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MAPPING PROPERTIES FOR CONVOLUTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

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For $\mu \geq 0$, we consider a linear operator $L_\mu : A \to A$ defined by the convolution $f_\mu * f$, where $f_\mu = (1-\mu)z{}_2F_1(a,b,c;z) + \mu z(z{}_2F_1(a,b,c;z))'$. Let $\varphi^*(A,B)$ denote the class of normalized functions f which are analytic in the open unit disk and satisfy the condition $zf'/f \prec (1+Az)/1+Bz$, $-1 \leq A < B \leq 1$, and let $R_\eta(\beta)$ denote the class of normalized analytic functions f for which there exits a number $\eta \in (-\pi/2, \pi/2)$ such that $\operatorname{Re}(e^{i\eta}(f'(z)-\beta)) > 0$, $(\beta < 1)$. The main object of this paper is to establish the connection between $R_\eta(\beta)$ and $\varphi^*(A,B)$ involving the operator $L_\mu(f)$. Furthermore, we treat the convolution $I = \int_0^z (f_\mu(t)/t)\,dt * f(z)$ for $f \in R_\eta(\beta)$.

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1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and S denotes the subclass of functions in A which are univalent in U. Moreover, let $S^*(\alpha)$ and $K(\alpha)$ be the subclasses of S consisting, respectively, of functions which are starlike of order α and convex of order α, where $0 \leq \alpha < 1$ in U. Clearly, we have $S^*(\alpha) \subseteq S^*(0) = S^*$, where S^* denotes the class of functions in A which are starlike in U and $K(\alpha) \subseteq K(0) = K$, where K denotes the class of functions in A which are convex in U, and we mention the well-known inclusion chain $K \subset S^*(1/2) \subset S^* \subset S$. For the analytic functions g and h on U with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w on U such that $w(0) = 0$, $|w(z)| < 1$, and $g(z) = h(w(z))$ for $z \in U$. We denote this subordinated relation by

$$g \prec h \qquad \text{or} \qquad g(z) \prec h(z) \quad (z \in U). \tag{1.2}$$

For $-1 \leq A < B \leq 1$, a function p, which is analytic in U with $p(0) = 1$, is said to belong to the class $P(A,B)$ if

$$p(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U). \tag{1.3}$$

The above condition means that p takes the values in the disk with a center $(1-AB)/(1-B^2)$ and a radius $|A-B|/(1-B^2)$. The boundary circle cuts the real axis at the points $(1+A)/(1+B)$ and $(1-A)/(1-B)$. A function $f \in A$ is said to be in $\varphi^*(A,B)$ if $zf'/f \in P(A,B)$, and in $K(A,B)$ if $zf' \in \varphi^*(A,B)$. The class $\varphi^*(A,B)$ was introduced by N. Shukla and P. Shukla [4]. Also, Janowski [2] introduced the class $P(A,B)$. For the fixed natural number n, the subclass $P_n(A,B)$ of $P(A,B)$ containing functions p of the form $p(z) = 1+p_nz^n+\cdots$, $z \in U$, was defined by Stankiewicz and Waniurski [7]. In addition, Stankiewicz and Trojnar-Spelina [6] investigated a function $p(z) = 1-p_nz^n-\cdots$ belongs to the class $R(n,A,B)$, where $A \in R$ and $B \in [0,1]$ if $p(z) \prec (1+Az)/(1-Bz)$. Let $R_\eta(\beta)$ denote the class of functions $f \in A$ for which there exists a number $\eta \in (-\pi/2,\pi/2)$ such that

$$\operatorname{Re}\left[e^{i\eta}(f'(z)-\beta)\right] > 0 \quad (z \in U,\ \beta < 1). \tag{1.4}$$

Clearly, we have $R_\eta(\beta) \subset S$ $(0 \le \beta < 1)$. Furthermore, if a function f of the form (1.1) belongs to the class $R_\eta(\beta)$, then

$$|a_n| \le \frac{2(1-\beta)\cos\eta}{n} \quad (n \in N\backslash\{1\}). \tag{1.5}$$

The class $R_\eta(\beta)$ was studied by Kanas and Srivastava [3].

