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ON THE WEAK UNIFORM ROTUNDITY OF BANACH SPACES

WEN D. CHANG and PING CHANG

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We prove that if X_i, $i = 1, 2, \ldots$, are Banach spaces that are weak* uniformly rotund, then their l_p product space ($p > 1$) is weak* uniformly rotund, and for any weak or weak* uniformly rotund Banach space, its quotient space is also weak or weak* uniformly rotund, respectively.

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1. Definitions and preliminaries. In this note, X and Y denote Banach spaces and X^* and Y^* denote the conjugate spaces of X and Y, respectively. Let $A \subset X$ be a closed subset and X/A denote the quotient space. We use $S(X)$ for the unit sphere in X and $P_{l_p}(X_i)$ for the l_p product space. We refer to [1, 3] for the following definitions and notations. For more recent treatment, one may see, for example, [2].

Definition 1.1. A Banach space X is $\mathrm{UR}^{A'}$, where A' is a nonempty subset of X^*, if and only if for any pair of sequences $\{x_n\}$ and $\{y_n\}$ in $S(X)$, if $\|x_n + y_n\| \to 2$, then $f(x_n - y_n) \to 0$ for all f in A'.

Definition 1.2. A Banach space X is WUR (weakly uniformly rotund) if and only if X is UR^{X^*}.

Definition 1.3. The conjugate space X^* is W*UR (weak* uniformly rotund) if and only if X is $\mathrm{UR}^{Q(X)}$, where $Q : X \to X^{**}$ is the canonical embedding.

2. Some results on the weak* and weak uniform rotundity. From the definition, we clearly have the following corollary.

Lemma 2.1. *The Banach space X is W*UR if and only if for any pair of sequences $\{x_n\}$ and $\{y_n\}$ in X, if $\|x_n\| - \|y_n\| \to 0$, $\{\|y_n\|\}$ is bounded, and $\|x_n\| + \|y_n\| - \|x_n + y_n\| \to 0$, then $x_n - y_n \xrightarrow{w^*} \theta$.*

Theorem 2.2. *Suppose that X_i, $i = 1, 2, \ldots$, are W*UR, then for $p > 1$, $P_{l_p}(X_i)$ is W*UR.*

Proof. Let $X_i = Y_i^*$, then $P_{l_p}(X_i) = [P_{l_q}(Y_i)]^*$ (where $1/p + 1/q = 1$) (see [2]). Let $\{x_n\} = \{(x_1^n, x_2^n, x_3^n, \ldots, x_m^n, \ldots)\} \in P_{l_p}(X_i)$, $\{y_n\} = \{(y_1^n, y_2^n, y_3^n, \ldots, y_m^n, \ldots)\} \in P_{l_p}(X_i)$, $\|x_n + y_n\| \to 2$. Using the properties of l_p norm and Minkowski inequality, one can see, for each i, that there exists a subsequence of $\{n\}$, $\{n_k^i\}$, such that $\lim_{k\to\infty}\|x_i^{n_k^i}\| = \lim_{k\to\infty}\|y_i^{n_k^i}\|$ and $\lim_{k\to\infty}\|x_i^{n_k^i}+y_i^{n_k^i}\| = \lim_{k\to\infty}[\|x_i^{n_k^i}\| + \|y_i^{n_k^i}\|]$. We now choose a subsequence with the diagonal method, without loss of generality, still use $\{n\}$ as the index such that for each i, we have $\lim_{n\to\infty}\|x_i^n\| - \lim_{n\to\infty}\|y_i^n\| = 0$ and $\lim_{k\to\infty}[\|x_i^n\| + \|y_i^n\| - \|x_i^n + y_i^n\|] = 0$. Since X_i is W*UR for each i, by the lemma, we have

$$x_i^n - y_i^n \xrightarrow{\text{w}^*} \theta. \tag{2.1}$$

Suppose that $P_{l_p}(X_i)$ is not W*UR, then there exist sequences $\{x_n\} \in S(P_{l_p}(X_i))$, $\{y_n\} \in S(P_{l_p}(X_i))$, $\|x_n + y_n\| \to 2$, but $x_n - y_n$ does not converge (w*) to θ. So, there must be an $a = (a_1, a_2, \ldots, a_i, \ldots)$ in $P_{l_q}(Y_i)$, with $a_i \in Y_i$, such that $|(x^n - y^n)(a)|$ does not converge to 0. Therefore, there exist $\epsilon > 0$ and a subsequence of $\{n\}$ (for simplicity, we still use $\{n\}$) such that $|(x^n - y^n)(a)| > \epsilon$, which implies that one can find an integer m, sufficiently large, so that

$$\sum_{i=1}^{m} |(x_i^n - y_i^n)(a_i)| > \frac{\epsilon}{2}. \tag{2.2}$$

Let (n_k) be the subsequence of $\{n\}$ such that (2.1) holds. By (2.2), we have

$$\sum_{i=1}^{m} |(x_i^{n_k} - y_i^{n_k})(a_i)| > \frac{\epsilon}{2}. \tag{2.3}$$

Let $k \to \infty$ in (2.3), we have a contradiction $0 > \epsilon/2$.

The proof is complete. □

Theorem 2.3. *Suppose that $X = Y^*$ and A is any w* closed subspace of X. If X is W* UR, then X/A is W* UR.*

Proof. Let $D = \{y \in Y \mid x(y) = 0$ for any $x \in A\}$, then

$$A = \{x \in X \mid x(y) = 0 \text{ for any } y \in D\}, \tag{2.4}$$

see [4]. So, We have $D^* \simeq X/A$.

Suppose that X/A is not W*UR, then there exist $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ in X/A such that $\|\tilde{x}_n\| = \|\tilde{y}_n\| = 1$, $\|\tilde{x}_n + \tilde{y}_n\| \to 2$, but $\tilde{x}_n - \tilde{y}_n$ does not converge (w*) to θ. Here, $\tilde{x} = \pi(x)$, where $\pi : X \to X/A$.

Now, for each n, take $x_n \in \tilde{x}_n$ and $y_n \in \tilde{y}_n$, $1 \le \|x_n\| \le 1 + 1/n$, $1 \le \|y_n\| \le 1 + 1/n$, then $\lim_{n\to\infty}\|x_n + y_n\| = 2$. Since X is W*UR, we have $x_n - y_n \xrightarrow{\text{w}^*} \theta$, π is w*-w* continuous. So, we must have $\tilde{x}_n - \tilde{y}_n \xrightarrow{\text{w}^*} \theta$. That contradicts the above, and the proof is complete. □

Theorem 2.4. *Suppose that A is a closed subspace of X and X is WUR (Definition 1.2), then X/A is WUR.*

Proof. The proof is similar to the proof of Theorem 2.3. □

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Wen D. Chang: Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36104, USA
E-mail address: wchang@asunet.alasu.edu

Ping Chang: Singapore Air Accounting Center in Beijing, Beijing, China
E-mail address: lovasun@charter.net