IJMMS 2003:46, 2897–2906
PII. S0161171203303035
http://ijmms.hindawi.com

α-COMPACTNESS IN SMOOTH TOPOLOGICAL SPACES

CHUN-KEE PARK, WON KEUN MIN, and MYEONG HWAN KIM

Received 3 March 2003

We introduce the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined by Demirci (1997) and obtain some of their structural properties.

2000 Mathematics Subject Classification: 54A40.

1. Introduction. Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar et al. [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some properties of them.

In this paper, we define the smooth α-closure and smooth α-interior of a fuzzy set and investigate some of their properties. In fact, the smooth α-closure and smooth α -interior of a fuzzy set coincide with the smooth closure and smooth interior of a fuzzy set defined in [3] when $\alpha = 0$. We also introduce the concepts of several types of α-compactness using smooth α-closure and smooth α-interior of a fuzzy set and investigate some of their properties.

2. Preliminaries. In this section, we give some notations and definitions which are to be used in the sequel. Let X be a set and let $I = [0,1]$ be the unit interval of the real line. Let I^X denote the set of all fuzzy sets of X. Let 0_X and 1_X denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [6] is an ordered pair (X,τ), where X is a nonempty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(1) $\tau(0_X) = \tau(1_X) = 1$;
(2) for all $A,B \in I^X$, $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J}\tau(A_i)$.

Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [6] if and only if the following three conditions are satisfied:

(1) $\tau^*(0_X) = \tau^*(1_X) = 1$;
(2) for all $A,B \in I^X$, $\tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [6].

For the s.t.s. (X,τ) and $\alpha \in [0,1]$, the family $\tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}$ defines a Chang's fuzzy topology (CFT) on X [2]. The family of all closed fuzzy sets with respect to τ_α is denoted by τ_α^* and we have $\tau_\alpha^* = \{A \in I^X : \tau^*(A) \geq \alpha\}$. For $A \in I^X$ and $\alpha \in [0,1]$, the τ_α-closure (resp., τ_α-interior) of A, denoted by $\mathrm{cl}_\alpha(A)$ (resp., $\mathrm{int}_\alpha(A)$), is defined by $\mathrm{cl}_\alpha(A) = \cap\{K \in \tau_\alpha^* : A \subseteq K\}$ (resp., $\mathrm{int}_\alpha(A) = \cup\{K \in \tau_\alpha : K \subseteq A\}$).

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows.

Let (X,τ) be an s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\bar{A}$ (resp., A^o), is defined by $\bar{A} = \cap\{K \in I^X : \tau^*(K) > 0,\ A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0,\ K \subseteq A\}$).

Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [6] if and only if $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [6] if and only if $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [6].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [6] if and only if $\tau(A) \leq \sigma(f(A))$ (resp., $\tau^*(A) \leq \sigma^*(f(A))$) for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] if and only if $\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B))$) for every $A,B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A,B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] if and only if $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A,B \in I^X$.

3. Smooth α-closure and smooth α-interior. In this section, we introduce the concepts of smooth α-closure and smooth α-interior of a fuzzy set in smooth topological spaces and investigate some properties of them.

Definition 3.1. Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A),\ K \subseteq A\}$).

Theorem 3.2. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A,B \in I^X$. Then*
(a) $\tau^*(\overline{A}_\alpha) \geq \alpha\tau^*(A)$,
(b) $\tau(A^o_\alpha) \geq \alpha\tau(A)$,
(c) $A \subseteq B$ *and* $\tau^*(A) \leq \tau^*(B) \Rightarrow \overline{A}_\alpha \subseteq \overline{B}_\alpha$,
(d) $A \subseteq B$ *and* $\tau(B) \leq \tau(A) \Rightarrow A^o_\alpha \subseteq B^o_\alpha$.

Proof. (a) and (b) follow directly from Definition 3.1.

(c) If $A \subseteq B$ and $\tau^*(A) \leq \tau^*(B)$, then $K \in \{K \in I^X : \tau^*(K) > \alpha\tau^*(B),\ B \subseteq K\} \Rightarrow K \in \{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}$. Hence $\overline{A}_\alpha \subseteq \overline{B}_\alpha$.

(d) The proof is similar to the proof of (c). □

Theorem 3.3. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A \in I^X$. Then*
(a) $(\overline{A}_\alpha)^c = (A^c)^o_\alpha$,
(b) $\overline{A}_\alpha = ((A^c)^o_\alpha)^c$,
(c) $(A^o_\alpha)^c = \overline{(A^c)}_\alpha$,
(d) $A^o_\alpha = (\overline{(A^c)}_\alpha)^c$.