The hypergeometric function ${}_2F_1(a,b,c;z)$ is given as a power series, converging in U, in the following way

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \tag{1.6}$$

where a, b, and c are complex numbers with $c \neq 0,-1,-2,\ldots$, and $(\lambda)_n$ denotes the Pochhammer symbol (or the generalized factorial since $(1)_n = n!$) defined, in terms of the Gamma function Γ, by

$$\begin{aligned}(\lambda)_n &:= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n \in N = \{1,2,\ldots\}. \end{cases}\end{aligned} \tag{1.7}$$

Note that ${}_2F_1(a,b,c;z)$, for $a = c$ and $b = 1$ (or, alternatively, for $a = 1$ and $b = c$), reduces to the relatively more familiar geometric function. We also note that ${}_2F_1(a,b,c;1)$ converges for $\operatorname{Re}(c-a-b)>0$ and is related to the Gamma functions by

$$ {}_2F_1(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.8}$$

The Hadamard product (or convolution) of two power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ is defined as the power series

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n. \tag{1.9}$$

N. Shukla and P. Shukla [4] studied the mapping properties of a function f_μ to be as given in

$$f_\mu(z)=(1-\mu)z{}_2F_1(a,b,c;z)+\mu z\left(z{}_2F_1(a,b,c;z)\right)' \quad (\mu\geq 0), \tag{1.10}$$

and investigated the geometric properties of an integral operator of the form

$$I(z)=\int_0^z\frac{f_\mu(t)}{t}dt. \tag{1.11}$$

We now consider a linear operator $L_\mu : A\to A$ defined by

$$L_\mu(f)=f_\mu(z)*f(z). \tag{1.12}$$

For $\mu=0$ in (1.12), $L_\mu(f)=[I_{a,b,c}(f)](z)$, which was introduced by Hohlov [1]. Also, Kanas and Srivastava [3], and Srivastava and Owa [5] showed that the operator $I_{a,b,c}(f)$ is the natural extensions of the Alexander, Libera, Bernardi, and Carlson-Shaffer operators. In this paper, we find a relation between $R_\eta(\beta)$ and $\varphi^*(A,B)$ involving the operator $L_\mu(f)$. Furthermore, we study to obtain some conditions for the starlikeness and convexity of the convolution of I and f, which are given by (1.11) and (1.1), respectively, for $f\in R_\eta(\beta)$.

2. Main results.

We make use of the following lemma.

Lemma 2.1 [4]. *Sufficient conditions for f of the form (1.1) to be in $\varphi^*(A,B)$ and $K(A,B)$ are*

$$\begin{aligned}\sum_{n=2}^{\infty}\left[(1+B)n-(A+1)\right]\left|a_n\right| &\leq B-A,\\ \sum_{n=2}^{\infty}n\left[(1+B)n-(A+1)\right]\left|a_n\right| &\leq B-A,\end{aligned} \tag{2.1}$$

respectively.

Theorem 2.2. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1+B)\left(1+\frac{\mu ab}{c-a-b-1}\right)-(A+1)\left(\mu-\frac{(\mu-1)(c-a-b)}{(a-1)(b-1)}\right)\right] \le (B-A)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right)+\frac{(A+1)(\mu-1)(c-1)}{(a-1)(b-1)} \tag{2.2}$$

is satisfied, then $L_\mu(f) \in \varphi^*(A,B)$.

Proof. By Lemma 2.1, it suffices to show that

$$T_1 := \sum_{n=2}^{\infty}[(1+B)n-(A+1)]\left|\frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le B-A. \tag{2.3}$$

Since $f \in R_\eta(\beta)$ and $|a_n| \le 2(1-\beta)\cos\eta/n$. Hence,

$$\begin{aligned}
T_1 &\le \sum_{n=2}^{\infty}[(1+B)n-(A+1)]\frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{2(1-\beta)\cos\eta}{n}\\
&= 2(1-\beta)\cos\eta\Bigg\{(1+B)\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right)\\
&\qquad -\frac{(A+1)(c-1)}{(a-1)(b-1)}\left(\sum_{n=0}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n}-1-\frac{(a-1)(b-1)}{c-1}\right)\\
&\qquad +\frac{(1+B)\mu ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\
&\qquad -(A+1)\mu\Bigg[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\\
&\qquad\qquad -\frac{c-1}{(a-1)(b-1)}\left(\sum_{n=0}^{\infty}\frac{(a-1)_n(b-1)_n}{(c)_n(1)_n}\right.\\
&\qquad\qquad\qquad \left.-1-\frac{(a-1)(b-1)}{c-1}\right)\Bigg]\Bigg\}\\
&= 2(1-\beta)\cos\eta\Bigg\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[(1+B)\left(1+\frac{\mu ab}{c-a-b-1}\right)\\
&\qquad\qquad +(A+1)\left(\mu-\frac{(\mu-1)(c-a-b)}{(a-1)(b-1)}\right)\Bigg]\\
&\qquad -\left[1+B-(A+1)\left(1-\frac{(\mu-1)(c-1)}{(a-1)(b-1)}\right)\right]\Bigg\}.
\end{aligned} \tag{2.4}$$

