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ON A STRUCTURE SATISFYING $F^K-(-)^{K+1}F=0$

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ABSTRACT. In this paper we shall obtain certain results on the structure defined by $F(K, -(-)^{K+1})$ and satisfying $F^K-(-)^{K+1}F=0$, where F is a non null tensor field of the type (1,1) Such a structure on an n-dimensional differentiable manifold M^n has been called $F(K, -(-)^{K+1})$ structure of rank "r", where the rank of F is constant on M^n and is equal to "r" In this case M^n is called an $F(K, -(-)^{K+1})$ manifold The case when K is odd has been considered in this paper

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1. INTRODUCTION.

Let F be a non zero tensor field of the type (1,1) and of class C^∞ on M^n such that [2]

$$F^K-(-)^{K+1}F=0 \qquad \text{and} \qquad F^\omega-(-)^{\omega+1}F\neq 0 \tag{1 1}$$

for $1<\omega<K$, where K is a fixed positive integer greater than 2 The degree of the manifold being K, $(K\geq 3)$. Let us define operators on M^n by:

$$\tilde{1} \overset{\text{def}}{=} (-)^{K+1}F^{K-1}, \qquad \tilde{m} \overset{\text{def}}{=} I-(-)^{K+1}F^{K-1} \tag{1.2}$$

where I denotes the identity operator on M^n. Thus from (1.1) and (1.2) the following results are obvious

$$\tilde{1}+\tilde{m}=I, \qquad \tilde{1}^2=\tilde{1}, \qquad \tilde{m}^2=\tilde{m}.$$

For F satisfying (1.1), there exists complementary distributions $\tilde{L}$ and $\tilde{M}$, corresponding to the projection operators $\tilde{1}$ and $\tilde{m}$ respectively. Now we state the following theorems [2].

THEOREM (1.1). We have

$$F\tilde{1}=\tilde{1}F=F \qquad \text{and} \qquad F\tilde{m}=\tilde{m}F=0 \tag{1.3}$$

THEOREM (1.2). Let the tensor field $F(\neq 0)$ satisfy (1 1) and let the operators $\tilde{1}$ and $\tilde{m}$ defined by (1.2). Then it admits an almost product structure on $\tilde{L}$ and null operator on $\tilde{M}$. That is

$$F^{k-1}\tilde{1}=\tilde{1} \qquad \text{and} \qquad F^{K-1}\tilde{m}=\tilde{m}F^{K-1}=0 \tag{1.4}$$

Then $F^{\frac{K-1}{2}}$ acts on $\tilde{L}$ as an almost product structure and on $\tilde{M}$ as a null operator.

THEOREM (1.3). If in M^n there is given a tensor field $F(F \neq 0, F^{K-1} \neq I)$ of type (1,1) and of class C^∞ such that $F^{K-1} - (-)^{K+1}F = 0$, then M^n admits an almost product structure $\overset{\circ}{\Psi} = 2(-)^{K+1}F^{K-1} - I$ where $\overset{\circ}{\Psi} \overset{\text{def}}{=} \tilde{1} - \tilde{m}$.

PROOF. We have

$$\begin{aligned}\overset{\circ}{\Psi} &\overset{\text{def}}{=} \tilde{1} - \tilde{m}, \\ &= 2(-)^{K+1}F^{K-1} - I\end{aligned}$$

Then

$$\overset{\circ}{\Psi} \neq I \qquad \text{if} \qquad F^{K-1} \neq I$$

Also,

$$\begin{aligned}\overset{\circ}{\Psi}{}^2 &= 4(-)^{2K+2}F^{2K-2} + I - 4(-)^{K+1}F^{K-1} \\ &= 4F^K F^{K-2} + I - 4(-)^{K+1}F^{K-1} \\ &= 4F^{K-1} + I - 4F^{K-1}, \qquad \text{from (1.1)} \\ &= I\,.\end{aligned}$$

Thus,

$$\overset{\circ}{\Psi} \neq I \qquad \text{if} \qquad F^{K-1} \neq I,$$

and

$$\overset{\circ}{\Psi}{}^2 = I \qquad \text{if} \qquad F^{K-1} \neq I\,.$$

Hence $\overset{\circ}{\Psi}$ is an almost product structure.