Proof. (a) From Definition 3.1, we have

$$\begin{aligned}
(\overline{A}_\alpha)^c &= \left(\cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}\right)^c \\
&= \cup\{K^c : K \in I^X, \tau(K^c) = \tau^*(K) > \alpha\tau^*(A) = \alpha\tau(A^c),\ K^c \subseteq A^c\} \\
&= \cup\{U \in I^X : \tau(U) > \alpha\tau(A^c),\ U \subseteq A^c\} \\
&= (A^c)^o_\alpha.
\end{aligned} \tag{3.1}$$

(b), (c), and (d) are easily obtained from (a). □

Theorem 3.4. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A,B \in I^X$. Then*
(a) $\overline{(0_X)}_\alpha = 0_X$,
(b) $A \subseteq \overline{A}_\alpha$,
(c) $\overline{A}_\alpha \subseteq \overline{(\overline{A}_\alpha)}_\alpha$,
(d) $\overline{A}_\alpha \cap \overline{B}_\alpha \subseteq \overline{(A \cup B)}_\alpha$.

Proof. (a) and (b) are easily obtained from Definition 3.1. (c) follows directly from (b).

(d) For every $A,B \in I^X$, we have

$$\begin{aligned}
\overline{(A \cup B)}_\alpha &= \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A \cup B),\ A \cup B \subseteq K\} \\
&\supseteq \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A) \wedge \alpha\tau^*(B),\ A \cup B \subseteq K\}
\end{aligned}$$

$$
\begin{aligned}
&= \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A) \\
&\qquad \text{or } \tau^*(K) > \alpha\tau^*(B),\ A \subseteq K,\ B \subseteq K\} \\
&= \cap\{K \in I^X : (\tau^*(K) > \alpha\tau^*(A),\ A \subseteq K,\ B \subseteq K) \\
&\qquad \text{or } (\tau^*(K) > \alpha\tau^*(B),\ A \subseteq K,\ B \subseteq K)\} \\
&\supseteq \cap[\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\} \\
&\qquad \cup \{K \in I^X : \tau^*(K) > \alpha\tau^*(B),\ B \subseteq K\}] \\
&= [\cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}] \\
&\quad \cap [\cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(B),\ B \subseteq K\}] \\
&= \overline{A}_\alpha \cap \overline{B}_\alpha .
\end{aligned}
\tag{3.2}
$$

□

THEOREM 3.5. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A,B \in I^X$. Then*
(a) $(1_X)^o_\alpha = 1_X$,
(b) $A^o_\alpha \subseteq A$,
(c) $(A^o_\alpha)^o_\alpha \subseteq A^o_\alpha$,
(d) $(A \cap B)^o_\alpha \subseteq A^o_\alpha \cup B^o_\alpha$.

PROOF. The proof is similar to the proof of Theorem 3.4. □

THEOREM 3.6. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A \in I^X$. Then*
(a) $\tau^*(A) > 0 \Rightarrow \overline{A}_\alpha = A$,
(b) $\tau(A) > 0 \Rightarrow A^o_\alpha = A$.

PROOF. (a) Let $\tau^*(A) > 0$. Then $A \in \{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}$. By Definition 3.1, $\overline{A}_\alpha \subseteq A$. By Theorem 3.4 , $A \subseteq \overline{A}_\alpha$. Hence $\overline{A}_\alpha = A$.

(b) Let $\tau(A) > 0$. Then $A \in \{K \in I^X : \tau(K) > \alpha\tau(A),\ K \subseteq A\}$. By Definition 3.1, $A \subseteq A^o_\alpha$. By Theorem 3.5, $A^o_\alpha \subseteq A$. Hence $A^o_\alpha = A$. □

REMARK 3.7. Let (X,τ) be an s.t.s., $\alpha_1,\alpha_2 \in [0,1)$ with $\alpha_1 \le \alpha_2$, and $A \in I^X$. Then $\overline{A}_{\alpha_1} \subseteq \overline{A}_{\alpha_2}$ and $A^o_{\alpha_2} \subseteq A^o_{\alpha_1}$.