Now, this last expression is bounded above by $B-A$ if (2.2) holds. □

If we take $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1-\frac{\alpha(c-a-b)}{(a-1)(b-1)}\right]&\\ \leq (1-\alpha)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right)-\frac{\alpha(c-1)}{(a-1)(b-1)}&\end{aligned} \tag{2.5}$$

is satisfied, then $z{}_2F_1(a,b,c;z)*f \in S^*(\alpha)$.

If we take $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in Corollary 2.3, we get the following corollary.

Corollary 2.4. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in S$, and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2} \tag{2.6}$$

is satisfied, then $z{}_2F_1(a,b,c;z)*f \in S^*$.

Theorem 2.5. *Let $a > 0$, $b > 0$, and $c > a+b+2$. If $f \in R_\eta(\beta)$, and the inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\bigg[B-A+\big((1+B)(1+2\mu)&\\ -(A+1)\mu\big)\frac{ab}{c-a-b-1}+\frac{(1+B)\mu(a)_2(b)_2}{(c-a-b-2)_2}\bigg]&\\ \leq (B-A)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right)&\end{aligned} \tag{2.7}$$

is satisfied, then $L_\mu(f) \in K(A,B)$.

Proof. The proof follows from Lemma 2.1. Using the method of the proof of Theorem 2.2, we omit the details involved. □

For $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$, Theorem 2.5 yields the following corollary.

Corollary 2.6. *Let $a>0$, $b>0$, and $c>a+b+2$. If $f\in R_\eta(\beta)$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1-\alpha+\frac{ab}{c-a-b-1}\right]\le(1-\alpha)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right) \tag{2.8}$$

is satisfied, then $z\,{}_2F_1(a,b,c;z)*f\in K(\alpha)$.

For $\alpha=0$, $\beta=0$, and $\eta=0$, Corollary 2.6 yields the following corollary.

Corollary 2.7. *Let $a>0$, $b>0$, and $c>a+b+1$. If $f\in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab}{c-a-b-1}\right]\le\frac{3}{2} \tag{2.9}$$

is satisfied, then $z\,{}_2F_1(a,b,c;z)*f\in K$.

In our next theorems, we find the sufficient conditions for $I*f$ to be in $\varphi^*(A,B)$ and $K(A,B)$. From the definition of I given by (1.11), we obtain

$$I(z)=z+\sum_{n=2}^{\infty}\frac{((1-\mu)+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n\quad(\mu\ge 0,\ z\in U). \tag{2.10}$$

Theorem 2.8. *Let $a>1$, $b>1$, and $c>a+b$. If $f\in R_\eta(\beta)$ and the inequality*

$$\begin{aligned}(1+B-(A+1)\mu)\,{}_2F_1(a,b,c;1)-(A+1)(1-\mu)\,{}_4F_3(a,b,1,1,c,2,2;1)\\ \le(B-A)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right)\end{aligned} \tag{2.11}$$

is satisfied, then $I*f\in\varphi^*(A,B)$.

