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POISSON STRUCTURES ON COTANGENT BUNDLES

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We make a study of Poisson structures of T^*M which are graded structures when restricted to the fiberwise polynomial algebra and we give examples. A class of more general graded bivector fields which induce a given Poisson structure w on the base manifold M is constructed. In particular, the *horizontal lifting* of a Poisson structure from M to T^*M via connections gives such bivector fields and we discuss the conditions for these lifts to be Poisson bivector fields and their compatibility with the canonical Poisson structure on T^*M. Finally, for a 2-form ω on a Riemannian manifold, we study the conditions for some associated 2-forms of ω on T^*M to define Poisson structures on cotangent bundles.

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1. Introduction. In this paper, we present the dual version of the subject discussed in [4] and study graded bivector fields and Poisson structures on the cotangent bundle of a manifold. Although this study is similar to the one in [4], it is motivated by the presence of specific aspects. Indeed, we do not have a natural almost tangent structure and semisprays anymore, but we have the canonical symplectic structure instead. This makes a separate exposition required. Another new aspect that we discuss is that of a base manifold which is a Riemannian space.

2. Graded Poisson structures on cotangent bundles. Let M be an n-dimensional differentiable manifold and $\pi : T^*M \to M$ its cotangent bundle. If (x^i) $(i = 1,\dots,n)$ are local coordinates on M, we denote by (p_i) the covector coordinates with respect to the cobasis (dx^i). (We assume that everything is C^∞ in this paper.)

In this section, we discuss *graded* Poisson structures W on the cotangent bundle T^*M obtained as *lifts* of Poisson structures w on the base manifold M, in the sense that the canonical projection π is a Poisson mapping (see [4]).

Denote by $S_k(TM)$ the space of k-contravariant symmetric tensor fields on M and by $\odot$ the symmetric tensor product on the algebra $S(TM) = \bigoplus_{k\geq 0} S_k(TM)$. The spaces of fiberwise homogeneous k-polynomials

$$\mathcal{HP}_k(T^*M) := \Big\{\tilde{Q} = Q^{i_1\cdots i_k} p_{i_1}\cdots p_{i_k} \mid Q = Q^{i_1\cdots i_k}\frac{\partial}{\partial x^{i_1}}\odot\cdots\odot\frac{\partial}{\partial x^{i_k}} \in S_k(TM)\Big\} \tag{2.1}$$

are interesting subspaces of the function space $C^\infty(T^*M)$ and play an important role in this paper.

The map

$$\sim : (S(TM), \odot) \longrightarrow (\mathcal{P}(T^*M), \cdot), \qquad \sim Q := \tilde{Q}, \tag{2.2}$$

where $\mathcal{P}(T^*M) := \oplus_k \mathcal{HP}_k(T^*M)$ is the *polynomial algebra* and the dot denotes the usual multiplication, is an isomorphism of algebras.

On T^*M we also have the spaces of (fiberwise) nonhomogeneous polynomials of degree less than or equal to k

$$\mathcal{P}_k(T^*M) := \bigoplus_{h=0}^{k} \mathcal{HP}_h. \tag{2.3}$$

For $k = 1$, $\mathcal{A}(T^*M) := \mathcal{P}_1(T^*M)$ is the space of *affine functions*, having the elements of the form

$$a(x,p) = f(x) + m(X), \tag{2.4}$$

where $f \in C^\infty(M)$, $X \in \chi(M)$ (the space of vector fields on M), and $m(X) :=\sim X$ is the *momentum* of X. (The momentum $m(X)$ is X regarded as a function on T^*M.)

The elements of the space $\mathcal{P}_2(T^*M)$ of nonhomogeneous quadratic polynomials are

$$t(x,p) = f(x) + m(X) + s(Q), \tag{2.5}$$

where $Q = Q^{ij}(\partial/\partial x^i) \odot (\partial/\partial x^j)$ is a symmetric contravariant tensor field on M and $s(Q) :=\sim Q$.

Hereafter, by a polynomial on T^*M, we always mean a fiberwise polynomial. Also, we write f for both f on M and $f \circ \pi$ on T^*M.

Definition 2.1. A Poisson structure W on T^*M is called *polynomially graded* if for all $Q, R \in \mathcal{P}(T^*M)$,

$$Q \in \mathcal{P}_h,\ R \in \mathcal{P}_k \Longrightarrow \{Q,R\}_W \in \mathcal{P}_{h+k}. \tag{2.6}$$

Proposition 2.2. *A polynomially graded Poisson structure W on T^*M induces a Poisson structure w on the base manifold M such that the projection $\pi : (T^*M, W) \to (M, w)$ is a Poisson mapping.*

Proof. Any function f on M is a polynomial $(f \circ \pi) \in \mathcal{P}_0(T^*M)$. By (2.6), for all $f, g \in C^\infty(M)$, $\{f \circ \pi, g \circ \pi\}_W \in C^\infty(M)$ and

$$\{f,g\}_w := \{f \circ \pi, g \circ \pi\}_W \tag{2.7}$$

defines a Poisson structure w on M. □

Hereafter, the bracket $\{\cdot,\cdot\}_W$ will be denoted simply by $\{\cdot,\cdot\}$.

If the local coordinate expression of the Poisson structure w introduced by Proposition 2.2 is

$$w = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \tag{2.8}$$

Definition 2.1 tells us that W must have the local coordinate expression

$$\begin{aligned} W = &\frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \left(\varphi^i_j(x) + p_a A^{ia}_j(x)\right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} \\ &+ \frac{1}{2}\left(\eta_{ij}(x) + p_a B^a_{ij}(x) + p_a p_b C^{ab}_{ij}(x)\right) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}, \end{aligned} \tag{2.9}$$

where w, φ, η, A, B, and C are local functions on M.

The Poisson structure W is completely determined by the brackets $\{f,g\}$, $\{m(X),f\}$, and $\{m(X),m(Y)\}$, where $f,g \in C^\infty(M)$ and $X,Y \in \chi(M)$ since the local coordinates x^i and p_i are functions of this type ($p_i = m(\partial/\partial x^i)$).

By (2.6), the bracket $\{m(X),f\}$ is in $\mathcal{P}_1(T^*M)$, that is,

$$\{m(X),f\} = Z_X f + m(\gamma_X f), \tag{2.10}$$

where $Z_X f \in C^\infty(M)$ and $\gamma_X f \in \chi(M)$.

The map $\{m(X),\cdot\}$ is a derivation of $C^\infty(M)$. Hence, Z_X is a vector field on M and the mapping $\gamma_X : C^\infty(M) \to \chi(M)$ also is a derivation. Therefore, $\gamma_X f$ depends only on df.