THEOREM 3.8. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and $A \in I^X$. Then*
(a) $\overline{A}_\alpha = \cap_{\beta > \alpha\tau^*(A)} \mathrm{cl}_\beta(A)$,
(b) $A^o_\alpha = \cup_{\beta > \alpha\tau(A)} \mathrm{int}_\beta(A)$.

PROOF. (a) For each $x \in X$, we have

$$
\begin{aligned}
\overline{A}_\alpha(x) &= [\cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\}](x) \\
&= \inf\{K(x) : K \in I^X,\ \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\} \\
&= \inf_{\beta > \alpha\tau^*(A)} \inf\{K(x) : K \in I^X,\ \tau^*(K) \ge \beta,\ A \subseteq K\}
\end{aligned}
$$

$$\begin{aligned}&=\inf_{\beta>\alpha\tau^*(A)}\left[\cap\{K\in I^X:\tau^*(K)\geq\beta,\ A\subseteq K\}\right](x)\\&=\inf_{\beta>\alpha\tau^*(A)}\mathrm{cl}_\beta(A)(x)\\&=\left[\cap_{\beta>\alpha\tau^*(A)}\mathrm{cl}_\beta(A)\right](x).\end{aligned}\tag{3.3}$$

Hence, $\overline{A}_\alpha=\cap_{\beta>\alpha\tau^*(A)}\mathrm{cl}_\beta(A)$.

(b) The proof is similar to that of (a). □

Remark 3.9. Let (X,τ) be an s.t.s., $\alpha\in[0,1)$, and $A\in I^X$. From Theorems 3.4, 3.5, and 3.8, we easily obtain the following:

(a) if there exists a $\beta\in(\alpha\tau^*(A),1]$ such that $A=\mathrm{cl}_\beta(A)$, then $A=\overline{A}_\alpha$;

(b) if there exists a $\beta\in(\alpha\tau(A),1]$ such that $A=\mathrm{int}_\beta(A)$, then $A=A^o_\alpha$.

Definition 3.10. Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha\in[0,1)$. A function $f:X\to Y$ is called smooth α-preserving (resp., strict smooth α-preserving) with respect to τ and σ if and only if $\sigma(A)\geq\alpha\sigma(B)\Leftrightarrow\tau(f^{-1}(A))\geq\alpha\tau(f^{-1}(B))$ (resp., $\sigma(A)>\alpha\sigma(B)\Leftrightarrow\tau(f^{-1}(A))>\alpha\tau(f^{-1}(B))$) for every $A,B\in I^Y$.

A function $f:X\to Y$ is called smooth open α-preserving (resp., strict smooth open α-preserving) with respect to τ and σ if and only if $\tau(A)\geq\alpha\tau(B)\Rightarrow\sigma(f(A))\geq\alpha\sigma(f(B))$ (resp., $\tau(A)>\alpha\tau(B)\Rightarrow\sigma(f(A))>\alpha\sigma(f(B))$) for every $A,B\in I^X$.

Theorem 3.11. *Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha\in[0,1)$. If $f:X\to Y$ is a smooth α-preserving function (resp., a strict smooth α-preserving function) with respect to τ and σ, then $\sigma^*(A)\geq\alpha\sigma^*(B)\Leftrightarrow\tau^*(f^{-1}(A))\geq\alpha\tau^*(f^{-1}(B))$ (resp., $\sigma^*(A)>\alpha\sigma^*(B)\Leftrightarrow\tau^*(f^{-1}(A))>\alpha\tau^*(f^{-1}(B))$) for every $A,B\in I^Y$.*

Proof. If $f:X\to Y$ is a smooth α-preserving function with respect to τ and σ, then

$$\begin{aligned}\sigma^*(A)\geq\alpha\sigma^*(B)&\Longleftrightarrow\sigma(A^c)\geq\alpha\sigma(B^c)\\&\Longleftrightarrow\tau\left(f^{-1}(A^c)\right)\geq\alpha\tau\left(f^{-1}(B^c)\right)\\&\Longleftrightarrow\tau\left(\left(f^{-1}(A)\right)^c\right)\geq\alpha\tau\left(\left(f^{-1}(B)\right)^c\right)\\&\Longleftrightarrow\tau^*\left(f^{-1}(A)\right)\geq\alpha\tau^*\left(f^{-1}(B)\right)\end{aligned}\tag{3.4}$$

for every $A,B\in I^Y$.

