Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 555-562



# ON A CLASS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS HAVING $\sum_{k=0}^{\infty} x_k \delta^{(k)}(t)$ AND $\sum_{k=0}^{m} x_k \delta^{(k)}(t)$ AS SOLUTIONS

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(Received March 18, 1994 and in revised form Janaury 25, 1995)

ABSTRACT. We introduced some linear homogeneous ordinary differential equations which have both formal and finite distributional solutions at the same time, where the finite solution is a partial sum of the formal one. In the nonhomogeneous case and sometimes in the homogeneous case we found formal rational and rational solutions for such differential equations and similarly the rational solution is a partial sum of the formal one.

KEY WORDS AND PHRASES. Linear Ordinary Differential Equation,Distributional Solution, Rational Solution, Laplace Transform.
1992 AMS SUBJECT CLASSIFICATION CODE. 34A30.

## 1. INTRODUCTION.

It is well known that normal linear homogeneous systems of ordinary differential equations (ODE) with infinitely smooth coefficients have no generalized-function solutions other than the classical ones. However distributional solutions may appear in the case of equations whose coefficients have singularities. The number m is called the order of the distribution

$$x_m(t) = \sum_{k=0}^{m} x_m \, \delta^{(k)}(t) \quad , x_m \neq 0. \qquad ...(1.1)$$

Solutions may appear also in the form of infinite series

$$x(t) = \sum_{k=0}^{\infty} x_k \delta^{(k)}(t). \qquad ...(1.2)$$

Zeitlin[5] introduced a class of linear ODE having $\sum_{k=0}^{\infty} x_k \, t^k$ and $\sum_{k=0}^{m} x_k \, t^k$ as solutions. Wiener[2] proved a necessary and sufficient condition for the existence of an m-order distributional solution (1.1) for a linear ODE. Wiener and Cooke [3] discovered necessary and sufficient conditions for the simultaneous existence of solutions to linear ODE in the form of rational functions and finite distributional solutions(1.1). Along the lines of [3] Wiener et al . [4] mentioned necessary and sufficient conditions for the simultaneous existence of solutions to linear ODE in the form of formal rational and formal distributional(1.2) solutions.

By the help of these results and Laplace transformation properties we introduce a class of linear ODE having $\sum_{k=0}^{\infty} x_k\, \delta^{(k)}(t)$ and $\sum_{k=0}^{m} x_k\, \delta^{(k)}(t)$ as solutions.

2. SOME KNOWN RESULTS.

In [5] it had been proved that, if F(p) is a solution of

$$\sum_{i=0}^{n} A_i\, F^{(i)}(p) = 0 \qquad n = 1,2,\ldots \quad , \qquad \ldots(2.1)$$

where $A_i$ , i = 0,1,...., n are constants, with $A_0 \neq 0$ and $A_n = 1$, *then* $F(p) = \sum_{k=0}^{\infty} x_k p^k$ and its partial sum $F_m(p) = \sum_{k=0}^{m} x_k p^k$ are both solutions of

$$[\prod_{j=0}^{n-1} (pD-(m-j))]\,(\sum_{i=0}^{n} A_i F^{(i)}(p)) = 0, \qquad m \geq n \qquad \ldots(2.2)$$

where $D \equiv \dfrac{d}{dp}$ is the differential operator.

In [2] it had been proved that if the equation

$$\sum_{i=0}^{n} t^i q_i(t) x^{(i)}(t) = 0 \qquad \ldots(2.3)$$

with coefficients $q_i(t) \in C^m$ and $q_n(0) \neq 0$ has a solution (1.1) of order m, then

$$\sum_{i=0}^{n} (-1)^i q_i(0)\,(m+i)! = 0. \qquad \ldots(2.4)$$

Conversely, if m is the smallest nonnegative integer root of relation(2.4) there exists an m-order solution (1.1) concentrated at t=0.

