$ has the development (4.2) then

$$\begin{aligned}
\langle f(px), \phi(x)\rangle &= \frac{1}{p}\langle f(x), \phi(x/p)\rangle \\
&= \frac{1}{p}\left[\sum_{j=1}^{m-1} \frac{a_j p^{\alpha_j}}{p^{1+\alpha_j} - 1} + \sum_{n=1}^{\infty} p^{-n}\left(\phi(p^{-n-1}) - \sum_{j=1}^{m} a_j p^{-(n+1)\alpha_j}\right) + \sum_{n=0}^{\infty} p^{n}\phi(p^{n})\right] \\
&= \sum_{j=1}^{m-1} \frac{a_j}{p^{1+\alpha_j}(p^{1+\alpha_j} - 1)} + \sum_{n=1}^{\infty} p^{-n}\left[\phi(p^{-n}) - \sum_{j=1}^{m} a_j p^{-n\alpha_j}\right] + \sum_{n=0}^{\infty} p^{n}\phi(p^{n}) + \sum_{j=1}^{m} a_j p^{-\alpha_j - 1} \\
&= \langle f(x), \phi(x)\rangle,
\end{aligned}$$

so that $f(x)$ is still a periodic function of $\ln x$ in the space $\mathcal{S}\{x^{\alpha_n}\}$. Hence we can write, as before,

$$f(x) = \frac{H(x)}{\ln p} + g(x), \tag{4.6}$$

where the oscillatory generalized function $g(x)$ has zero mean.

Notice, however, that the formula $H(\lambda x) = H(x)$ ceases to hold in the space $\mathcal{S}\{x^{\alpha_n}\}$. Indeed, if $\phi \in \mathcal{S}\{x^{\alpha_n}\}$, the integral $\int_0^\infty \phi(x)\,dx$ is generally divergent and thus it is necessary to consider its finite part $\mathrm{Fp}\int_0^\infty \phi(x)\,dx$. But if $\alpha_m = -1$, then

$$\mathrm{Fp}\int_0^\infty \phi(\varepsilon x)\,dx = \frac{1}{\varepsilon}\left[\mathrm{Fp}\int_0^\infty \phi(x)\,dx - a_m \ln\varepsilon\right], \tag{4.7}$$

so that

$$H(\lambda x) = H(x) + \frac{\ln\lambda}{\lambda}\delta_m(x), \tag{4.8}$$

where $\delta_m(x)$ is the functional defined in (2.5), so that $\langle \delta_m, \phi\rangle = a_m$.

The moment asymptotic expansion of $f_-(\lambda x)$ in the space $\mathcal{S}\{x^{\alpha_n}\}$ takes the form

$$f_-(\lambda x) \sim \sum_{j=1}^{m-1} \frac{\delta_j(x)}{(p^{1+\alpha_j} - 1)\lambda^{\alpha_j}} + \sum_{j=m+1}^{\infty} \frac{\delta_j(x)}{(p^{1+\alpha_j} - 1)\lambda^{\alpha_j}}, \qquad \text{as } \lambda \to \infty. \tag{4.9}$$

If we now use (4.6), (4.8) and (4.9), we obtain the expansion of $f_+(\lambda x)$ as

$$f_+(\lambda x) \sim -\sum_{j=1}^{m-1} \frac{\delta_j(x)}{(p^{1+\alpha_j}-1)\lambda^{\alpha_j}} + \frac{\ln\lambda}{\lambda\ln p}\delta_m(x) + \frac{1}{\lambda}\left[\frac{H(x)}{\ln p} + g(\lambda x)\right] - \sum_{j=m+1}^{\infty} \frac{\delta_j(x)}{(p^{1+\alpha_j}-1)\lambda^{\alpha_j}}. \tag{4.10}$$

Set $\varepsilon = \lambda^{-1}$. Then evaluation at $\phi \in \mathcal{S}\{x^{\alpha_n}\}$ yields

$$\sum_{n=0}^{\infty} p^n \phi(\varepsilon p^n) \sim \sum_{j=1}^{m-1} \frac{a_j \varepsilon^{\alpha_j}}{1-p^{1+\alpha_j}} - \frac{a_m \ln\varepsilon}{\varepsilon\ln p} + \frac{1}{\varepsilon}\left(\mathrm{Fp}\int_0^\infty \frac{\phi(x)\,dx}{\ln p} + \psi(\varepsilon)\right) + \sum_{j=m+1}^{\infty} \frac{a_j\varepsilon^{\alpha_j}}{1-p^{1+\alpha_j}}. \tag{4.11}$$

as $\varepsilon \to 0^+$, if $\phi(x) \sim a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots$, as $x \to 0^+$. Here $\alpha_m = -1$ and the oscillatory component $\psi(\varepsilon) = \psi(\phi;\varepsilon)$ is a periodic function of $\ln\varepsilon$ of period $\ln p$ and zero mean.

