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STATISTICALLY STRONGLY REGULAR MATRICES AND SOME CORE THEOREMS

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We define the statistically strongly regular matrices analogous to the strongly regular matrices, and further we use these matrices to establish necessary and sufficient conditions to prove some core theorems.

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1. Banach core. Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)_1^\infty$ (real or complex). Note that the functional

$$q(x) = \limsup_p \sup_n \frac{1}{p+1} \sum_{j=0}^{p} x_{n+j} \tag{1.1}$$

is a sublinear functional on ℓ_∞. In case $q(x) = -q(-x) = L$, x is called *almost convergent* to L [12], and we write $f - \lim x = L$. Let f denote the set of all almost convergent sequences. Note that

$$f := \left\{ x \in \ell_\infty : \lim_p \frac{1}{p+1} \sum_{j=0}^{p} x_{n+j} = L, \text{ uniformly in } n \right\}. \tag{1.2}$$

The *Knopp core* (or K-core) of a real bounded sequence x is defined to be the closed interval $[\liminf x, \limsup x]$ and analogously the *Banach core* (or B-core) is defined as $[-q(-x), q(x)]$.

Shcherbakov [16] has shown that for every complex bounded x,

$$\text{K-core}\{x\} = \bigcap_{z \in \mathbb{C}} K_x^*(z), \tag{1.3}$$

where

$$K_x^*(z) := \left\{ w \in \mathbb{C} : |w - z| \le \limsup_k |x_k - z| \right\}. \tag{1.4}$$

It is natural to extend this definition for B-core, that is, for every complex bounded x,

$$\text{B-core}\{x\} = \bigcap_{z \in \mathbb{C}} B_x^*(z), \tag{1.5}$$

where

$$B_x^*(z) := \Big\{ w \in \mathbb{C} : |w - z| \le \limsup_p \sup_n \,|\, t_{pn}(x) - z \,|\, \Big\},$$
$$t_{pn}(x) = \frac{1}{p+1} \sum_{j=0}^{p} x_{n+j}. \tag{1.6}$$

Note that $q(x) \le \limsup x$ for all $x \in \ell_\infty$. Hence,

$$\text{B-core}\{x\} \subseteq \text{K-core}\{x\}. \tag{1.7}$$

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex entries a_{nk}. By $Ax = (A_n(x))$, we denote the A-transform of the sequence $x = (x_k)_1^{\infty}$, where

$$A_n(x) = \sum_k a_{nk} x_k. \tag{1.8}$$

For any two sequence spaces X and Y, we denote by (X, Y) a class of matrices A such that $Ax \in Y$ for $x \in X$, provided that the series on the right of (1.8) converges for each n. If, in addition, $\lim Ax = \lim x$, then we denote such a class by $(X, Y; P)$ or $(X, Y)_{\text{reg}}$.

The matrix A is said to be *regular* if $A \in (c, c)_{\text{reg}}$, that is, $Ax \in c$ for $x \in c$ with $\lim Ax = \lim x$. The matrix A is said to be *strongly regular* if $Ax \in c$ with $\lim Ax = f - \lim x$ for each $x \in f$ and we write it as $A \in (f, c)_{\text{reg}}$. Well-known conditions for regularity and strong regularity can be found in [3] and [12], respectively.

2. Statistical core. The concept of statistical convergence was introduced by Fast [4] and further studied by Šalát [15], Fridy [6], Connor [2], Kolk [9, 10] and others.

Let $\mathbb{N}$ and $\mathbb{C}$ be the set of natural numbers and complex numbers, respectively. If $E \subseteq \mathbb{N}$, then the *natural density* of E (see Freedman and Sember [5]) is denoted by

$$\delta(E) := \lim_n \frac{1}{n} \,|\, \{k \le n : k \in E\} \,|\,, \tag{2.1}$$

where the vertical bars denote the cardinality of the enclosed set. The sequence x is said to be *statistically convergent* to L, denoted by st-$\lim x = L$, if for every $\varepsilon > 0$, the set

$$\{k : |\, x_k - L \,| \ge \varepsilon\} \tag{2.2}$$

has natural density zero. We will denote the set of all statistically convergent sequences by st.