3. Linear ODE having $\sum_{k=0}^{\infty} x_k \delta^{(k)}(t)$ and $\sum_{k=0}^{m} x_k \delta^{(k)}(t)$ as solutions.

It is well known that the ODE

$$\left(D^2+1\right) F = 0$$

has the solution

$$F(p) = C_1 \cos p + C_2 \sin p,$$

where $C_1$ and $C_2$ are arbitrary constants.

From Eq (2.2), the differential equation

$$(pD-(m-1))(pD-m)\left(D^2+1\right)F = 0 \qquad \ldots(2.9)$$

has the solution

$$F(p) = C_1 \cos p + C_2 \sin p,$$

or,in series form,

$$F(p) = C_1 \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k}}{(2k)!} + C_2 \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k+1}}{(2k+1)!}, \qquad \ldots(2.10)$$

and the solution

$$F_m(p)=C_1\sum_{k=0}^{[m/2]}\frac{(-1)^k p^{2k}}{(2k)!}+C_2\sum_{k=0}^{[(m-1)/2]}\frac{(-1)^k p^{2k+1}}{(2k+1)!}. \quad ...(2.11)$$

Along the lines of Eq.(2.9) and inverse Laplace transform properties we introduce the following theorem.

THEOREM 3.1. The equation

$$t^2\left(t^2+1\right)x''+t[2(\nu+3)t^2+2(\nu+1)]x'+[(\nu+2)(\nu+3)t^2+\nu(\nu+1)]x=0 \quad ...(3.1)$$

has an m-order distributional solution (1.1) if and only if

$$\nu=\text{m or } \nu=\text{m+1}. \quad ...(3.2)$$

(i) If $\nu$=m, then this solution is given by the formula

$$x_m(t)=C_1\sum_{k=0}^{[m/2]}\frac{(-1)^k\delta^{(2k)}(t)}{(2k)!}+C_2\sum_{k=0}^{[(m-1)/2]}\frac{(-1)^k\delta^{(2k+1)}(t)}{(2k+1)!}. \quad ...(3.3)$$

(ii) If $\nu$ = m+1, then

(a) for even m this solution is given by the formula

$$x_m(t)=C_1\sum_{k=0}^{m/2}\frac{(-1)^k\delta^{(2k)}(t)}{(2k)!}; \quad ...(3.3')$$

(b) for odd m this soluion is given by the formula

$$x_m(t)=C_2\sum_{k=0}^{(m-1)/2}\frac{(-1)^k\delta^{(2k+1)}(t)}{(2k+1)!}. \quad ...(3.3'')$$

Also the equation has the formal distributional solution

$$x(t)=C_1\sum_{k=0}^{\infty}\frac{(-1)^k\delta^{(2k)}(t)}{(2k)!}+C_2\sum_{k=0}^{\infty}\frac{(-1)^k\delta^{(2k+1)}(t)}{(2k+1)!}. \quad ...(3.4)$$

PROOF. In the case of Eq(3.1), formula (2.4) takes the form

$$(m+2)(m+1)-2(\nu+1)(m+1)+\nu(\nu+1)=0,$$

that is

$$(m-\nu)(m-\nu+1)=0,$$

implies that (3.2) is a necessary and sufficient condition for the existence of an m-order solution (1.1) to (3.1).

Substituting (1.1) in (3.1) and taking into account

$$t^j\delta^{(k)}(t)=\begin{cases}(-1)^j k!\delta^{(k-j)}(t)/(k-j)!, & k\ge j\\ 0 & k<j\end{cases}, \quad ...(3.5)$$

gives

$$\sum_{k=0}^{m-2}x_{k+2}(k+4)(k+3)(k+2)(k+1)\delta^{(k)}(t)+\sum_{k=0}^{m}x_k(k+2)(k+1)\delta^{(k)}(t)$$
$$-2(\nu+3)\sum_{k=0}^{m-2}x_{k+2}(k+3)(k+2)(k+1)\delta^{(k)}(t)-2(\nu+1)\sum_{k=0}^{m}x_k(k+1)\delta^{(k)}(t)$$
$$+(\nu+2)(\nu+3)\sum_{k=0}^{m-2}x_{k+2}(k+2)(k+1)\delta^{(k)}(t)+\nu(\nu+1)\sum_{k=0}^{m}x_k\delta^{(k)}(t)=0.$$