From the Leibniz rule, we get that $Z_{hX} = hZ_X$ ($h \in C^\infty(M)$) and γ must satisfy

$$\gamma_{hX} f = h\gamma_X f + (X^w_h f)X. \tag{2.11}$$

The bracket of two affine functions has an expression of the form

$$\{m(X),m(Y)\} = \beta(X,Y) + m(V(X,Y)) + s(\Psi(X,Y)), \tag{2.12}$$

where $\beta(X,Y) \in C^\infty(M)$, $V(X,Y) \in \chi(M)$, and $\Psi(X,Y) \in S_2(TM)$ are skew-symmetric operators. If we replace Y by fY in (2.12), the Leibniz rule gives that β is a 2-form on M and

$$\begin{aligned} V(X,fY) &= fV(X,Y) + (Z_X f)Y, \\ \Psi(X,fY) &= f\Psi(X,Y) + (\gamma_X f) \odot Y. \end{aligned} \tag{2.13}$$

Definition 2.3. A polynomially graded Poisson structure W on T^*M is said to be a *graded structure* if for all $Q \in \mathcal{HP}_h$ and for all $R \in \mathcal{HP}_k$, it follows $\{Q,R\}_W \in \mathcal{HP}_{h+k}$.

Remark that a polynomially graded structure on T^*M is graded if and only if $Z_X = 0$, $\beta = 0$, and $V = 0$. In this case, (2.9) reduces to

$$W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + p_a A_j^{ia}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} + \frac{1}{2} p_a p_b C_{ij}^{ab}(x) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}. \tag{2.14}$$

As in [4], a bivector field W on T^*M which is locally of the form (2.9) (resp., (2.14)) is called a *polynomially graded* (resp., *graded*) *bivector field*.

Proposition 2.4. *If W is a graded bivector field on T^*M which is π-related with a Poisson structure w on M, there exists a contravariant connection D on the Poisson manifold (M, w) such that*

$$\{m(X), f\} = -m(D_{df}X), \quad X \in \chi(M),\ f \in C^\infty(M). \tag{2.15}$$

Moreover, if W is a graded Poisson structure on T^*M, then the connection D is flat.

Proof. A contravariant connection on (M, w) is a contravariant derivative on TM with respect to the Poisson structure [8].

The required connection is defined by

$$D_{df}X := -\gamma_X f. \tag{2.16}$$

That we really get a connection, which is flat in the Poisson case, follows in exactly the same way as in [4]. □

The relation (2.15) extends to the following proposition.

Proposition 2.5. *If Q is a symmetric contravariant tensor field on M and $\tilde{Q}$ is its corresponding polynomial, then for any graded Poisson bivector field W on T^*M,*

$$\{\tilde{Q}, f\}_W = -\widetilde{D_{df}Q}. \tag{2.17}$$

Proof. The contravariant connection D_{df} of (2.17) is extended to $S(TM)$ by

$$\begin{aligned}(D_{df}Q)(\alpha_1, \dots, \alpha_k) = {} & X_f^w(Q(\alpha_1, \dots, \alpha_k)) \\ & - \sum_{i=1}^{k} Q(\alpha_1, \dots, D_{df}\alpha_i, \dots, \alpha_k),\end{aligned} \tag{2.18}$$

where $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$, and $D_{df}\alpha$ is defined by

$$\langle D_{df}\alpha, X\rangle = X_f^w\langle \alpha, X\rangle - \langle \alpha, D_{df}X\rangle, \quad X \in \chi(M). \tag{2.19}$$

We put

$$D_{dx^i}\frac{\partial}{\partial x^j} = -\Gamma_j^{ik}\frac{\partial}{\partial x^k}, \tag{2.20}$$

and by a straightforward computation we get for $\{\tilde{Q}, f\}$ and $-(\widetilde{D_{df}Q})$ the same local coordinate expression. (See [4] for the complete proof in the case of a symmetric covariant tensor field on M.) □

In order to discuss the next two Jacobi identities, we make some remarks concerning the operator Ψ of (2.12), which is given in the case of a graded Poisson structure on T^*M by

$$\{m(X), m(Y)\} = s(\Psi(X,Y)), \quad X, Y \in \chi(M). \tag{2.21}$$

With (2.16), the second relation (2.13) becomes

$$\Psi(X, fY) = f\Psi(X,Y) - \frac{1}{2}(D_{df}X \otimes Y + Y \otimes D_{df}X) \tag{2.22}$$

and this allows us to derive the local coordinate expression of Ψ. If $X = X^i(\partial/\partial x^i)$ and $Y = Y^j(\partial/\partial x^j)$, we obtain

$$\begin{aligned}\Psi(X,Y) = X^iY^j\Psi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + \left(X^h\frac{\partial Y^j}{\partial x^k}\Gamma_h^{ki} - Y^h\frac{\partial X^i}{\partial x^k}\Gamma_h^{kj}\right)\frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j} \\ + w^{kh}\frac{\partial X^i}{\partial x^k}\frac{\partial Y^j}{\partial x^h}\frac{\partial}{\partial x^i} \odot \frac{\partial}{\partial x^j}.\end{aligned} \tag{2.23}$$

Remark that $\Psi : TM \times TM \to \odot^2 TM$ is a bidifferential operator of the first order.

Proposition 2.6. *If the operator D_{df} acts on Ψ by*

$$(D_{df}\Psi)(X,Y) := D_{df}(\Psi(X,Y)) - \Psi(D_{df}X, Y) - \Psi(X, D_{df}Y), \tag{2.24}$$

the Jacobi identity

$$\{\{m(X), m(Y)\}, f\} + \{\{m(Y), f\}, m(X)\} + \{\{f, m(X)\}, m(Y)\} = 0 \tag{2.25}$$

has the equivalent form

$$(D_{df}\Psi)(X,Y) = 0, \quad \forall X, Y \in \chi(M). \tag{2.26}$$

Proof. Using (2.15), (2.17), and (2.21) for $Q = \Psi(X,Y)$, (2.25) becomes (2.26). □

We also find

$$(D_{df}\Psi)(X,hY) = h(D_{df}\Psi)(X,Y) - [C_D(df,dh)X]\odot Y, \tag{2.27}$$

and hence we see that (2.26) is invariant by $X \mapsto fX$, $Y \mapsto gY$ ($f,g \in C^\infty(M)$) if and only if the curvature $C_D = 0$.

Concerning the Jacobi identity

$$\sum_{(X,Y,Z)} \{\{m(X),m(Y)\},m(Z)\} = 0, \tag{2.28}$$

(putting indices between parentheses denotes that summation is on cyclic permutations of these indices) remark that one must have an operator Θ such that

$$\{s(G),m(X)\} = \widetilde{\Theta(G,X)}, \quad X \in \chi(M),\ G \in S_2(M), \tag{2.29}$$

and $\Theta(G,X)$ is a symmetric 3-contravariant tensor field on M.