The proof is similar when $f:X\to Y$ is a strict smooth α-preserving function with respect to τ and σ. □

Theorem 3.12. *Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is injective and strict smooth α-preserving with respect to τ and σ, then $f(\overline{A}_\alpha) \subseteq \overline{(f(A))}_\alpha$ for every $A \in I^X$.*

Proof. For every $A \in I^X$, we have

$$\begin{aligned}
f^{-1}(\overline{(f(A))}_\alpha) &= f^{-1}[\cap\{U \in I^Y : \sigma^*(U) > \alpha\sigma^*(f(A)),\ f(A) \subseteq U\}] \\
&\supseteq f^{-1}[\cap\{U \in I^Y : \tau^*(f^{-1}(U)) > \alpha\tau^*(A),\ A \subseteq f^{-1}(U)\}] \\
&= \cap\{f^{-1}(U) \in I^X : \tau^*(f^{-1}(U)) > \alpha\tau^*(A),\ A \subseteq f^{-1}(U)\} \\
&\supseteq \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A),\ A \subseteq K\} \\
&= \overline{A}_\alpha.
\end{aligned} \tag{3.5}$$

Hence, $f(\overline{A}_\alpha) \subseteq \overline{(f(A))}_\alpha$. □

Theorem 3.13. *Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is strict smooth α-preserving with respect to τ and σ, then*

(a) $\overline{(f^{-1}(A))}_\alpha \subseteq f^{-1}(\overline{A}_\alpha)$ *for every* $A \in I^Y$,

(b) $f^{-1}(A^o_\alpha) \subseteq (f^{-1}(A))^o_\alpha$ *for every* $A \in I^Y$.

Proof. (a) For every $A \in I^Y$, we have

$$\begin{aligned}
f^{-1}(\overline{A}_\alpha) &= f^{-1}[\cap\{U \in I^Y : \sigma^*(U) > \alpha\sigma^*(A),\ A \subseteq U\}] \\
&\supseteq f^{-1}[\cap\{U \in I^Y : \tau^*(f^{-1}(U)) > \alpha\tau^*(f^{-1}(A)),\ f^{-1}(A) \subseteq f^{-1}(U)\}] \\
&= \cap\{f^{-1}(U) \in I^X : \tau^*(f^{-1}(U)) > \alpha\tau^*(f^{-1}(A)),\ f^{-1}(A) \subseteq f^{-1}(U)\} \\
&\supseteq \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(f^{-1}(A)),\ f^{-1}(A) \subseteq K\} \\
&= \overline{(f^{-1}(A))}_\alpha.
\end{aligned} \tag{3.6}$$

(b) For every $A \in I^Y$, we have

$$\begin{aligned}
f^{-1}(A^o_\alpha) &= f^{-1}[\cup\{U \in I^Y : \sigma(U) > \alpha\sigma(A),\ U \subseteq A\}] \\
&\subseteq f^{-1}[\cup\{U \in I^Y : \tau(f^{-1}(U)) > \alpha\tau(f^{-1}(A)),\ f^{-1}(U) \subseteq f^{-1}(A)\}] \\
&= \cup\{f^{-1}(U) \in I^X : \tau(f^{-1}(U)) > \alpha\tau(f^{-1}(A)),\ f^{-1}(U) \subseteq f^{-1}(A)\} \\
&\subseteq \cup\{K \in I^X : \tau(K) > \alpha\tau(f^{-1}(A)),\ K \subseteq f^{-1}(A)\} \\
&= (f^{-1}(A))^o_\alpha.
\end{aligned} \tag{3.7}$$

□

Theorem 3.14. *Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is strict smooth open α-preserving with respect to τ and σ, then $f(A_\alpha^o) \subseteq (f(A))_\alpha^o$ for every $A \in I^X$.*

Proof. For every $A \in I^X$, we have

$$\begin{aligned} f(A_\alpha^o) &= f[\cup\{U \in I^X : \tau(U) > \alpha\tau(A),\ U \subseteq A\}] \\ &\subseteq f[\cup\{U \in I^X : \sigma(f(U)) > \alpha\sigma(f(A)),\ f(U) \subseteq f(A)\}] \\ &= \cup\{f(U) \in I^Y : \sigma(f(U)) > \alpha\sigma(f(A)),\ f(U) \subseteq f(A)\} \\ &= \cup\{K \in I^Y : \sigma(K) > \alpha\sigma(f(A)),\ K \subseteq f(A)\} \\ &= (f(A))_\alpha^o. \end{aligned} \tag{3.8}$$

□

4. Types of smooth α-compactness. In this section, we introduce the concepts of several types of smooth α-compactness in smooth topological spaces and investigate some properties of them.