2. METRIC FOR $F(K, -(-)^{K+1})$ STRUCTURE.

THEOREM (2.1). Let M^n be an $F(K, -(-)^{K+1})$ manifold of degree K defined by $F^K - (-)^{K+1}F = 0$ and $F^\omega - (-)^{\omega+1}F \neq 0$ for $1 < \omega < K$,and K is a fixed positive integer greater than 2, then:

there exists a positive definite Riemannian metric g with respect to which $\tilde{L}$ and $\tilde{M}$ are orthogonal and such that:

$$H_j^t H_i^s g_{ts} + \tilde{m}_j^t g_{ti} = g_{ji}\,,$$

$$H_{ji} = H_{ij}\,,$$

where

$$H = F^{\frac{K-1}{2}} \qquad \text{and} \qquad H_{ji} = H_j^t g_{ti}$$

and the rank of F is odd.

PROOF. Let us consider local coordinate system in the manifold M^n and let us denote the local components of the tensor ϕ in the set $\{F, \tilde{1}, \tilde{m}, H\}$ by ϕ_i^p. Here we consider r-mutally orthogonal unit vectors $u_a^p(a, b, c, ... = 1, 2, 3, ..., r)$ in $\tilde{L}$ and $(n-r)$ mutually orthogonal unit vectors

$$u_A^p(A, B, C, ... = r+1, r+2, ..., n) \qquad \text{in} \qquad \widetilde{M}^n\,.$$

(ω_i^a, ω_i^A) denotes the inverse matrix of (u_b^p, u_B^p).

Then ω_i^a and ω_i^A are both components of linearly independent covariant vectors. Let

$$\tilde{m}_{ji} = \tilde{m}^t_j a_{ti} \ ,$$
$$a_{ji} = \omega^a_j \omega^a_i + \omega_j^1 \omega_i^1$$
$$g_{ji} = \frac{1}{2}(a_{ji} + \tilde{m}_{ji} + H^t_j H^s_i a_{st}) \ ,$$
$$F_{it} = F_i^s g_{st}$$

If $\phi \in \{a, m, g\}$ then we put

$$\phi(X, y) = \phi_{st} X^s X^t$$

Now we can show that

$$\omega^a_s \omega^s_A = 0 \ , \qquad \omega^A_t \omega^t_a = 0$$

$$\tilde{m}^p_s u^s_A = u^p_A \qquad \text{and} \qquad a(u^A, u_a) = 0 \ . \tag{2 2}$$

From $F\tilde{m} = 0$ we have $F^p_i u^i_s = 0$ and hence, $H^p_s u^s_A = 0$ As $\tilde{m}(U_A, u_a) = 0$ by (2 1), we get $g(u_A, u_a) = 0$ This gives us that $\tilde{L}$ and $\tilde{M}$ are orthogonal with respect to g and a From $F\tilde{m} = \tilde{m}F = 0$ we have

$$F^t_j \tilde{m}^i_t = 0 \ , \qquad F^t_i \omega^A_t = 0 \ , \qquad H^t_i \omega^A_t = 0 \ , \tag{2 3}$$

$$\tilde{m}^p_j \tilde{m}^q_i a_{pq} = \tilde{m}_{ji} \tag{2 4}$$

By virtue of (1 2), we have

$$H^t_j H^s_t = \delta^j_s - \tilde{m}^s_j \tag{2 5}$$

From (2 4), (2 5) and $F^t_j \tilde{m}^i_t = 0$.

$$F^t_i \omega^A_t = 0 \ , \qquad H^t_i \omega^A_t = 0 \ , \qquad \text{we get}$$

$$H^t_j H^s_i g_{ts} + \tilde{m}_{ji} = g_{ji} \ , \qquad \text{we obtain} \tag{2 6}$$

$$H^t_j H^i_t + \tilde{m}^i_j = \delta^i_j$$

Let $H^s_i g_{st} = H_{it}$, then we get

$$H^t_j H_{ti} + \tilde{m}_{ji} = g_{ji} \tag{2 7}$$

From (2 6) and (2 7) we get

$$H^t_j H_{ti} = H^t_j H^s_i g_{ts}$$

or

$$H^t_j (H_{ti} - H_{it}) = 0$$

which shows that H is symmetric.