Therefore

$$[(m+2)(m+1)-2(\nu+1)(m+1)+\nu(\nu+1)]x_m=0, \quad ...(3.6)$$

$$[(m+1)m-2(\nu+1)m+\nu(\nu+1)]x_{m-1}=0, \quad ...(3.7)$$

$$\begin{aligned}&[(k+2)(k+1)-2(\nu+1)(k+1)+\nu(\nu+1)]x_k\\&=-(k+2)(k+1)[(k+4)(k+3)-2(\nu+3)(k+3)+(\nu+2)(\nu+3)]x_{k+2}\end{aligned} \quad ...(3.8)$$

(i) If $v = m$, we choose from (3.6) and (3.7) arbitrary $x_m$ and $x_{m-1}$ and successively find all $x_k$ $(k < m-1)$.
From (3.8) we find

$$x_k = -(k+2)(k+1)x_{k+2}, \qquad ...(3.9)$$

and multiplying these relations for k=m-2, m-4,...,m-2j, we have

$$x_{m-2j} = \frac{(-1)^j m!}{(m-2j)!} x_m. \qquad ...(3.10)$$

Again multiplying the relations in (3.9) for k=m-3, m-5,..., m-2j-1, we have

$$x_{m-2j-1} = \frac{(-1)^j (m-1)!}{(m-2j-1)!} x_{m-1}. \qquad ...(3.11)$$

Taking into account the following properties:

$$\sum_{k=0}^{[(m-1)/2]} \frac{(-1)^k \delta^{(2k+1)}(t)}{(2k+1)!} = \begin{cases} \displaystyle\sum_{k=0}^{(m-2)/2} \frac{(-1)^{(m-2)/2-k} \delta^{(m-2k-1)}(t)}{(m-2k-1)!}, & if\ m\ is\ even \\ \displaystyle\sum_{k=0}^{(m-1)/2} \frac{(-1)^{(m-1)/2-k} \delta^{(m-2k)}(t)}{(m-2k)!}, & if\ m\ is\ odd \end{cases}, \qquad ...(3.12)$$

$$\sum_{k=0}^{[m/2]} \frac{(-1)^k \delta^{(2k)}(t)}{(2k)!} = \begin{cases} \displaystyle\sum_{k=0}^{m/2} \frac{(-1)^{m/2-k} \delta^{(m-2k)}(t)}{(m-2k)!}, & if\ m\ is\ even \\ \displaystyle\sum_{k=0}^{(m-1)/2} \frac{(-1)^{(m-1)/2-k} \delta^{(m-2k-1)}(t)}{(m-2k-1)!}, & if\ m\ is\ odd \end{cases}. \qquad ...(3.13)$$

If m is even, choose

$$x_m = C_1(-1)^{m/2} / m! \qquad ...(3.14)$$

and

$$x_{m-1} = C_2(-1)^{(m-2)/2} / (m-1)!, \qquad ...(3.15)$$

to get

$$x_{m-2j} = C_1 \frac{(-1)^{m/2-j}}{(m-2j)!}, \quad j = 0,1,...,m/2 \qquad ...(3.16)$$

and

$$x_{m-2j-1} = C_2 \frac{(-1)^{(m-2)/2-j}}{(m-2j-1)!} \qquad j = 0,1,...,(m-2)/2. \qquad ...(3.17)$$