Definition 4.1 [5]. An s.t.s. (X,τ) is called smooth compact if and only if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$.

Theorem 4.2 [4]. *Let (X,τ) and (Y,σ) be two smooth topological spaces and $f : X \to Y$ a surjective weakly smooth continuous function with respect to τ and σ. If (X,τ) is smooth compact, then so is (Y,σ).*

Definition 4.3. Let $\alpha \in [0,1)$. An s.t.s. (X,τ) is called smooth nearly α-compact if and only if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} (\overline{(A_i)}_\alpha)_\alpha^o = 1_X$.

Definition 4.4. Let $\alpha \in [0,1)$. An s.t.s. (X,τ) is called smooth almost α-compact if and only if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} \overline{(A_i)}_\alpha = 1_X$.

Definition 4.5. Let $\alpha \in [0,1)$. An s.t.s. (X,τ) is called smooth α-regular if and only if each fuzzy set $A \in I^X$ satisfying $\tau(A) > 0$ can be written as $A = \cup\{K \in I^X : \tau(K) \geq \tau(A),\ \overline{K}_\alpha \subseteq A\}$.

Definition 4.6. A smooth topology $\tau : I^X \to I$ on X is called monotonic increasing (resp., monotonic decreasing) if and only if $A \subseteq B \Rightarrow \tau(A) \leq \tau(B)$ (resp., $A \subseteq B \Rightarrow \tau(A) \geq \tau(B)$) for every $A,B \in I^X$.

Theorem 4.7. *Let (X,τ) be an s.t.s., $\alpha \in [0,1)$, and τ a monotonic decreasing smooth topology. If (X,τ) is smooth compact, then (X,τ) is smooth nearly α-compact.*

Proof. Let (X,τ) be a smooth compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$. Since $\tau(A_i) > 0$ for each $i \in J$, we have $A_i = (A_i)^o_\alpha$ for each $i \in J$ by Theorem 3.6. Since τ is monotonic decreasing and $A_i \subseteq \overline{(A_i)}_\alpha$ for each $i \in J$, we have $\tau(A_i) \geq \tau(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence from Theorem 3.2, we have $A_i = (A_i)^o_\alpha \subseteq (\overline{(A_i)}_\alpha)^o_\alpha$ for each $i \in J$. Thus $1_X = \cup_{i\in J_0} A_i \subseteq \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha$, that is, $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_X$. Hence (X,τ) is smooth nearly α-compact. □

Theorem 4.8. *Let $\alpha \in [0,1)$. Then a smooth nearly α-compact s.t.s. (X,τ) is smooth almost α-compact.*

Proof. Let (X,τ) be a smooth nearly α-compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_X$. Since $(\overline{(A_i)}_\alpha)^o_\alpha \subseteq \overline{(A_i)}_\alpha$ for each $i \in J$ by Theorem 3.5, $1_X = \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha \subseteq \cup_{i\in J_0}\overline{(A_i)}_\alpha$. So $\cup_{i\in J_0}\overline{(A_i)}_\alpha = 1_X$. Hence (X,τ) is smooth almost α-compact. □

Theorem 4.9. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$, and $f : X \to Y$ a surjective, weakly smooth continuous, and strict smooth α-preserving function with respect to τ and σ. If (X,τ) is smooth almost α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^Y : \sigma(A) > 0\}$ covering Y, that is, $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is weakly smooth continuous, $\tau(f^{-1}(A_i)) > 0$ for each $i \in J$. Since (X,τ) is smooth almost α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{(f^{-1}(A_i))}_\alpha = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}\overline{(f^{-1}(A_i))}_\alpha) = \cup_{i\in J_0} f(\overline{(f^{-1}(A_i))}_\alpha)$. Since $f : X \to Y$ is strict smooth α-preserving with respect to τ and σ, from Theorem 3.13 we have $\overline{(f^{-1}(A_i))}_\alpha \subseteq f^{-1}(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence we have $1_Y = \cup_{i\in J_0} f(\overline{(f^{-1}(A_i))}_\alpha) \subseteq \cup_{i\in J_0} f(f^{-1}(\overline{(A_i)}_\alpha)) = \cup_{i\in J_0}\overline{(A_i)}_\alpha$, that is, $\cup_{i\in J_0}\overline{(A_i)}_\alpha = 1_Y$. Thus (Y,σ) is smooth almost α-compact. □

We obtain the following corollary from Theorems 4.8 and 4.9.