Proof. By Lemma 2.1, it satisfies to show that

$$T_2:=\sum_{n=2}^{\infty}((1+B)n-(A+1))\left|\frac{(1-\mu+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}a_n\right|\le B-A. \tag{2.12}$$

Suppose that $f \in R_\eta(\beta)$. Then by (1.5) we observe that

$$\begin{aligned}
T_2 &\le \sum_{n=2}^{\infty}((1+B)n-(A+1))\frac{(1-\mu+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\frac{2(1-\beta)\cos\eta}{n} \\
&= 2(1-\beta)\cos\eta\bigg\{((1+B)(1-\mu)-(A+1)\mu)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \\
&\qquad -(A+1)(1-\mu)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n+1}} \\
&\qquad +(1+B)\mu\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\bigg\} \\
&= 2(1-\beta)\cos\eta\bigg\{((1+B)(1-\mu)-(A+1)\mu)\left(\frac{c-1}{(a-1)(b-1)}+{}_2F_1(a,b,c;1)\right) \\
&\qquad -(A+1)(1-\mu)\,{}_4F_3(a,b,1,1,c,2,2;1) \\
&\qquad +(1+B)\mu\,{}_2F_1(a,b,c;1) \\
&\qquad -\left[((1+B)(1-\mu)-(A+1)\mu)\frac{c-1}{(a-1)(b-1)}+B-A\right]\bigg\} \\
&\le B-A
\end{aligned} \tag{2.13}$$

by (2.11). This completes the proof. □

Taking $\mu = 0$, $A = 2\alpha-1$, and $B = 1$ in Theorem 2.8, we see the following corollary.

Corollary 2.9. *Let $a > 1$, $b > 1$, and $c > a+b$. If $f \in R_\eta(\beta)$ and the inequality*

$$ {}_2F_1(a,b,c;1)-\alpha\,{}_4F_3(a,b,1,1,c,2,2;1) \le (1-\alpha)\left(\frac{1}{2(1-\beta)\cos\eta}+1\right) \tag{2.14}$$

is satisfied, then $\int_0^z {}_2F_1(a,b,c;t)\,dt * f \in S^*(\alpha)$.

Taking $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in Corollary 2.9, we get the following corollary.

Corollary 2.10. *Let $a > 1$, $b > 1$, and $c > a+b$. If $f \in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \le \frac{3}{2} \tag{2.15}$$

is satisfied, then $\int_0^z {}_2F_1(a,b,c;t)\,dt * f \in S^*$.

Theorem 2.11. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[&(1+B)\Big(1+\frac{\mu ab}{c-a-b-1}\Big)\\&+(A+1)\Big(\mu\Big(\frac{c-a-b}{(a-1)(b-1)}-1\Big)-\frac{c-a-b}{(a-1)(b-1)}\Big)\Big]\\&\le (B-A)\Big(\frac{1}{2(1-\beta)\cos\eta}+1\Big)-\frac{(1-\mu)(A+1)(c-1)}{(a-1)(b-1)}\end{aligned} \tag{2.16}$$

is satisfied, then $I*f \in K(A,B)$.

Proof. The proof follows from Lemma 2.1 and by applying similar method as in the proof of Theorem 2.8; we omit the details involved. □

If we let $\mu = 0$, $A = 2\alpha-1$, and $B = 1$ in Theorem 2.11, we get the following corollary.

Corollary 2.12. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[1-\frac{\alpha(c-a-b)}{(a-1)(b-1)}\Big]&\\ \le (1-\alpha)\Big(\frac{1}{2(1-\beta)\cos\eta}+1\Big)-\frac{\alpha(c-1)}{(a-1)(b-1)}&\end{aligned} \tag{2.17}$$

is satisfied, then $\int_0^z {}_2F_1(a,b,c;t)\,dt * f \in K(\alpha)$.

If we let $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in Corollary 2.12, we have the following corollary.

Corollary 2.13. *Let $a > 1$, $b > 1$, and $c > a+b+1$. If $f \in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \le \frac{3}{2} \tag{2.18}$$

is satisfied, then $\int_0^z {}_2F_1(a,b,c;t)\,dt * f \in K$.

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References

[1] Ju. E. Hohlov, *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. **10 (197)** (1978), 83–89 (Russian).

[2] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math. **28** (1973), 297–326.

[3] S. Kanas and H. M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transform. Spec. Funct. **9** (2000), no. 2, 121–132.

[4] N. Shukla and P. Shukla, *Mapping properties of analytic function defined by hypergeometric function. II*, Soochow J. Math. **25** (1999), no. 1, 29–36.
[5] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing, New Jersey, 1992.
[6] J. Stankiewicz and L. Trojnar-Spelina, *Some parametric family of functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz. **14** (1992), 45–54.
[7] J. Stankiewicz and J. Waniurski, *Some classes of functions subordinate to linear transformation and their applications*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **28** (1974), 85–94.

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