Let us now replace $\phi(x)$ by $\phi(x)/x$ in formula (4.11). Since $\phi(x)/x$ has the expansion $a_1 x^{\alpha_1-1} + a_2 x^{\alpha_2-1} + a_3 x^{\alpha_3-1} + \cdots$, as $x \to 0^+$, we obtain the formula

$$\sum_{n=0}^{\infty} \phi(\varepsilon p^n) \sim \sum_{j=1}^{k-1} \frac{a_j\varepsilon^{\alpha_j}}{1-p^{\alpha_j}} - \frac{a_k\ln\varepsilon}{\ln p} + \frac{\mathrm{Fp}\int_0^\infty \phi(x)x^{-1}\,dx}{\ln p} + \psi_1(\varepsilon) + \sum_{j=k+1}^{\infty} \frac{a_j\varepsilon^{\alpha_j}}{1-p^{\alpha_j}}. \tag{4.12}$$

where $\alpha_k = 0$ and where $\psi_1(\varepsilon)$ is a periodic function of $\ln\varepsilon$ of period $\ln p$ and zero mean.

In particular, if $\phi \in \mathcal{S}$ formula (4.12) takes the form

$$\sum_{n=0}^{\infty} \phi(\varepsilon p^n) \sim \frac{-\phi(0)\ln\varepsilon}{\ln p} + \frac{\mathrm{Fp}\int_0^\infty \phi(x)x^{-1}\,dx}{\ln p} + \psi_1(\varepsilon) + \sum_{n=1}^{\infty} \frac{\phi^{(n)}(0)\varepsilon^n}{n!(1-p^n)}. \tag{4.13}$$

5. REGULARIZATION IN $\mathcal{S}'\{x^{\alpha_n}\ln x, x^{\alpha_n}\}$.

The same regularization procedure can be applied to the space $\mathcal{S}'\{x^{\alpha_n}\ln x, x^{\alpha_n}\}$. Indeed, we use the finite part values

$$\langle f_-(x), x^{-1}\ln x\rangle = -\ln p\,\mathrm{Fp}\sum_{n=1}^{\infty} n = 0. \tag{5.1}$$

$$\langle f_-(x), x^{\alpha}\ln x\rangle = -\ln p\sum_{n=1}^{\infty} np^{-n(\alpha+1)} = \frac{-p^{1+\alpha}\ln p}{(p^{1+\alpha}-1)^2}, \qquad \alpha < -1. \tag{5.2}$$

to obtain the regularization of $f_-(x)$ and, consequently, of $f(x)$. The generalized function $f(x)$ remains a periodic function of $\ln x$, of period $\ln p$.

Formula (4.7) becomes

$$\mathrm{Fp}\int_0^\infty \phi(\varepsilon x)\,dx = \frac{1}{\varepsilon}\left[\mathrm{Fp}\int_0^\infty \phi(x)\,dx - a_m\ln\varepsilon - a'_m\frac{(\ln\varepsilon)^2}{2}\right]. \tag{5.3}$$

if $\phi(x) \sim \sum_{j=1}^{\infty}(a'_j\ln x + a_j)x^{\alpha_j}$, as $x \to 0^+$ and $\alpha_m = -1$.

The asymptotic formula then takes the form

$$\begin{aligned}\sum_{n=0}^{\infty}\phi(\varepsilon p^n) \sim{}& \sum_{j=1}^{k-1}\left[\frac{a'_j\ln\varepsilon + a_j}{1-p^{\alpha_j}} + \frac{a'_j p^{\alpha_j}\ln p}{(1-p^{\alpha_j})^2}\right]\varepsilon^{\alpha_j} \\ &+ \frac{\mathrm{Fp}\int_0^\infty \phi(x)x^{-1}\,dx}{\ln p} - \frac{a_k\ln\varepsilon}{\ln p} - \frac{a'_k(\ln\varepsilon)^2}{2\ln p} + \psi(\varepsilon) \\ &+ \sum_{j=k+1}^{\infty}\left[\frac{a'_j\ln\varepsilon + a_j}{1-p^{\alpha_j}} + \frac{a'_k p^{\alpha_j}\ln p}{(1-p^{\alpha_j})^2}\right]\varepsilon^{\alpha_j}, \qquad \text{as } \varepsilon \to 0^+.\end{aligned} \tag{5.4}$$

where $\alpha_k = 0$.