3. CONFORMAL DIFFEOMORPHISM OF $F(K, -(-)^{K+1})$ MANIFOLD.

Let M^n be a C^∞ differentiable manifold $\mathfrak{F}(M^n)$ be the ring of real valued differentiable function on M^n and $\mathfrak{X}(M^n)$ be the moduli of derivatives of $\mathfrak{F}(M^n)$ Then $\mathfrak{X}(M^n)$ is a Lie algebra over the real numbers and the elements of $\mathfrak{X}(M^n)$ are called vector fields

Let (M^n, g) and $(\overset{\circ}{M}{}^n, \overset{\circ}{g})$ be two Riemannian manifolds and $\Psi : M^n \rightarrow \overset{\circ}{M}{}^n$ be diffeomorphism Let $X \in \mathfrak{X}(M^n)$, $\overset{\circ}{X} \in \mathfrak{X}(\overset{\circ}{M}{}^n)$ be the vector fields on M^n and $\overset{\circ}{M}{}^n$ respectively $\overset{\circ}{X}$ corresponds to the X induced by Ψ Then diffeomorphism Ψ is called conformal diffeomorphism provided there exists

$$\rho \in \mathfrak{F}(M^n) \qquad \text{such that}$$

$$\overset{\circ}{g}(X^{\circ}, Y^{\circ}) * \Psi = e^{2\rho} g(X, Y) \qquad \text{for all} \qquad X, Y \in \mathfrak{X}(M^n) . \tag{3.1}$$

for $\sigma \in \mathfrak{F}(M^n)$ defined grad $\sigma \in \mathfrak{X}(M^n)$ by·

$$g(\text{grad}\ \sigma, X) = X(\sigma) \qquad \text{for all} \qquad X \in \mathfrak{X}(M^n) \tag{3 2}$$

In addition to (3.1) and (3 2) if

$$\Psi : M^n \to \overset{\circ}{M}{}^n , \qquad \text{preserves} \qquad F(K, \ - \ (- \ )^{K+1}) \qquad \text{structure i e}$$

$$F^{\circ} X^{\circ} = (FX)^{\circ} \tag{3 3}$$

where F and F° are (1,1) tensor fields with respect to M^n and $\overset{\circ}{M}$ If g° be the Riemannian metric in $\overset{\circ}{M}{}^n$, its metric satisfies the following·

$$\overset{\circ}{g}(F^{\circ} X^{\circ}, F^{\circ} Y^{\circ}) = \overset{\circ}{g}(X \ , Y \ ) , \tag{3 4}$$

for all X°, Y° in $\tilde{L}^{\circ}$ that is g° restricted to $\tilde{L}^{\circ}$ is an almost product structure with respect to F°. The Nijenhuis tensor $N(X, Y)$ of F in M^n is expressed as follows, for all X, $Y \in \mathfrak{X}(M^n)$

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] \tag{3.5}$$

We have [3]

$$[X^{\circ}, Y^{\circ}] = \{[X, Y]\}^{\circ} \tag{3.6}$$

By means of (3.3), (3.6) we get

$$N^{\circ}(X^{\circ}, Y^{\circ}) = \{N(X, Y)\}^{\circ} \qquad \text{for all} \qquad X, Y \in \mathfrak{X}(M^n) , \tag{3.7}$$

where N° is the Nijenhuis tensor corresponding to F° in $\overset{\circ}{M}{}^n$.

Since $\overset{\circ}{M}{}^n$ is also an $F(K, \ - \ (- \ )^{K+1})$ structure manifold therefore we can define complementary distribution corresponding to the projection operators $\tilde{1}$ and $\tilde{m}$. Let $\tilde{1}^{\circ}$ and $\tilde{m}^{\circ}$ be the projection operators in $\overset{\circ}{M}{}^n$ corresponding to the structure $F(K, \ - \ (- \ )^{K+1})$ which is defined as follows:

$$\tilde{1}^{\circ} \overset{\text{def}}{=} ((- \ )^{K+1} F^{K-1})^{\circ}, \qquad \tilde{m}^{\circ} \overset{\text{def}}{=} (I - (- \ )^{K+1} F^{K-1})^{\circ}$$

or,

$$\tilde{1}^{\circ} \overset{\text{def}}{=} (- \ )^{K+1} F^{(K-1)^{\circ}} ,$$

$$\tilde{m}^{\circ} \overset{\text{def}}{=} I^{\circ} - (- \ )^{K+1} F^{(K-1)^{\circ}}$$