Puting $x_{m-2j}$ and $x_{m-2j-1}$ as given by (3.16) and (3.17) in (1.1) and keeping in mind m is even, we get $x_m(t)$ as given in formula (3.3) to be a solution of (3.1).
If m is odd, we choose in (3.10) and (3.11)

$$x_m = C_2(-1)^{(m-1)/2} / m!$$

and

$$x_{m-1} = C_1(-1)^{(m-1)/2} / (m-1)!,$$

to get

$$x_{m-2j} = \frac{C_2(-1)^{(m-1)/2-j}}{(m-2j)!}, \qquad j = 0,1,...,(m-1)/2 \qquad ...(3.18)$$

and

$$x_{m-2j-1}=\frac{C_1(-1)^{(m-1)/2-j}}{(m-2j-1)!},\qquad j=0,1,\ldots,(m-1)/2. \qquad ...(3.19)$$

Puting $x_{m-2j}$ and $x_{m-2j-1}$ as given by (3.18), and (3.19) in (1.1) and keeping in mind m is odd, we get $x_m(t)$ as given in formula (3.3) to be a solutions of (3.1).

(ii) If $v$ = m+1, from (3.6) we choose arbitrary $x_m$ , from(3.7) we find $x_{m-1}=0$ and successively find all $x_k(k<m-1)$.

Relations (3.9) and (3.10) still hold true, but

$$x_{m-2j-1}=0,\qquad j=0,1,\ldots,[(m-1)/2].$$

(a) If m is even, put $x_{m-2j}$ as given by (3.16) in (1.1), we get $x_m$(t) as given in formula(3.3') to be a solution of (3.1).

(b) If m is odd, put $x_{m-2j}$ as given by (3.18) in (1.1), we get $x_m(t)$ as given in formula(3.3") to be a solution of (3.1).

Substituting (1.2) in (3.1), taking into account formula (3.5), gives

$$\sum_{k=0}^{\infty}x_{k+2}(k+4)(k+3)(k+2)(k+1)\delta^{(k)}(t)+\sum_{k=0}^{\infty}x_k(k+2)(k+1)\delta^{(k)}(t)$$
$$-2(v+3)\sum_{k=0}^{\infty}x_{k+2}(k+3)(k+2)(k+1)\delta^{(k)}(t)-2(v+1)\sum_{k=0}^{\infty}x_k(k+1)\delta^{(k)}(t)$$
$$+(v+2)(v+3)\sum_{k=0}^{\infty}x_{k+2}(k+2)(k+1)\delta^{(k)}(t)+v(v+1)\sum_{k=0}^{\infty}x_k\delta^{(k)}(t)=0.$$

Therefore

$$x_k=-(k+2)(k+1)x_{k+2}\qquad k=0,1,2,\ldots \qquad ...(3.20)$$

multiplying these relations for k=0,2,...2(j-1),then

$$x_o=(-1)^{j}(2j)!\;x_{2j},$$

let $x_o=C_1$, then

$$x_{2j}=C_1(-1)^j/(2j)!. \qquad ...(3.21)$$

Again multiplying the relations in (3.20) for k=1,3,... 2j-1, then

$$x_1=(-1)^j(2j+1)!\,x_{2j+1},$$

let $x_1=C_2$ ,then

$$x_{2j+1}=C_2(-1)^j/(2j+1)!. \qquad ...(3.22)$$

Puting $x_{2j}$ and $x_{2j+1}$ as given by (3.21)and (3.22) in (1.2), we get $x(t)$ as given in formula (3.4) to be a solution of (3.1).