The real number sequence x is said to be *statistically bounded* if there is a number B such that

$$\delta\{k : |x_k| > B\} = 0. \tag{2.3}$$

If x is a statistically bounded sequence, then the *statistical core* of x is the closed interval [st-$\liminf x$,st-$\limsup x$], (see [8]). It is noted that

$$\liminf x \le \text{st-}\liminf x \le \text{st-}\limsup x \le \limsup x \tag{2.4}$$

and consequently

$$\text{st-core}\{x\} \subseteq \text{K-core}\{x\}. \tag{2.5}$$

Fridy and Orhan [7] established, for a statistically bounded complex sequence x,

$$\text{st-core}\{x\} = \bigcap_{z\in\mathbb{C}} S_x^*(z), \tag{2.6}$$

where

$$S_x^*(z) := \left\{w \in \mathbb{C} : |w-z| \le \text{st-}\limsup_k |x_k - z|\right\}. \tag{2.7}$$

The famous Knopp's core theorem states that (see [3, 13]), in order that $L(Ax) \le L(x)$ for every $x \in \ell_\infty$, it is necessary and sufficient that A should be regular and $\lim_n \sum_k |a_{nk}| = 1$, where $L(x) = \limsup x$.

In [7], the authors have obtained the necessary and sufficient conditions for the inclusion

$$\text{K-core}\{Ax\} \subseteq \text{st-core}\{x\}. \tag{2.8}$$

In [11], Li and Fridy have obtained the necessary and sufficient conditions for the inclusion

$$\text{st-core}\{Ax\} \subseteq \text{K-core}\{x\}. \tag{2.9}$$

In this paper, we establish the inequality

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{x\} \tag{2.10}$$

and further, for a normal matrix T,

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{Tx\}. \tag{2.11}$$

Remark 2.1. Almost convergence and statistical convergence are incomparable. For example, the sequence $x = (x_k)$ with

$$x_k = \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even,} \end{cases} \tag{2.12}$$

is almost convergent to $1/2$ but not statistically convergent.

In [14], an example of a sequence was given, which is statistically convergent but not almost convergent.

3. Main results.

First, we note here some useful results which will be used in establishing our main theorems (see [9, 10]).

Lemma 3.1. *A sequence $x = (x_k)$ is statistically convergent to ℓ if and only if there is an index set $K = \{k_i\}$ such that $\delta(K) = 1$ and $\lim_i x_{k_i} = \ell$.*

Lemma 3.2. *Let X be a sequence space. Then $A \in (X, \text{st})$ if and only if for every $x \in X$, there exists an index set $N = \{n_i\}$ such that $\delta(N) = 1$ and $A^{[N]}x \in c$, where $A^{[N]} = (d_{ik})$ for all $k \in \mathbb{N}$ and $d_{ik} = a_{n_ik}$ for all $i = 1, 2, \ldots$.*

Lemma 3.3. *The necessary and sufficient conditions for a matrix $A \in (c, \text{st} \cap \ell_\infty)_{\text{reg}}$ are:*

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$ *and there exists $N = \{n_i\}$ such that $\delta(N) = 1$,*

(ii) $\lim_i a_{n_ik} = 0$ $(k \in \mathbb{N})$,

(iii) $\lim_i \sum_k a_{n_ik} = 1$.

We call such matrices statistically regular matrices.

Similarly, we define the following.

Definition 3.4. An infinite matrix $A = (a_{nk})$ is said to be *statistically strongly regular* if $Ax \in \text{st} \cap \ell_\infty$ for all $x \in f$, with $f - \lim x = \text{st-}\lim Ax$, that is, $A \in (f, \text{st} \cap \ell_\infty)_{\text{reg}}$.

By Lemma 3.3 and strong regularity, we can easily have the following theorem.