where I° is the identity operator in $\overset{\circ}{M}{}^n$. Now from (1 2), (3.3) and (3.8), it follows that in $F(K, \ - \ (- \ )^{K+1})$ structure manifold, we have:

$$\begin{aligned} \tilde{1}^{\circ} X^{\circ} &= (1)^{K+1} F^{(K-1)^{\circ}} X^{\circ} \\ &= ((- \ )^{K+1} F^{K-1} X)^{\circ} \\ &= (\tilde{1} X)^{\circ} . \end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned} \tilde{m}^{\circ} X^{\circ} &= X^{\circ} - (- \ )^{K+1} F^{(K-1)^{\circ}} X^{\circ} \\ &= (X - (- \ )^{K+1} F^{K-1} X)^{\circ} \\ &= (\tilde{m} X)^{\circ} \end{aligned}$$

which shows that $\tilde{l}$, $\tilde{m}$ preserves the structure

THEOREM (3.1). If $\tilde{L}$ and $\tilde{M}$ be the distributions corresponding to the projection operators $\tilde{l}$ and $\tilde{m}$ in $\overset{\circ}{M}^{n}$ then we have

$$N\ (X\ ,Y\ ) = \{N(\tilde{l}X,\ \tilde{l}Y) + N(\tilde{l}X,\ \tilde{m}Y) + N(\tilde{m}X,\ \tilde{l}Y) + N(\tilde{m}X,\ \tilde{m}Y)\} \tag{3 10}$$

$$N\ (X\ ,Y\ ) = \{\tilde{l}N(\tilde{l}X,\ \tilde{l}Y) + N(\tilde{l}X,\ \tilde{m}Y) + \tilde{l}N(\tilde{m}X,\ \tilde{m}Y) + \tilde{m}N(\tilde{l}X,\ \tilde{l}Y)$$
$$+ N(\tilde{m}X,\ \tilde{l}Y) + \tilde{m}N(\tilde{m}X,\ \tilde{m}Y)\} \tag{3 11}$$

PROOF. We have in consequence of (3 10)

$$N(\tilde{l}X,\ \tilde{l}Y) = [F\tilde{l}X, F\tilde{l}Y] - F[F\tilde{l}X,\ \tilde{l}Y] - F[\tilde{l}X,\ F\tilde{l}Y] + F^{2}[\tilde{l}X,\ \tilde{l}Y] \tag{3 12}$$

$$N(\tilde{l}X,\ \tilde{m}Y) = [F\tilde{l}X, F\tilde{m}Y] - F[F\tilde{l}X,\ \tilde{m}Y] - F[\tilde{l}X,\ F\tilde{m}Y] + F^{2}[\tilde{l}X,\ \tilde{m}Y] \tag{3 13}$$

$$N(\tilde{m}X,\ \tilde{l}Y) = [F\tilde{m}X, F\tilde{l}Y] - F[F\tilde{m}X,\ \tilde{l}Y] - F[\tilde{m}X,\ F\tilde{l}Y] + F^{2}[\tilde{m}X,\ \tilde{l}Y] \tag{3 14}$$

$$N(\tilde{m}X,\ \tilde{m}Y) = [F\tilde{m}X, F\tilde{m}Y] - F[F\tilde{m}X,\ \tilde{m}Y] - F[\tilde{m}X,\ F\tilde{m}Y] + F^{2}[\tilde{m}X,\ \tilde{m}Y] \tag{3 15}$$

Adding (3 12), (3 13), (3 14) and (3 15) we get

$$N(\tilde{l}X,\ \tilde{l}Y) + N(\tilde{l}X,\ \tilde{m}Y) + N(\tilde{m}X,\ \tilde{l}Y) + N(\tilde{m}X,\ \tilde{m}Y) = N(X,Y) \tag{3 16}$$

So in consequence of (3 7) we get

$$N^{\circ}(X^{\circ},Y^{\circ}) = \{N(\tilde{l}X,\ \tilde{l}Y) + N(\tilde{l}X,\ \tilde{m}Y) + N(\tilde{m}X,\ \tilde{l}Y)$$
$$+ N(\tilde{m}X,\ \tilde{m}Y)\}^{\circ} = \{N(X,Y)\}^{\circ}$$