By using theorem 2.1 [3] and theorem 2.5 [4] there exist, constants $A_0,A_1,A_2$ and $A_3$such that:

(i) If $v$ = m , then the ODE

$$t^2\left(t^2+1\right)x''+t\left[2(m+3)t^2+2(m+1)\right]x'$$
$$+\left[(m+2)(m+3)t^2+m(m+1)\right]x=A_0+A_1t+A_2t^2+A_3t^2+A_3t^3 \qquad ...(3.23)$$

has the formal solution

$$x(t)=C_1\sum_{k=0}^{\infty}t^{-2k-1}+C_2\sum_{k=0}^{\infty}t^{-2k-2}, \qquad ...(3.24)$$

and the rational solution

$$x_m(t) = C_1 \sum_{k=0}^{[m/2]} t^{-2k-1} + C_2 \sum_{k=0}^{[(m-1)/2]} t^{-2k-2}. \qquad ...(3.25)$$

Substitutng (3.25) in (3.23), we find that

$$A_0 = m(m+1)C_2, A_1 = (m+1)(m+2)C_1 \text{ and } A_2 = A_3 = 0,$$

and it is easly seen that (3.24) is also a solution of (3.23) with the same constants we determined above.

(ii) If $v$ = m+1, consider the ODE

$$t^2(t^2+1)x'' + t[2(m+4)t^2 + 2(m+2)]x'$$
$$+[(m+3)(m+4)t^2 + (m+1)(m+2)]x = A_0 + A_1 t + A_2 t^2 + A_3 t^3. \quad ..(3.23')$$

(a) If m is even, then Eq.(3.23') has the formal solution (3.24) and the rational solution

$$x_m(t) = C_1 \sum_{k=0}^{m/2} t^{-2k-1} \qquad ..(3.25')$$

with $A_0 = A_2 = A_3 = 0$ and $A_1 = (m+2)(m+3)C_1$.

(b) If m is odd, then Eq.(3.23') has the formal solution (3.24) and the rational solution

$$x_m(t) = C_2 \sum_{k=0}^{(m-1)/2} t^{-2k-2} \qquad ..(3.25'')$$

with $A_1 = A_2 = A_3 = 0$ and $A_0 = (m+1)(m+2)C_2$.

Remark: By the same steps, which had been used to introduce theorem 3.1, we can introduce many other theorems in which we find solutions of the form (1.1) and (1.2) for some ODE. All what we need is to begin with an ODE of the form (2.1) which has a solution $F(p) = \sum_{k=0}^{\infty} x_k p^k$ , then by using Eq.(2.2) and the Laplace transformation properties we reach our purpose.

For example, consider the differential equation

$$F' + A F = 0, \qquad ...(3.26)$$

where $A \neq 0$ is a constant, which has the solution

$$F(p) = e^{-Ap} = \sum_{k=0}^{\infty} (-A)^k p^k / k!, \qquad ...(3.27)$$

then

$$(pD - m)(D + A)F = 0$$

has both F(p) given by(3.27) and its partial sum

$$F_m(p) = \sum_{k=0}^{m} (-A)^k p^k / k! \qquad ...(3.28)$$

as solutions.

From which we introduce the following theorem.

THEOREM 3.2. the equation

$$t(t-A)x' + [(v+2)t - A(v+1)]x = 0 \qquad ...(3.29)$$

has an m-order distributional solution(1.1) if and only if

$$v = m, \qquad ...(3.30)$$

this solution is given by the formula

$$x_m(t) = \sum_{k=0}^{m} (-A)^k \delta^{(k)}(t) / k! . \qquad ...(3.31)$$

Also it has the formal solution

$$x(t)=\sum_{k=0}^{\infty}(-A)^k\delta^{(k)}(t)/k! \quad . \qquad ...(3.32)$$

Proof comes by the same method used to prove theorem 3.1 or by direct application of the Laplace transform.

In [5] it was proved that for $R(p)$ analytic at $p$=0, with $R'(p)\neq 0$ , if $F(p)$ is a solution of

$$\left(D-R'(p)-a\frac{R'(p)}{R(p)}\right)F=0,$$

where $a$ is an arbitrary constant, then

$$F(P)=R^a(p)e^{R(p)} \ \ and\, F_m(p)=\sum_{k=0}^{m}R^{a+k}(p)/k!$$

are both solutions of the differential equation

$$\left(D-\frac{R''(p)}{R'(p)}-(m+a)\frac{R'(p)}{R(p)}\right)\left(D-R'(p)-a\frac{R'(p)}{R(p)}\right)F=0. \qquad ...(3.33)$$

Consider the case $R(p)=p^r$ where r is a positive integer one can easily find that

$$\left(D-rp^{r-1}-\frac{ar}{p}\right)F=0$$

has the solution $F(P)=p^{ar}e^{p^r}$.