Theorem 3.5. *The necessary and sufficient conditions for a matrix $A \in (f, \text{st} \cap \ell_\infty)_{\text{reg}}$ are:*

(a) *A is statistically regular, that is, $A \in (c, \text{st} \cap \ell_\infty)_{\text{reg}}$,*

(b) *there exists an index set $N = \{n_i\}$ such that $\delta(N) = 1$ and*

$$\lim_i \sum_k \left| a_{n_ik} - a_{n_i,k+1} \right| = 0. \tag{3.1}$$

Theorem 3.6. *If $\|A\| < \infty$, then for every $x \in \ell_\infty$,*

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{x\} \tag{3.2}$$

if and only if

$$A \in (f, \mathrm{st} \cap \ell_\infty)_{\mathrm{reg}}, \tag{3.3}$$

$$\text{st-}\lim_n \sum_{k \in E} |a_{nk}| = 1, \quad \textit{whenever } \mathbb{N}\backslash E \textit{ is finite}, \quad \textit{for } E \subseteq \mathbb{N}. \tag{3.4}$$

Proof

Necessity. Let (3.2) hold and let x be almost convergent to ℓ. Then,

$$\{\ell\} = \text{B-core}\{x\} \supseteq \text{st-core}\{Ax\}. \tag{3.5}$$

Since $\|A\| < \infty$ implies $Ax \in \ell_\infty$ for $x \in \ell_\infty$, Ax has at least one statistical cluster point and therefore the set of statistical cluster points is in st-core$\{Ax\}$ (see [11]). Therefore, st-core$\{Ax\} \neq \varnothing$, and so st-core$\{Ax\} = \{\ell\}$ so that st-lim Ax $= f - \lim x = \ell$. Hence, $A \in (f, \mathrm{st} \cap \ell_\infty)_{\mathrm{reg}}$, that is, condition (3.3) holds.

To prove (3.4), define $x = (x_k) \in \ell_\infty$ by

$$x_k = \begin{cases} 1, & \text{if } k \in E, \\ 0, & \text{otherwise}, \end{cases} \tag{3.6}$$

where $E \subseteq \mathbb{N}$ such that $\mathbb{N}\backslash E$ is finite. Then

$$\text{B-core}\{x\} = \{1\}. \tag{3.7}$$

Since $Ax \in \ell_\infty$, Ax has at least a statistical cluster point. Therefore, by [11, Proposition 4], st-core$\{Ax\} \neq \varnothing$. Since

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{x\} = \{1\}, \tag{3.8}$$

we have st-core$\{Ax\} = \{1\}$ and 1 is the only statistical cluster point of Ax. Hence,

$$\text{st-}\lim Ax = 1. \tag{3.9}$$

Therefore, by [17, Corollary 12], we get

$$\text{st-}\lim_n \sum_{k \in E} |a_{nk}| = 1, \quad \text{whenever } \mathbb{N}\backslash E \text{ is finite}. \tag{3.10}$$

Therefore, (3.4) holds.

Sufficiency. Suppose that conditions (3.3) and (3.4) hold and $w \in$ st-core$\{Ax\}$. Then for any $z \in \mathbb{C}$, we have

$$\begin{aligned} |w-z| &\le \text{st-}\limsup_n |z-A_n(x)| = \text{st-}\limsup_n \left| z - \sum_k a_{nk}x_k \right| \\ &\le \text{st-}\limsup_n \left| \sum_k a_{nk}(z-x_k) \right| + \text{st-}\limsup_n |z| \left| 1-\sum_k a_{nk} \right| \\ &= \text{st-}\limsup_n \left| \sum_k a_{nk}(z-x_k) \right|, \quad \text{by (3.3)}. \end{aligned} \tag{3.11}$$

Therefore, for an index set $N = \{n_i\}$ such that $\delta(N) = 1$,

$$|w-z| \le \limsup_i \left| \sum_k a_{n_ik}(z-x_k) \right|. \tag{3.12}$$

Therefore, by using (3.3), we obtain

$$\limsup_i \sum_k a_{n_ik}(z-x_k) = \limsup_i \sum_k a_{n_ik}(z-t_{pk}(x)). \tag{3.13}$$