We get the formula

$$\Theta(fG,hX) = fh\Theta(G,X) - f(D_{dh}G)\odot X + hG\odot D_{df}X + \{f,h\}_w G\odot X, \tag{2.30}$$

and then the local coordinate expression

$$\begin{aligned}\Theta(G,X) = {} & G^{ij}X^k\Theta\left(\frac{\partial}{\partial x^i}\odot\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\right)\\ & + \frac{1}{3}\sum_{(i,j,k)}\left(G^{hj}\frac{\partial X^k}{\partial x^a}\Gamma_h^{ai} + G^{ih}\frac{\partial X^k}{\partial x^a}\Gamma_h^{aj} - \frac{\partial G^{ij}}{\partial x^a}X^h\Gamma_h^{ak}\right.\\ & \left. + w^{ab}\frac{\partial G^{ij}}{\partial x^a}\frac{\partial X^k}{\partial x^b}\right)\frac{\partial}{\partial x^i}\odot\frac{\partial}{\partial x^j}\odot\frac{\partial}{\partial x^k}.\end{aligned} \tag{2.31}$$

Using the operator Θ, the Jacobi identity (2.28) becomes

$$\sum_{(X,Y,Z)} \Theta(\Psi(X,Y),Z) = 0, \tag{2.32}$$

and we may summarize our analysis concerning the graded Poisson structures on T^*M in the following proposition.

Proposition 2.7. *A graded Poisson structure W on T^*M with the bracket $\{\cdot,\cdot\}$ is defined by*

(a) *a Poisson structure w on the base manifold M such that*

$$\{f,g\}_W = \{f,g\}_w, \quad f,g \in C^\infty(M); \tag{2.33}$$

(b) *a flat contravariant connection D on (M,w) such that*

$$\{m(X),f\} = -m(D_{df}X), \quad X \in C^\infty(M); \tag{2.34}$$

(c) *an operator $\Psi : TM \times TM \to \odot^2 TM$ such that*

$$\{m(X),m(Y)\} = s(\Psi(X,Y)), \quad X,Y \in \chi(M), \tag{2.35}$$

and formula (2.26) holds;

(d) *an operator Θ defined by (2.29), satisfying (2.32).*

To give examples, we consider the following situation similar to [4].

Let (M,w) be an n-dimensional Poisson manifold and suppose that its symplectic foliation S is contained in a regular foliation $\mathcal{F}$ on M such that $T\mathcal{F}$ is a *foliated bundle*, that is, there are local bases $\{Y_u\}$ $(u = 1,\dots,p,\ p = \operatorname{rank}\mathcal{F})$ of $T\mathcal{F}$ with transition functions constant along the leaves of $\mathcal{F}$. Consider a decomposition

$$TM = T\mathcal{F} \oplus \nu\mathcal{F}, \tag{2.36}$$

where $\nu\mathcal{F}$ is a complementary subbundle of $T\mathcal{F}$, and $\mathcal{F}$-adapted local coordinates (x^a, y^u) $(a = 1,\dots,n-p)$ on M [7].

The Poisson bivector w has the form

$$w = \frac{1}{2} w^{uv}(x,y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \quad (w^{vu} = -w^{uv}) \tag{2.37}$$

since $S \subseteq \mathcal{F}$.

If $\{\beta^u\}$, $\{\tilde{\beta}^v\}$ $(u,v = 1,\dots,p)$ are the dual cobases of $\{Y_u\}$, $\{\tilde{Y}_v\}$ $(\beta^u(Y_v) = \delta^u_v)$, then their transition functions are constant along the leaves of $\mathcal{F}$.

Now, for all $\alpha \in T^*M$, $\alpha = \zeta_a dx^a + \varepsilon_u \beta^u$ and we may consider $(x^a, y^u, \zeta_a, \varepsilon_u)$ as *distinguished local coordinates* on T^*M. The transition functions are

$$\tilde{x}^a = \tilde{x}^a(x), \qquad \tilde{y}^u = \tilde{y}^u(x,y), \qquad \tilde{\zeta}_u = \frac{\partial x^a}{\partial \tilde{x}^u}\zeta_a, \qquad \tilde{\varepsilon}_u = a^v_u(x)\varepsilon_v. \tag{2.38}$$

Proposition 2.8. *Under the previous hypotheses, W given with respect to the distinguished local coordinates by*

$$W = \frac{1}{2} w^{uv}(x,y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \tag{2.39}$$

defines a graded Poisson bivector on T^*M.

Proof. From (2.38) it follows that W of (2.39) is a global tensor field on T^*M. The Schouten-Nijenhuis bracket $[W,W]$ has the same expression as $[w,w]$ on M, and thus the Poisson condition $[W,W] = 0$ holds.

To prove that W is graded, we also consider natural coordinates and show that the expression of W with respect to these coordinates becomes of the form (2.14) (see [4]). □

There are some interesting particular cases of Proposition 2.8.

(a) The Poisson structure w is regular, and the bundle TS is a foliated bundle; in this case we may take $\mathcal{F} = S$.

(b) The symplectic foliation S is contained in a regular foliation $\mathcal{F}$ which admits adapted local coordinates (x^a, y^u) with local transition functions

$$\tilde{y}^v = p^v_u(x)y^u + q^v(x). \tag{2.40}$$

(The foliation $\mathcal{F}$ is a leaf-wise, locally affine and regular.) In this case, $(\partial/\partial y^u) = \sum_v a^v_u(x)(\partial/\partial\tilde{y}^v)$ and we may use the local vector fields $Y_u = \partial/\partial y^u$.

(c) There exists a flat linear connection ∇ (possibly with torsion) on the Poisson manifold (M,w). In this case, we may consider as leaves of $\mathcal{F}$ the connected components of M, and the local ∇-parallel vector fields have constant transition functions along these leaves. Therefore, we may take them as Y_i $(i = 1,\dots,n)$.

In particular, we have the result of (c) for a locally affine manifold M (where ∇ has no torsion), using as Y_i local ∇-parallel vector fields, and also for a parallelizable manifold M (where we have global vector fields Y_i).

As a consequence, Proposition 2.8 holds for the Lie-Poisson structure [8] of any dual $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$, the graded Poisson structure being defined on $T^*\mathcal{G}^* = \mathcal{G}^* \times \mathcal{G}$.

3. Graded bivector fields on cotangent bundles.

In this section, we discuss graded bivector fields on a cotangent bundle T^*M, which may be seen as lifts of a given Poisson structure w on M, that satisfy less restrictive existence conditions than in the case of graded Poisson structures.