Corollary 4.10. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$, and $f : X \to Y$ a surjective, weakly smooth continuous, and strict smooth α-preserving function with respect to τ and σ. If (X,τ) is smooth nearly α-compact, then (Y,σ) is smooth almost α-compact.*

Theorem 4.11. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$, and $f : X \to Y$ a surjective, weakly smooth continuous, strict smooth α-preserving, and strict smooth open α-preserving function with respect to τ and σ. If (X,τ) is smooth nearly α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^Y : \sigma(A) > 0\}$ covering Y, that is, $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is weakly smooth continuous, $\tau(f^{-1}(A_i)) > 0$ for each $i \in J$. Since (X,τ) is smooth nearly α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha = 1_X$.

From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}(\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha) = \cup_{i\in J_0} f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha)$. Since $f : X \to Y$ is strict smooth open α-preserving with respect to τ and σ, from Theorem 3.14 we have $f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha) \subseteq (f(\overline{(f^{-1}(A_i))}_\alpha))^o_\alpha$ for each $i \in J$. Since $f : X \to Y$ is strict smooth α -preserving with respect to τ and σ, from Theorem 3.13 we have $\overline{(f^{-1}(A_i))}_\alpha \subseteq f^{-1}(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence, we have

$$\begin{aligned} 1_Y &= \cup_{i\in J_0} f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha) \\ &\subseteq \cup_{i\in J_0}(f(\overline{(f^{-1}(A_i))}_\alpha))^o_\alpha \\ &\subseteq \cup_{i\in J_0}(f(f^{-1}(\overline{(A_i)}_\alpha)))^o_\alpha \\ &= \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha. \end{aligned} \tag{4.1}$$

Hence, $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_Y$. Thus (Y,σ) is smooth nearly α-compact. □

Theorem 4.12. *Let $\alpha \in [0,1)$. Then a smooth almost α-compact smooth α-regular s.t.s. (X,τ) is smooth compact.*

Proof. Let $\{A_i : i \in J\}$ be a family in $\{A \in I^X : \sigma(A) > 0\}$ covering X, that is, $\cup_{i\in J} A_i = 1_X$. Since (X,τ) is smooth α-regular, $A_i = \cup_{j_i\in J_i}\{K_{j_i} \in I^X : \tau(K_{j_i}) \geq \tau(A_i),\ \overline{(K_{j_i})}_\alpha \subseteq A_i\}$ for each $i \in J$. Since $\cup_{i\in J} A_i = \cup_{i\in J}[\cup_{j_i\in J_i} K_{j_i}]$ $= 1_X$ and (X,τ) is smooth almost α-compact, there exists a finite subfamily $\{K_l \in I^X : \tau(K_l) > 0,\ l \in L\}$ such that $\cup_{l\in L}\overline{(K_l)}_\alpha = 1_X$. Since for each $l \in L$ there exists $i \in J$ such that $\overline{(K_l)}_\alpha \subseteq A_i$, we have $\cup_{i\in J_0} A_i = 1_X$, where J_0 is a finite subset of J. Hence (X,τ) is smooth compact. □

We obtain the following corollary from Theorems 4.8 and 4.12.

Corollary 4.13. *Let $\alpha \in [0,1)$. Then a smooth nearly α-compact smooth α-regular s.t.s. (X,τ) is smooth compact.*

References

[1] R. Badard, *Smooth axiomatics*, First IFSA Congress, Palma de Mallorca, 1986.

[2] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[3] M. Demirci, *On several types of compactness in smooth topological spaces*, Fuzzy Sets and Systems **90** (1997), no. 1, 83–88.

[4] ———, *Three topological structures of smooth topological spaces*, Fuzzy Sets and Systems **101** (1999), no. 1, 185–190.

[5] M. K. El Gayyar, E. E. Kerre, and A. A. Ramadan, *Almost compactness and near compactness in smooth topological spaces*, Fuzzy Sets and Systems **62** (1994), no. 2, 193–202.

[6] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48** (1992), no. 3, 371–375.

Chun-Kee Park: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
E-mail address: ckpark@kangwon.ac.kr

Won Keun Min: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
E-mail address: wkmin@cc.kangwon.ac.kr

Myeong Hwan Kim: Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
E-mail address: kimmw@kangwon.ac.kr