6. AN ILLUSTRATION.

There are some Tauberian theorems, which go back to Hardy and Littlewood [9], for the differentiation of asymptotic approximations which are valid under appropiate monotonicity conditions [2]. For instance, if

$$G(x) \sim e^{\lambda x}, \qquad \text{as} \quad x \to \infty, \tag{6.1}$$

where $\lambda \neq 0$, and if G' is increasing, then

$$G'(x) \sim e^{\lambda x}. \tag{6.2}$$

The corresponding Tauberian theorems for comparison with a power function also hold, namely, if

$$F(x) \sim (x-a)^{\alpha}, \qquad \text{as} \quad x \to a^+, \tag{6.3}$$

and if F' is increasing and $\alpha > 1$ or $\alpha < 0$ or F' is decreasing and $0 < \alpha < 1$, then

$$F'(x) \sim \alpha(x-a)^{\alpha-1}, \qquad \text{as} \quad x \to a^+. \tag{6.4}$$

Therefore, it is surprising that the results for comparison with a logarithmic function are false [2]. A simple counterexample can be constructed by using the asymptotic approximations of the previous sections. Indeed, let $p > 1$ and set

$$f(x) = \sum_{n=0}^{\infty} x^{p^n}, \qquad 0 < x < 1. \tag{6.5}$$

The behavior of $f(x)$ as $x \to 1^-$ can be found from (4.13) by taking $x = e^{-\varepsilon}$, $\varepsilon = \ln(1/x)$ and $\phi(x) = e^{-x}$. The result is

$$f(x) \sim \frac{-\ln\ln(1/x)}{\ln p} - \gamma + \omega(\ln\ln(1/x)) + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n!(1-p^n)}, \qquad \text{as} \quad x \to 1^-, \tag{6.6}$$

where we have used the value $\gamma = \int_0^{\infty} \frac{e^{-x}}{x}\, dx$, Euler's constant, and where ω is a periodic function of period $\ln p$ and zero mean.

To the first order, (6.6) takes the form

$$f(x) \sim \frac{\ln\left(\frac{1}{1-x}\right)}{\ln p}, \qquad \text{as} \quad x \to 1^-. \tag{6.7}$$

But $f'(x)$, though increasing, is not asymptotically equivalent to $\frac{1}{(1-x)\ln p} = \frac{d}{dx}\left[\frac{\ln\left(\frac{1}{1-x}\right)}{\ln p}\right]$.

Actually, using (4.11) we find the approximation

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} p^n x^{p^n-1} = \frac{1}{x}\sum_{n=0}^{\infty} p^n x^{p^n} \\ &\sim \frac{1}{x}\left[\frac{1}{\ln p \ln(1/x)} + \omega_1(\ln\ln(1/x)) + \sum_{n=0}^{\infty} \frac{(\ln x)^n}{n!(1-p^{n+1})}\right], \qquad \text{as} \quad x \to 1^-. \end{aligned} \tag{6.8}$$

To the first order, then,

$$f'(x) \sim \frac{1}{1-x}\left[\frac{1}{\ln p} + \omega_1\left(\ln\ln\frac{1}{x}\right)\right], \tag{6.9}$$

so that $f'(x)(1-x)\ln p$ does not tend to 1, but rather it oscillates about it since ω_1 is a periodic function with zero mean.

REFERENCES

1. BRÜNING, J. On the asymptotic expansion of some integrals. Arch. Math. 58 (1984), 253-259.

2. DURÁN, A. L. El recílproco de la regla de L'Hòpital. Rev. Cienc. Tecn. (C.R.) 16 (1992), 111-119.

3. ESTRADA, R. The asymptotic expansion of certain series considered by Ramanujan, Appl. Anal. 43 (1992), 191-228.

4. ESTRADA, R., GRACIA-BONDIA, J. M. and VARILLY, J. C., On asymptotic expansions of twisted products, J. Math. Phys. 30 (1989), 2789-2796.

5. ESTRADA, R. and KANWAL, R. P. Regularization, pseudofunction and Hadamard finite part, J. Math. Anal. Appls. 141 (1989), 195-207.

6. ESTRADA, R. and KANWAL, R. P. A distributional theory for asymptotic expansions, Proc. Roy. Soc. London A 428 (1990), 399-430.

7. ESTRADA, R. and KANWAL, R. P. The asymptotic expansion of certain multidimensional generalized functions, J. Math. Anal. Appl. 163 (1992), 264-283.

8. FLAJOLET, P. and MARTIN, G. N. Probabilistic counting algorithms for database applications, J. Comp. Syst. Sci. 31 (1985), 182-209.

9. HARDY, G. H. Divergent Series, Clarendon Press, Oxford, 1949.

10. HARDY, G. H. Ramanujan: Twelve lectures on Subjects suggested by his Life and Work, Chelsea, New York, 1978.

11. KANWAL, R. P. Generalized Functions: Theory and Technique, Academic Press, New York, 1983.

12. PILIPOVIC, S. Quasiasymptotic expansion and Laplace transform, Appl. Anal. 35 (1990), 247-261.

13. SCHWARTZ, L. Thèorie des Distributions, Hermann, Paris, 1966.

14. VLADIMIROV, V. S., DROZHINOV, Y. N. and ZAV'ALOV, B. I., Multidimensional Tauberian Theorems for Generalized Functions, Nauka, Moskow, 1986.

15. WONG, R. Asymptotic Approximation of Integrals, Academic Press, New York, 1989.