Recall the following definition from [4]. Let $\mathcal{F}$ be an arbitrary regular foliation, with p-dimensional leaves, on an n-dimensional manifold N. We denote by $C^\infty_{\text{fol}}(N)$ the space of *foliated functions* (the functions on N which are constant along the leaves of $\mathcal{F}$). A *transversal Poisson structure* of $(N,\mathcal{F})$ is a bivector field w on N such that

$$\{f,g\} := w(df,dg), \quad f,g \in C^\infty_{\text{fol}}(N) \tag{3.1}$$

is a Lie algebra bracket on $C^\infty_{\text{fol}}(N)$. A bivector field w on N defines a transversal Poisson structure of $(N,\mathcal{F})$ if and only if [4]

$$(\mathcal{L}_Y w)\big|_{\operatorname{Ann} T\mathcal{F}} = 0, \qquad [w,w]\big|_{\operatorname{Ann} T\mathcal{F}} = 0, \tag{3.2}$$

for all $Y \in \Gamma(T\mathcal{F})$ (the space of global cross sections of $T\mathcal{F}$), where $\operatorname{Ann} T\mathcal{F} \subseteq \Omega^1(N)$ is the annihilator space of $T\mathcal{F}$. ($\Omega^1(N)$ denotes the space of Pfaff forms on N.)

The cotangent bundle T^*M of any manifold M has the vertical foliation $\mathcal{F}$ by fibers with the tangent distribution $V := T\mathcal{F}$.

Obviously, the set of foliated functions on T^*M may be identified with $C^\infty(M)$.

Proposition 3.1. *Any polynomially graded bivector field W on T^*M, which is π-related with a Poisson structure of M, is a transversal Poisson structure of (T^*M,V).*

Proof. The local coordinate expression of W is of the form (2.9), and W is π-related with the bivector field w defined on M by the first term of (2.9). Then, (3.2) holds because w is a Poisson bivector on M. □

Definition 3.2. A transversal Poisson structure of the vertical foliation of T^*M will be called a *semi-Poisson structure* on T^*M.

Remark 3.3. The structures W of Proposition 3.1 are polynomially graded semi-Poisson structures on T^*M.

In what follows, we discuss some interesting classes of graded semi-Poisson structures of T^*M. Then, we give a method to construct all the graded semi-Poisson bivector fields on T^*M, which induce the same Poisson structure w on the base manifold M.

Let D be a contravariant derivative on a Poisson manifold (M,w). First, for all $Q \in S_k(TM)$, define ${}^sDQ \in S_{k+1}(TM)$ by

$$({}^sDQ)(\alpha_1,\dots,\alpha_{k+1}) = \frac{1}{k+1}\sum_{i=1}^{k+1}(D_{\alpha_i}Q)(\alpha_1,\dots,\hat{\alpha}_i,\dots,\alpha_{k+1}), \tag{3.3}$$

where $\alpha_1,\dots,\alpha_{k+1} \in \Omega^1(M)$ and the hat denotes the absence of the corresponding factor.

If $X = X^i(\partial/\partial x^i) \in \chi(M)$, then DX, defined by $(DX)(\alpha_1,\alpha_2) = (D_{\alpha_1}X)\alpha_2$, is a 2-contravariant tensor field on M, and

$$DX = D^iX^j\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}, \tag{3.4}$$

where $D^iX^j = (D_{dx^i}X)dx^j = D_{dx^i}X^j - X(D_{dx^i}dx^j)$. According to (2.20), we must have

$$D_{dx^i}dx^j = \Gamma_k^{ij}dx^k \tag{3.5}$$

and obtain

$$D^iX^j = (dx^i)^\sharp X^j - \Gamma_k^{ij}dx^k = \{x^i,X^j\}_w - \Gamma_k^{ij}X^k. \tag{3.6}$$

Then

$$ {}^sDX = \frac{1}{2}(D^iX^j + D^jX^i)\frac{\partial}{\partial x^i}\odot\frac{\partial}{\partial x^j} \tag{3.7}$$

and we get

$$ {}^sDX = \frac{1}{2}\left[\{x^i,X^j\}_w + \{x^j,X^i\}_w - \Gamma_k^{ij}X^k - \Gamma_k^{ji}X^k\right]\frac{\partial}{\partial x^i}\odot\frac{\partial}{\partial x^j}. \tag{3.8}$$

Proposition 3.4. *Let (M,w) be a Poisson manifold and D a contravariant derivative of (M,w). The bivector field W_1 on T^*M, of bracket $\{\cdot,\cdot\}_{W_1}$ defined by the conditions*

$$\{f,g\}_{W_1} := \{f,g\}_w, \tag{3.9}$$

$$\{m(X),f\}_{W_1} := -m(D_{df}X), \tag{3.10}$$

$$\{m(X),m(Y)\}_{W_1} = \frac{1}{2}s\left[{}^sD\langle X,Y\rangle - \langle {}^sDX,Y\rangle - \langle X,{}^sDY\rangle\right], \tag{3.11}$$

where $f,g\in C^\infty(M)$, $X,Y\in\chi(M)$, and $\langle\cdot,\cdot\rangle$ is the Schouten-Nijenhuis bracket of symmetric tensor fields (defined by the natural Lie algebroid of M) [1, 4], *defines a graded semi-Poisson structure on T^*M which is π-related with w.*

Proof. If the local coordinate expression of w is (2.8), using (3.8) and the properties of $\langle\cdot,\cdot\rangle$ [1, 4], we get

$$\begin{aligned} W_1 = {} & \frac{1}{2}w^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} - p_a\Gamma_j^{ia}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial p_j} \\ & - \frac{1}{4}p_ap_b\left[\frac{\partial}{\partial x^j}(\Gamma_i^{ab}+\Gamma_i^{ba}) - \frac{\partial}{\partial x^i}(\Gamma_j^{ab}+\Gamma_j^{ba})\right]\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j}. \end{aligned} \tag{3.12}$$

□

Remark 3.5. The relation (3.11) provides us with the expression of the operator Ψ_{W_1} associated to W_1 (see (2.21)):

$$\Psi_{W_1}(X,Y) = \frac{1}{2}\left({}^sD\langle X,Y\rangle - \langle {}^sDX,Y\rangle - \langle X,{}^sDY\rangle\right). \tag{3.13}$$

Now, instead of D we consider a linear connection ∇ on a Poisson manifold (M,w) and define the vector field K on T^*M by

$$K(\alpha) = (\sharp_w\alpha)^H_\alpha, \quad \alpha\in T^*M, \tag{3.14}$$

where $\sharp_w : T^*M \to TM$ is defined by $\beta(\alpha^\sharp) = w(\alpha,\beta)$ for all $\beta \in \Omega^1(M)$, and the upper index H denotes the horizontal lift with respect to ∇ (see [2, 9]). In local coordinates, we get

$$K = p_a w^{ai}\frac{\partial}{\partial x^i} + \frac{1}{2}p_a p_b\left(w^{ak}\Gamma^b_{ki} + w^{bk}\Gamma^a_{ki}\right)\frac{\partial}{\partial p_i}. \tag{3.15}$$

On T^*M, we have the canonical symplectic form $\omega = d\lambda = dp_i \wedge dx^i$, where $\lambda = p_i dx^i$ is the Liouville form, and the vector bundle isomorphism

$$\sharp_\omega : T^*M \longrightarrow TM, \qquad i_X\omega \in T^*M \longmapsto X \in TM \tag{3.16}$$

leads to the canonical Poisson bivector $W_0 := \sharp_\omega \omega$ on T^*M. It follows that

$$W_0(dF, dG) = \omega\left(\sharp(dF), \sharp(dG)\right), \quad F, G \in C^\infty(T^*M), \tag{3.17}$$

and, locally, one has

$$W_0 = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i}. \tag{3.18}$$

Proposition 3.6. *If (M, w) is a Poisson manifold, then the bivector field*

$$W_2 = \frac{1}{2}\mathcal{L}_K W_0 \tag{3.19}$$

defines a graded semi-Poisson structure on T^*M which is π-related with w.