This proves the first part of the theorem The proof of the second part follows from (1 2)

4. INTEGRABILITY CONDITIONS OF $F(K,\ -(-)^{K+1})$ STRUCTURE

If the distribution $\tilde{L}$ in M^n is integrable then $N(\tilde{l}X,\ \tilde{l}Y)$ is exactly the Nijenhuis tensor of $F^{*} = \frac{F}{L}$

THEOREM (4.1). For any two vector fields X and Y we have

(i) the distribution $\tilde{L}$ is integrable in M^n iff the distribution $\tilde{L}^{\circ}$ is integrable in $\overset{\circ}{M}^{n}$

(ii) the distribution $\tilde{M}$ is integrable in M^n iff the distribution $\tilde{M}^{\circ}$ is integrable in $\tilde{M}^{n}$

PROOF. We know that the distribution $\tilde{L}$ is integrable in M^n iff $\tilde{m}[\tilde{l}X,\ \tilde{l}Y] = 0$ and the distribution $\tilde{M}$ is integrable in M^n iff $\tilde{l}[\tilde{m}X,\ \tilde{m}Y] = 0$, for any two vector fields $X,\ Y \in \mathfrak{X}(M^n)$ Hence in view of (3.6) and (3 7) and by means of integrability conditions of $\tilde{L}$ and $\tilde{M}$ [4] we obtain the proof of the theorem (4 1) (i) and (ii).

THEOREM (4.2). The distribution $\tilde{L}$ and $\tilde{M}$ are both integrable in M^n iff $\tilde{L}^{\circ}$ and $\tilde{M}^{\circ}$ are integrable in $\overset{\circ}{M}^{n}$

PROOF. The proof follows directly with the help of (4 1) (i) and (ii) and (3 10)

THEOREM (4.3). If the distribution $\tilde{L}$ is integrable inM^n then the almost product structure defined by $F^{*} \overset{\text{def}}{=} \frac{F}{L}$ on each integral manifold of $\tilde{L}$ is integrable in M^n iff the almost product structure defined by $\overset{\circ}{F}^{*} \overset{\text{def}}{=} \frac{F^{\circ}}{\tilde{L}^{\circ}}$ on each integral manifold of $\tilde{L}^{\circ}$ is integrable in $\overset{\circ}{M}^{n}$ provided $\tilde{L}^{\circ}$ is integrable in $\overset{\circ}{M}^{n}$.

PROOF. We suppose that the distribution $\tilde{L}$ is integrable in M^n then F induces on each integral manifold of $\tilde{L}$ an almost product structure if F is $F(K,\ -(-)^{K+1})$ structure In both the cases the structure is integrable iff the Nijenhuis tensor of M^nvanishes i e, $N(\tilde{1}X, \tilde{1}Y) = 0$, or equivalently $\tilde{1}N(\tilde{1}X, \tilde{1}Y) = 0$ for any two vector fields X and Y

In view of (3 10) and $\tilde{1}\tilde{m} = \tilde{m}\tilde{1} = 0$ we get

$$\overset{\circ}{N}(\overset{\circ}{\tilde{1}}\overset{\circ}{X}, \overset{\circ}{\tilde{1}}\overset{\circ}{Y}) = \{N(\tilde{1}X, \tilde{1}Y)\}^{\circ} .$$

DEFINITION (4.1). We say that an $F(K, -(-)^{K+1})$ structure in M^n endowed with (1,1) tensor field F satisfying $F^K - (-)^{K+1}F = 0$ is p-partially integrable and the almost product structure $F^* \overset{\text{def}}{=} \frac{F}{L}$ is integrable

THEOREM (4.4). The $F(K, -(-)^{K+1})$ structure is p-partially integrable in M^n iff it is also p-partially integrable in $\overset{\circ}{M}{}^n$

PROOF. The proof follows in view of Def (4 1), Theorems (4 1) (i) and (4 3)

DEFINITION (4.2). We say that $F(K, -(-)^{K+1})$ structure to be partially integrable iff it is p-partially integrable and the distribution of $\tilde{M}$ is integrable

THEOREM (4.5). The structure $F(K, -(-)^{K+1})$ is partially integrable in M^n iff it is so in $\overset{\circ}{M}{}^n$

PROOF. The proof of the theorem follows from Definition (4 2) and Theorems (4.4) and (4.1) (i).

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