Using equation (3.33) , we find that

$$\left(D-\frac{(r-1)}{p}-\frac{(m+a)r}{p}\right)\left(D-rp^{r-1}-\frac{ar}{p}\right)F=0 \qquad ...(3.34)$$

has the solutions

$$F(p)=p^{ar}e^{p^r}=\sum_{k=0}^{\infty}p^{(a+k)r}/k!, \qquad ...(3.35)$$

and

$$F_m(P)=\sum_{k=0}^{m}P^{(a+k)r}/k!. \qquad ...(3.36)$$

We can write eq (3.34) in the form

$$p^2F''-p\left[rp^r+r(m+2a+1)-1\right]F+\left[r^2p^r(m+a)+ar^2(m+a+1)\right]F=0. \qquad ...(3.37)$$

If $a$ is a nonnegative interger, applying the inverse Laplace transfrom to the latter equation with $L^{-1}\{F(p)\}=x(t)$,we get

$$rtx^{(r+1)}+r\left[r(m+a+1)+1\right]x^{(r)}+t^2x''+t\left[r(m+2a+1)+3\right]x'$$
$$+(ar+1)\left[r(m+a+1)+1\right]x=0. \qquad ...(3.38)$$

where we used $(tx)^{(r+1)}=tx^{(r+1)}+(r+1)x^{(r)}$ for r = 1,2,... ,
which will have the formal distributional solution

$$x(t)=\sum_{k=0}^{\infty}\delta^{((a+k)r)}(t)/k! \qquad ...(3.39)$$

and the finite distributional solution

$$x_m(t)=\sum_{k=0}^{m}\delta^{((a+k)r)}(t)/k!. \qquad ...(3.40)$$

Again by using theorem 2.1 [3] and theorem 2.5 [4] we find that Eq.(3.38) has the formal solution

$$x(t)=\sum_{k=0}^{\infty}(-1)^k t^{-(a+k)r-1} \qquad ...(3.41)$$

and the rational solution

$$x_m(t)=\sum_{k=0}^{m}(-1)^k t^{-(a+k)r-1}. \qquad ...(3.42)$$

As a special case consider $a$=0 and r = 1 in (3.38), to get

$$\left(t^2+t\right)x''+\left[(m+4)t+(m+2)\right]x'+(m+2)x=0, \qquad ...(3.43)$$

which has the formal distributional solution

$$x(t)=\sum_{k=0}^{\infty}\delta^{(k)}(t)/k!, \qquad ...(3.44)$$

and the finite distributional solution

$$x_m(t)=\sum_{k=0}^{m}\delta^{(k)}(t)/k!. \qquad ...(3.45)$$

Let

$$\varphi=\left(t^2+t\right)x'+\left[(m+2)t+(m+1)\right]x, \qquad ...(3.46)$$

then we can write (3.43) in the form $\varphi'=0$ , that is $\varphi$=C , C is an arbitrary constant, choosing C = 0 to get $\varphi$ = 0. In other words,

$$\left(t^2+t\right)x'+\left[(m+2)t+(m+1)\right]x=0 \qquad ...(3.47)$$

has the solutions (3.44) and (3.45).

It is clear that Eq.(3.29) is equivalent to Eq.(3.47) whenever $A$= -1 and formula (3.30) holds true.

Remark: In [3] it was shown that Eq.(3.47) has the solutions (3.44) and (3.45) and we found it as a special case of Eq.(3.29) and similarly of Eq.(3.38).

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