Proof. We get

$$\begin{aligned} W_2 = {} & \frac{1}{2}w^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{2}p_a\left(\nabla_j w^{ai} + 2w^{ik}\Gamma^a_{kj}\right)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} \\ & + \frac{1}{4}p_a p_b\left[\frac{\partial}{\partial x^j}\left(w^{ak}\Gamma^b_{ki} + w^{bk}\Gamma^a_{ki}\right) - \frac{\partial}{\partial x^i}\left(w^{ak}\Gamma^b_{kj} + w^{bk}\Gamma^a_{kj}\right)\right]\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}, \end{aligned} \tag{3.20}$$

where $\nabla_j w^{ai}$ are the components of the $(2,1)$-tensor field on M defined by $X \mapsto \nabla_X w$, $X \in \chi(M)$. □

We will say that W_2 of (3.19) is the *graded ∇-lift* of the Poisson structure w of M.

Using local coordinates and the notation of (2.2), we get

$$\mathcal{L}_K \tilde{Q} = \widetilde{{}^sDQ}, \tag{3.21}$$

where D is the contravariant derivative induced by the linear connection ∇, defined by $D_{df} = \nabla_{(df)^\sharp}$ (see [8]).

From (3.19) we have

$$\{F_1,F_2\}_{W_2} := W_2(dF_1,dF_2) = \frac{1}{2}\Big(\mathcal{L}_K\big(\{F_1,F_2\}_{W_0}\big) - \{\mathcal{L}_K F_1,F_2\}_{W_0} - \{F_1,\mathcal{L}_K F_2\}_{W_0}\Big), \tag{3.22}$$

where $F_1,F_2 \in C^\infty(T^*M)$.

If $Q_1,Q_2 \in S(TM)$, using (3.21) and the relation

$$\{\tilde{Q},\tilde{H}\}_{W_0} := \widetilde{\langle Q,H\rangle}, \quad Q,H \in S(TM) \tag{3.23}$$

(see [1, 4]), we get the explicit formula

$$\{\tilde{Q}_1,\tilde{Q}_2\}_{W_2} = \frac{1}{2} \sim [{}^sD\langle Q_1,Q_2\rangle - \langle {}^sDQ_1,Q_2\rangle - \langle Q_1,{}^sDQ_2\rangle]. \tag{3.24}$$

Proposition 3.7. *The graded ∇-lift W_2 of w is characterized by the following:*

(i) *the Poisson structure induced on M by W_2 is w, that is,*

$$\{f,g\}_{W_2} = \{f,g\}_w, \quad \forall f,g \in C^\infty(M); \tag{3.25}$$

(ii) *for every $f \in C^\infty(M)$ and $X \in \chi(M)$,*

$$\{m(X),f\}_{W_2} = -m(\bar{D}_{df}X), \tag{3.26}$$

where $\bar{D}$ is the contravariant derivative of (M,w) defined by

$$\bar{D}_\alpha\beta = D_\alpha\beta + \frac{1}{2}(\nabla.w)(\alpha,\beta), \quad \alpha,\beta \in \Omega^1(M), \tag{3.27}$$

where the contravariant derivative D is induced by ∇ and $(\nabla.w)(\alpha,\beta)$ is the 1-form $X \mapsto (\nabla_X w)(\alpha,\beta)$;

(iii) *for any vector fields X and Y of M,*

$$\{m(X),m(Y)\}_{W_2} = \frac{1}{2}s({}^sD\langle X,Y\rangle - \langle {}^sDX,Y\rangle - \langle X,{}^sDY\rangle). \tag{3.28}$$

Proof. (i) If $f \in C^\infty(M)$, then $Df = -X_f^w$ and from (3.22), (3.23), and the formula

$$\langle Q,f\rangle = i(df)Q, \quad f \in C^\infty(M),\ Q \in S_p(TM), \tag{3.29}$$

we get

$$\{f,g\}_{W_2} = -\frac{1}{2}(\langle Df,g\rangle + \langle f,Dg\rangle) = \frac{1}{2}(X_f^w g - X_g^w f) = \{f,g\}_w. \tag{3.30}$$

(ii) As W_2 is graded, the bracket $\{m(X), f\}_{W_2}$ must be of the form (3.26). Denoting

$$\bar{D}_{dx^i} dx^j = \bar{\Gamma}_k^{ij} dx^k, \tag{3.31}$$

(3.20) gives us

$$\bar{\Gamma}_k^{ij} = \Gamma_k^{ij} + \frac{1}{2} \nabla_k w^{ij}, \tag{3.32}$$

where

$$\Gamma_k^{ij} = -w^{ih} \Gamma_{hk}^j, \tag{3.33}$$

(Γ_{jk}^i are the coefficients of the linear connection ∇) and hence (3.27).

(iii) Equation (3.28) is a direct consequence of (3.24). □

Notice from (3.28) that the operator Ψ_{W_2} associated to W_2 has the same expression as Ψ_{W_1} of (3.13), but in the case of W_1, the contravariant derivative D is induced by a linear connection ∇ on M.

Proposition 3.8. *If the graded semi-Poisson structure W_1 is defined by a linear connection on (M, w), then it coincides with W_2 if and only if w is ∇-parallel.*

Proof. Compare the characteristic conditions of Propositions 3.4 and 3.7 (or the coefficients of $(\partial/\partial x^i) \wedge (\partial/\partial p_j)$ of (3.12) and of (3.20), using (3.33)). □

We prove now the following proposition.

Proposition 3.9. *Let (M, w) be a Poisson manifold and $\pi : T^*M \to M$ its cotangent bundle. The graded semi-Poisson structures W on T^*M which are π-related with w are defined by the relations*

$$\begin{gathered} \{f, g\}_W = \{f, g\}_w, \qquad \{m(X), f\}_W = -m(D_{df} X), \\ \{m(X), m(Y)\}_W = s(\Psi(X, Y)), \quad f, g \in C^\infty(M),\ X, Y \in \chi(M), \end{gathered} \tag{3.34}$$

where D is an arbitrary contravariant connection of (M, w) and the operator Ψ is given by

$$\Psi = \Psi_0 + A + T, \tag{3.35}$$

where Ψ_0 is the operator Ψ of a fixed graded semi-Poisson structure and $A : TM \times TM \to \odot^2 TM$ is a skew-symmetric, first-order, bidifferential operator such that

$$A(X, fY) = fA(X, Y) - \tau(df, X) \odot Y, \tag{3.36}$$

where τ is a $(2,1)$-tensor field on M and T is a $(2,2)$-tensor field on M with the properties $T(Y,X) = -T(X,Y)$ and $T(X,Y) \in S_2(TM)$ for all $X,Y \in \chi(M)$.