Now, let $r = \limsup_p \sup_k |t_{pk}(x)-z|$ and $E = \{k : |t_{pk}(x)-z| > r+\varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ as E is finite. Therefore,

$$\begin{aligned} \left| \sum_k a_{n_ik}(z-x_k) \right| &= \left| \sum_k a_{n_ik}(z-t_{pk}(x)) \right| \\ &\le \sup_k |z-t_{pk}(x)| \left| \sum_{k\in E} a_{n_ik} \right| + (r+\varepsilon) \left| \sum_{k\notin E} a_{n_ik} \right|. \end{aligned} \tag{3.14}$$

From (3.3) and (3.4), we get

$$\limsup_i \left| \sum_k a_{n_ik}(z-x_k) \right| \le r+\varepsilon \tag{3.15}$$

and so by (3.12), we have

$$|w-z| \le r+\varepsilon. \tag{3.16}$$

Since ε is arbitrary,

$$|w-z| \le r = \limsup_p \sup_k |t_{pk}(x)-z|, \tag{3.17}$$

that is, $w \in B_x^*(z)$. Hence, $w \in$ B-core$\{x\}$ so that

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{x\}. \tag{3.18}$$

This completes the proof of the theorem. □

Remark 3.7. In the following example, we see that condition (3.4) cannot be replaced by

$$\text{st-}\lim_n \sum_{k\in E} |a_{nk}| = 1, \quad \text{for any set } E \subseteq \mathbb{N} \text{ such that } \delta(E) = 1. \tag{3.19}$$

Example 3.8. Let $A = (a_{nk})$ be defined as

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a nonsquare and } k = n^2 \text{ or } n^2+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

Then

$$\sum_k a_{nk} = \begin{cases} 1, & \text{if } n \text{ is a nonsquare,} \\ 0, & \text{otherwise,} \end{cases} \tag{3.21}$$

and $A \in (f, \text{st} \cap \ell_\infty)_{\text{reg}}$. Further, for any set $E \subseteq \mathbb{N}$ such that $\mathbb{N}\backslash E$ is finite, we have

$$\text{st-}\lim_n \sum_{k\in E} |a_{nk}| = 1. \tag{3.22}$$

Then, for any bounded sequence x, we have

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{x\}. \tag{3.23}$$

Now, let $E = \{k \in \mathbb{N} : k \neq n^2 \text{ and } k \neq n^2+1\}$. Then $\delta(E) = 1$ and we have

$$\sum_{k\in E} |a_{nk}| = 0, \quad \forall n. \tag{3.24}$$

Hence,

$$\text{st-}\lim_n \sum_{k\in E} |a_{nk}| = 0. \tag{3.25}$$

Further, for any bounded sequence, say $x = (1,0,1,0,\dots)$, we have B-core$\{x\}$ $= \{1/2\}$ and

$$\sum_k a_{nk}x_k = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a nonsquare,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.26}$$

Therefore, st-lim $Ax = 1/2$,

$$\text{st-core}\{Ax\} = \left\{\frac{1}{2}\right\} = \text{B-core}\{x\}, \tag{3.27}$$

but (3.19) does not hold.

The following theorem is an analogue of [7, Theorem 2].

Theorem 3.9. *Let $T = (t_{jk})$ be a normal matrix (i.e., triangular with nonzero diagonal entries) and denote its triangular inverse by $T^{-1} = (t^{-1}_{jk})$. Let $A = (a_{nj})$ be any matrix. In order that whenever Tx is bounded, Ax should exist, be bounded, and satisfy*

$$\text{st-core}\{Ax\} \subseteq \text{B-core}\{Tx\}, \tag{3.28}$$

it is necessary and sufficient that

(1) $(c_{nk}) = C = AT^{-1}$ *exists,*
(2) $C \in (f, \text{st} \cap \ell_\infty)_{\text{reg}}$,
(3) $\text{st-lim}_n \sum_{k\in E} |c_{nk}| = 1$ *whenever* $\mathbb{N}\backslash E$ *is finite,*
(4) *for any fixed* n,

$$\sum_{k=0}^{m} \left| \sum_{j=m+1}^{\infty} a_{nj} t^{-1}_{jk} \right| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{3.29}$$

The proof follows easily by using [1, Theorem 3.2 and Lemma 2].

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