Proof. If two graded semi-Poisson bivector fields, π-related with w, have associated the same contravariant connection D, it follows from (2.22) that the difference $\Psi' - \Psi$ is a tensor field T, as in Proposition 3.8. To change D means to pass to a contravariant connection $D' = D + \tau$, where τ is a $(2,1)$-tensor field on M and from (2.22) again, it follows that $A = \Psi' - \Psi$ becomes a bidifferential operator with the property (3.35). □

4. Horizontal lifts of Poisson structures. In this section, we define and study an interesting class of semi-Poisson structures on T^*M which are produced by a process of *horizontal lifting* of Poisson structures from M to T^*M via connections.

On T^*M, we distinguish the vertical distribution V, tangent to the fibers of the projection π and, by complementing V by a distribution H, called *horizontal*, we define a *nonlinear connection* on T^*M [5, 6].

We have (*adapted*) bases of the form

$$\mathrm{V} = \operatorname{span}\left\{\frac{\partial}{\partial p_i}\right\}, \qquad \mathrm{H} = \operatorname{span}\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{ij}\frac{\partial}{\partial p_j}\right\}, \tag{4.1}$$

and N_{ij} are the *coefficients of the connection* defined by H.

Equivalently, a nonlinear connection may be seen as an almost product structure Γ on T^*M such that the eigendistribution corresponding to the eigenvalue -1 is the vertical distribution V [6].

We assume that the nonlinear connection above is symmetric, that is, $N_{ji} = N_{ij}$. This condition is independent [6] of the local coordinates.

The complete integrability of H, in the sense of the Frobenius theorem, is equivalent to the vanishing of the curvature tensor field

$$R = R_{kij}dx^i \wedge dx^j \otimes \frac{\partial}{\partial p_k}, \quad R_{kij} = \frac{\delta N_{kj}}{\delta x^i} - \frac{\delta N_{ki}}{\delta x^j}. \tag{4.2}$$

For a later utilization, we also notice the formulas [5, 6]

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R_{kij}\frac{\partial}{\partial p_k}, \qquad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_j}\right] = -\Phi^j_{ik}\frac{\partial}{\partial p_k}, \quad \Phi^j_{ik} = -\frac{\partial N_{ik}}{\partial p_j}. \tag{4.3}$$

Let w be a bivector on M with the local coordinate expression (2.8).

Definition 4.1. The *horizontal lift* of w to the cotangent bundle T^*M is the (global) bivector field w^H defined by

$$w^H = \frac{1}{2}w^{ij}(x)\frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j}. \tag{4.4}$$

Proposition 4.2. *Let (M, w) be a Poisson manifold. If the connection Γ on T^*M is defined by a linear connection ∇ on M, the bivector w^H defines a graded semi-Poisson structure on T^*M.*

Proof. In this case, the coefficients of Γ are

$$N_{ij} = -p_k \Gamma^k_{ij}, \tag{4.5}$$

where Γ^k_{ij} are the coefficients of ∇ and, with respect to the bases $\{\partial/\partial x^i, \partial/\partial p_j\}$, the local expression of w^H becomes

$$\begin{aligned} W = {} & \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + w^{ik} \Gamma^a_{kj} p_a \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_j} \\ & + \frac{1}{2} w^{kh} \Gamma^a_{ki} \Gamma^b_{hj} p_a p_b \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j}. \end{aligned} \tag{4.6}$$

□

Proposition 4.3. *The horizontal lift w^H is a Poisson bivector on the cotangent bundle T^*M if and only if w is a Poisson bivector on the base manifold M and*

$$R(X^H_f, X^H_g) = 0, \quad \forall f, g \in C^\infty(M), \tag{4.7}$$

where X^H_f denotes the usual horizontal lift [2, 9]*, from M to T^*M, of the w-Hamiltonian vector field X_f on M.*

In this case, the projection $\pi : (T^*M, w^H) \to (M, w)$ is a Poisson mapping.

Proof. We compute the bracket $[w^H, w^H]$ with respect to the bases (4.1) and get that the Poisson condition $[w^H, w^H] = 0$ is equivalent with the pair of conditions

$$\sum_{(i,j,k)} w^{hk} \frac{\partial w^{ij}}{\partial x^h} = 0, \qquad w^{il} w^{jh} R_{klh} = 0. \tag{4.8}$$

(Putting indices between parentheses denotes that summation is on cyclic permutations of these indices.)

The first condition in (4.8) is equivalent to $[w, w] = 0$ and the second is the local coordinate expression of (4.7). □

Notice that the condition (4.7) has the equivalent form

$$R\big((\sharp\alpha)^H, (\sharp\beta)^H\big) = 0, \quad \forall \alpha, \beta \in \Omega^1(M). \tag{4.9}$$

Remark 4.4. If w is defined by a symplectic form on M, condition (4.8) becomes $R = 0$.

Corollary 4.5. *If (M,w) is a Poisson manifold and the connection Γ on T^*M is defined by a linear connection ∇ on M, the bivector w^H defines a Poisson structure on T^*M if and only if the curvature C_D of the contravariant connection induced by ∇ on TM vanishes. In this case, w^H is a graded Poisson structure on T^*M.*

Proof. If R^h_{kij} are the components of the curvature R_∇, then

$$R_{kij} = -p_h R^h_{kij} \tag{4.10}$$

and (4.9) becomes

$$R_\nabla(\sharp\alpha,\sharp\beta)Z = 0, \quad \forall\alpha,\beta \in \Omega^1(M),\ \forall Z \in \chi(M), \tag{4.11}$$

or, equivalently,

$$R_\nabla(X_f,X_g)Z = 0, \quad \forall f,g \in C^\infty(M),\ \forall Z \in \chi(M). \tag{4.12}$$

This is equivalent to $C_D = 0$. □

In the case where w^H is a Poisson bivector, it is interesting to study its compatibility with the canonical Poisson structure W_0 of (3.17).

Proposition 4.6. *If w^H is a Poisson bivector, then it is compatible with W_0 if and only if*

$$\frac{\partial w^{ij}}{\partial x^k} + w^{ih}\Phi^j_{hk} - w^{jh}\Phi^i_{hk} = 0, \qquad w^{ih}R_{hjk} = 0. \tag{4.13}$$

Proof. By a straightforward computation, we get that the compatibility condition $[w^H,W] = 0$ is equivalent to (4.13). □

The Bianchi identity [6]

$$R_{kij} + R_{ijk} + R_{jki} = 0 \tag{4.14}$$

shows that the second relation in (4.13) implies (4.7). Then we have the following corollary.

Corollary 4.7. *If (M,w) is a Poisson manifold and the cotangent bundle T^*M is endowed with a symmetric nonlinear connection, then w^H is a Poisson bivector on T^*M compatible with W_0 if and only if conditions (4.13) hold.*

Remark 4.8. Considering the isomorphism

$$\Psi : \mathrm{V}_u \longrightarrow \mathrm{H}^*_u, \quad \Psi\left(X_k\frac{\partial}{\partial p_k}\right) = X_k dq^k, \tag{4.15}$$

where $u \in T^*M$ and H_u^* is the dual space of H_u, the second condition in (4.13) may be written in the equivalent form

$$[\Psi(R(X,Y))](\sharp_w \alpha)^H = 0, \quad \forall X,Y \in \chi(T^*M),\ \forall \alpha \in \Omega^1(M). \tag{4.16}$$

We recall that a symmetric linear connection ∇ on a Poisson manifold (M,w) is called a *Poisson connection* if $\nabla w = 0$. Such connections exist if and only if w is regular, that is, $\operatorname{rank} w = \text{const}$ (see [8]).

Proposition 4.9. *Let (M,w) be a regular Poisson manifold with a Poisson connection ∇. Then the bivector w^H, defined with respect to ∇, is a Poisson structure on T^*M compatible with the canonical Poisson structure W_0 if and only if the 2-form*

$$(X,Y) \longrightarrow R_\nabla(X,Y)(\sharp_w \alpha), \quad X,Y \in \chi(M) \tag{4.17}$$

vanishes for every Pfaff form α on M.

Proof. With (4.5), the first condition in (4.13) becomes $\nabla w = 0$, which we took as a hypothesis. The second condition in (4.13) becomes

$$w^{ih} R^l_{hjk} = 0, \tag{4.18}$$

and we get the required conditions. □

Remark 4.10. If w is defined by a symplectic structure of M, then (4.17) means $R_\nabla = 0$.

5. Poisson structures derived from differential forms. If ω is a 2-form on a Riemannian manifold (M,g), we associate with it a 2-form $\Theta(\omega)$ on the cotangent bundle $\pi : T^*M \to M$, and considering (pseudo-)Riemannian metrics on T^*M related to g, we study the conditions for $\Theta(\omega)$ to produce a Poisson structure on this bundle.

Let (M,g) be an n-dimensional manifold and ∇ its Levi-Civita connection. If Γ^k_{ij} are the local coefficients of ∇, a connection Γ with the coefficients (4.5) is obtained on T^*M.

The system of local 1-forms $(dx^i, \delta p_i)$ $(i = 1,\dots,n)$, where

$$\delta p_i := dp_i + N_{ij} dx^j, \tag{5.1}$$

defines the dual bases of the bases $\{\delta/\delta x^i, \partial/\partial p_i\}$.

The components of the curvature form are given by (4.2). Since the connection is symmetric, the Bianchi identity (4.14) holds. The elements Φ^k_{ij} of (4.3) are

$$\Phi^k_{ij} = \Gamma^k_{ij}. \tag{5.2}$$

The Riemannian metric g provides the "musical" isomorphism $\sharp_g : T^*M \to TM$ and the codifferential

$$\delta_g : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \quad (\delta_g \alpha)_{i_1 \cdots i_{k-1}} = -g^{st} \nabla_t \alpha_{s i_1 \cdots i_{k-1}}, \tag{5.3}$$

where $k \geq 1$,

$$\alpha = \frac{1}{k!} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(M), \tag{5.4}$$

and (g^{st}) are the entries of the inverse of the matrix (g_{ij}) [8].

Let

$$\omega = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j, \quad \omega_{ji} = -\omega_{ij}, \tag{5.5}$$

be a 2-form on M.

Definition 5.1. The 2-form $\Theta(\omega)$ on T^*M given by

$$\Theta(\omega) = \pi^* \omega - d\lambda, \tag{5.6}$$

where λ is the Liouville form, is said to be the *associated* 2*-form* of ω.

With respect to the cobases $(dx^i, \delta p_i)$, we get

$$\Theta(\omega) = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j + dx^i \wedge \delta p_i. \tag{5.7}$$

Now, we consider two (pseudo-)Riemannian metrics G_1 and G_2 on T^*M and study the conditions for the bivectors $W_i = \sharp_{G_i} \Theta(\omega)$ $(i = 1, 2)$ to define Poisson structures on T^*M. The Poisson condition $[W_i, W_i] = 0$, $i = 1, 2$, is equivalent to [8]

$$\delta_{G_i}\big(\Theta(\omega) \wedge \Theta(\omega)\big) = 2\Theta(\omega) \wedge \delta_{G_i} \Theta(\omega), \quad i = 1, 2. \tag{5.8}$$

First, consider [5, 6] the pseudo-Riemannian metric G_1 of signature (n, n)

$$G_1 = 2\delta p_i \odot dx^i. \tag{5.9}$$

To find the condition which ensures that (5.8) holds, we need the local expression of the codifferential δ_{G_1} of G_1. Denote by $\tilde{\nabla}$ the Levi-Civita connection of G_1, and for simplicity we put

$$\tilde{\nabla}_i = \tilde{\nabla}_{\delta/\delta x^i}, \qquad \tilde{\nabla}^i = \tilde{\nabla}_{\partial/\partial p_i}. \tag{5.10}$$

The connection $\tilde{\nabla}$ is defined by [6]

$$\begin{aligned}
&\tilde{\nabla}^i \frac{\partial}{\partial p_j} = 0, \qquad \tilde{\nabla}_i \frac{\partial}{\partial p_j} = -\Gamma^j_{ik} \frac{\partial}{\partial p_k}, \\
&\tilde{\nabla}^i \frac{\delta}{\delta q^j} = 0, \qquad \tilde{\nabla}_i \frac{\delta}{\delta q^j} = \Gamma^k_{ij} \frac{\delta}{\delta q^k} - p_h R^h_{ijk} \frac{\partial}{\partial p_k}.
\end{aligned} \tag{5.11}$$

Proposition 5.2. *The bivector $\sharp_{G_1}\Theta(\omega)$ defines a Poisson structure on the cotangent bundle T^*M if and only if ω is a closed 2-form on M and $\Gamma^a_{ai} = 0$, for all $i = 1,\dots,n$. In this case, $\Theta(\omega)$ is a symplectic form.*

Proof. The proof is by a long computation in local coordinates. After computing the exterior product $\Theta(\omega)\wedge\Theta(\omega)$, we get

$$\delta_{G_1}\big(\Theta(\omega)\wedge\Theta(\omega)\big) = \frac{2}{3!}\sum_{(i,j,k)} \nabla_i \omega_{jk} dx^i \wedge dx^j \wedge dx^k. \tag{5.12}$$

Then we compute $\delta_{G_1}\Theta(\omega)$ and obtain

$$\begin{aligned}
\Theta(\omega)\wedge\delta_{G_1}\Theta(\omega) = \frac{2}{3!}\sum_{(i,j,k)} &\omega_{ij}\Gamma^a_{ak} dx^i \wedge dx^j \wedge dx^k \\
&+ \big(\delta^k_j \Gamma^a_{ai} - \delta^k_i \Gamma^a_{aj}\big) dx^i \wedge dx^j \wedge \delta p_k.
\end{aligned} \tag{5.13}$$

Equation (5.8) implies

$$\delta^k_j \Gamma^a_{ai} - \delta^k_i \Gamma^a_{aj} = 0, \quad \forall i,j,k = 1,\dots,n. \tag{5.14}$$

Making the contraction $k = j$, it follows that $\Gamma^a_{ai} = 0$. Conversely, if $\Gamma^a_{ai} = 0$, then (5.14) holds. Also, since ∇ is symmetric, we get

$$\sum_{(i,j,k)} \frac{\partial \omega_{jk}}{\partial x^i} = \sum_{(i,j,k)} \nabla_i \omega_{jk}. \tag{5.15}$$

Therefore, the condition $\sum_{(i,j,k)} \nabla_i \omega_{jk} = 0$ is equivalent to $d\omega = 0$. □

We consider now the Riemannian metric of Sasaki type

$$G_2 = g_{ij} dx^i \odot dx^j + g^{ij} \delta p_i \odot \delta p_j \tag{5.16}$$

(see [3] for the Sasaki metric).

Lemma 5.3. *The local coordinate expression of the Levi-Civita connection $\bar{\nabla}$ of G_2 is*

$$\begin{aligned}
&\bar{\nabla}^i \frac{\partial}{\partial p_j} = 0, \qquad \bar{\nabla}_i \frac{\partial}{\partial p_j} = -\frac{1}{2} R^{jk}{}_i \frac{\delta}{\delta q^k} - \Gamma^j_{ik} \frac{\partial}{\partial p_k}, \\
&\bar{\nabla}^i \frac{\delta}{\delta q^j} = \frac{1}{2} R^i{}_j{}^k \frac{\delta}{\delta q^k}, \qquad \bar{\nabla}_i \frac{\delta}{\delta q^j} = \Gamma^k_{ij} \frac{\delta}{\delta q^k} - \frac{1}{2} R_{kij} \frac{\partial}{\partial p_k},
\end{aligned} \tag{5.17}$$

where the notations of (5.10) are used again and $R^{jk}_{\ \ i}$ *(also* $R^{i\ \ k}_{\ j}$*) are obtained from* R_{kij} *by the operation of lifting the indices, that is,*

$$R^{jk}_{\ \ i} = g^{ja}g^{kb}R_{abi}, \qquad R^{i\ \ k}_{\ j} = g^{ia}g^{kb}R_{ajb}. \tag{5.18}$$

Proof. The result is proved by a straightforward computation. □

Proposition 5.4. *The bivector* $\delta_{G_2}\Theta(\omega)$ *defines a Poisson structure on the cotangent bundle* T^*M *if and only if*

$$\nabla\omega = 0, \qquad g^{ab}R^k_{abi} = 0, \qquad \omega^{ab}R^k_{iab} = 0, \tag{5.19}$$

where $\omega^{ab} = g^{ai}g^{bj}\omega_{ij}$ *are the components of the bivector* $w = \sharp_g\omega$ *on* M*.*

Proof. By a new long computation again, we get

$$\begin{aligned}
\frac{1}{2}\delta_{G_2}(\Theta(\omega)\wedge\Theta(\omega)) &= \frac{1}{3!}g^{ab}\nabla_a\Bigg(\sum_{(i,j,k)}\omega_{ij}\omega_{kb}\Bigg)dx^i\wedge dx^j\wedge dx^k\\
&\quad - g^{ab}\sum_{(i,j,k)}(\nabla_a\omega_{ij}\delta^k_b)dx^i\wedge dx^j\wedge\delta p_k\\
&\quad + \frac{1}{2}\omega_{ab}(R^{kab}\delta^j_i - R^{jab}\delta^k_i)dx^i\wedge\delta p_j\wedge\delta p_k,\\
\Theta(\omega)\wedge\delta_{G_2}\Theta(\omega) &= \frac{1}{3!}\sum_{(i,j,k)}(\delta_{G_2}\Theta(\omega))_k dx^i\wedge dx^j\wedge dx^k\\
&\quad + \frac{1}{2!}\Big[\delta^k_i(\delta_{G_2}\Theta(\omega))_j - \delta^k_j(\delta_{G_2}\Theta(\omega))_i\Big]dx^i\wedge dx^j\wedge\delta p_k,
\end{aligned} \tag{5.20}$$

where

$$\delta_{G_2}\Theta(\omega) = (\delta_{G_2}\Theta(\omega))_k dx^k = g^{ab}\Big(\nabla_a\omega_{kb} - \frac{1}{2}R_{abk}\Big)dx^k. \tag{5.21}$$

Identifying the coefficients, the Poisson condition (5.8) for W_2 becomes

$$g^{ab}\sum_{(i,j,k)}\omega_{ij}R^h_{abk} = 0, \qquad g^{ab}\sum_{(i,j,k)}(\nabla_a\omega_{ij})\omega_{kb} = 0, \tag{5.22}$$

$$\nabla\omega = 0, \qquad g^{ab}R^k_{abi} = 0, \tag{5.23}$$

$$\omega^{ab}R^k_{iab} = 0. \tag{5.24}$$

We remark that the conditions (5.23) imply (5.22) because if $\nabla\omega = 0$, then $\nabla_a\omega_{ij} = 0$, and $g^{ab}R^k_{abi} = 0$ implies $g^{ab}\omega_{ij}R^h_{abk} = 0$. □

Remark 5.5. If the bivector $\sharp_{G_2}\Theta(\omega)$ defines a Poisson structure on T^*M, then $w = \sharp_g \omega$ defines a Poisson structure on M, as the second condition in (5.22) is equivalent to the Poisson condition [8]

$$\sum_{(i,j,k)} w^{ia}\nabla_a w^{jk} = 0. \tag{5.25}$$

(The local coordinate expression of w is (2.8).)

Corollary 5.6. *If $\sharp_{G_2}\Theta(\omega)$ is a Poisson bivector on T^*M, then the scalar curvature r of (M,g) vanishes.*

Proof. The expression of r is $r = g^{ab}R_{ab}$, where $R_{ba} = R^k_{akb} = R_{ab}$ are the components of the Ricci tensor, and if we make the contraction $k = i$ in the second relation in (5.19), we get $g^{ab}R^k_{akb} = 0$, and whence $r = 0$. □

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