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UNIQUENESS AND RADIAL SYMMETRY FOR AN INVERSE ELLIPTIC EQUATION

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We consider an inverse rearrangement semilinear partial differential equation in a 2-dimensional ball and show that it has a unique maximizing energy solution. The solution represents a confined steady flow containing a vortex and passing over a seamount. Our approach is based on a rearrangement variational principle extensively developed by G. R. Burton.

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1. Introduction. This paper is concerned with the following problem in a bounded domain Ω:

$$\begin{gathered} -\Delta u = \phi(u) + h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0, \qquad -\Delta u \in \mathcal{F} + h, \end{gathered} \tag{1.1}$$

where Ω is some bounded domain in $\mathbb{R}^2$. In (1.1), the *nonlinearity* ϕ is unknown, and $\mathcal{F}$ is a family of functions which are *rearrangements* of a prescribed function, hence problem (1.1) is named an *inverse rearrangement semilinear elliptic equation*. Therefore, by a solution for (1.1) we mean a pair (u, ϕ) which satisfies all conditions (in some sense) of (1.1). Here we are concerned with special types of solutions for (1.1); namely, the *energy maximizing solutions*. To state the definition of such solutions, we first need some preparations.

Henceforth p is a fixed number in $(2, \infty)$ and q is its conjugate exponent, so $1/p + 1/q = 1$. The so-called *height* function h is some nonnegative function in $L^p(\Omega)$. We let $K : L^p(\Omega) \to H_0^1(\Omega)$ denote the standard inverse of $-\Delta$ with Dirichlet homogeneous boundary conditions in Ω. We recall that K is *continuous* and *positive*; that is,

$$\int_\Omega \zeta K \zeta > 0 \quad \forall \zeta \in L^p(\Omega). \tag{1.2}$$

Finally note that K is *symmetric*:

$$\int_\Omega \zeta K \zeta' = \int_\Omega \zeta' K \zeta \quad \forall \zeta, \zeta' \in L^p(\Omega). \tag{1.3}$$

Now we can set up the *energy functional* associated with (1.1). We define $\Psi : L^p(\Omega) \to \infty$ as follows:

$$\Psi(\zeta) = \frac{1}{2}\int_\Omega \zeta K\zeta + \int_\Omega \eta\zeta, \tag{1.4}$$

where $\eta = Kh$. Next we define the variational problem

$$\sup_{\zeta\in\mathcal{F}} \Psi(\zeta), \tag{1.5}$$

where $\mathcal{F}$ denotes the set of rearrangements of some nonnegative function $\zeta_0 \in L^p(\Omega)$. We recall that ζ is a *rearrangement* of ζ_0 whenever the sets

$$\{x \in \Omega : \zeta(x) \geq \alpha\}, \qquad \{x \in \Omega : \zeta_0(x) \geq \alpha\} \tag{1.6}$$

have the same Lebesgue measures for every positive α. Note that all members $\zeta \in \mathcal{F}$ satisfy

$$\|\zeta\|_p = \|\zeta_0\|_p, \tag{1.7}$$

where $\|\cdot\|_p$ denotes the usual norm in $L^p(\Omega)$. The solution set for (1.5) is denoted Σ.

Definition 1.1. The pair (u,ϕ) is called a maximizing energy solution of (1.1) whenever the following conditions are satisfied:

(i) $u \in K(\Sigma) + h$,

(ii) (u,ϕ) is a solution of (1.1).

In (i) we have

$$K(\Sigma) = \{K\zeta : \zeta \in \Sigma\}. \tag{1.8}$$

The main result of this paper is the following theorem.

Theorem 1.2. *If Ω is a ball centered at the origin, then there exists a unique u and there exists an increasing function ϕ such that (u,ϕ) is a maximizing energy solution for (1.1).*

We end this section with some history of problem (1.1). This problem was first considered in an unbounded domain, precisely in the whole of $\mathbb{R}^2$, by Emamizadeh and Nycander [7]. Later Emamizadeh and Bahrami [6] considered the problem in the half-plane. In the case of unbounded domain, we usually face the *lack of compactness* which causes unavailability of the direct method in the analysis. In the present situation, we do not need to worry about the existence of a solution since this will readily be provided using results of Burton [2] about maximization of convex functionals over the sets of rearrangements. However, the point here is the uniqueness that we usually do not obtain when dealing with unbounded domains. The reader could also be referred to [3, 4, 5] for similar problems in unbounded domains.

2. Preliminary results. In this section, we state some lemmas which will be used in the proof of Theorem 1.2.

Lemma 2.1. *Let $\Phi : L^p(B) \to \infty$ be strictly convex, weakly sequentially continuous, and Gateaux differentiable. Then the variational problem*

$$\sup_{\zeta \in \mathcal{F}} \Phi(\zeta) \tag{2.1}$$

is solvable. Moreover, if $\hat{\zeta} \in \mathcal{F}$ is any such solution, then

$$\hat{\zeta} = \phi \circ \Phi'(\hat{\zeta}) \tag{2.2}$$

for some increasing function ϕ unknown a priori.

If $u \in H_0^1(\mathbb{R}^n)$ is nonnegative, u^* will denote the essentially unique spherically symmetric radially decreasing rearrangement of u; then $u^* \in H_0^1(\mathbb{R}^n)$ also, and the inequality

$$\int_{\mathbb{R}^n} |\nabla u^*|^2 \le \int_{\mathbb{R}^n} |\nabla u|^2 \tag{2.3}$$

is standard. The case of equality has been studied by Brothers and Ziemer [1]; they proved results from which the following lemma can be deduced.

Lemma 2.2. *Let $u \in H_0^1(\mathbb{R}^n)$ be nonnegative and have compact support, and let $M = \operatorname{ess\,sup} u$ (which may be infinite). Suppose that*

$$\int_{\mathbb{R}^n} |\nabla u^*|^2 = \int_{\mathbb{R}^n} |\nabla u|^2. \tag{2.4}$$

Then,

(1) *for $0 \le \alpha < M$, $u^{-1}(\alpha, \infty)$ is a translate of the ball $(u^*)^{-1}(\alpha, \infty)$, apart from a set of measure zero;*

(2) *if additionally*

$$\{x \in \mathbb{R}^n : \nabla u^*(x) = 0,\ 0 < u^*(x) < M\} \tag{2.5}$$

is a set of zero measure, then u is a translate of u^*.

The following lemma is an immediate consequence of [1, Lemma 2.3(v) and the succeeding remark].

Lemma 2.3. *Let $u \in H_0^1(\mathbb{R}^n)$ be nonnegative and $M = \operatorname{ess\,sup} u$. If*

$$\{x \in \mathbb{R}^n : \nabla u(x) = 0,\ 0 < u(x) < M\} \tag{2.6}$$

has zero measure, then

$$\{x \in \mathbb{R}^n : \nabla u^*(x) = 0,\ 0 < u^*(x) < M\} \tag{2.7}$$

also has zero measure.

3. Proof of the theorem. We begin by considering the solvability of (1.5). Indeed, using elliptic regularity theory, it is clear that $K : L^p(B) \to W^{2,p}(B)$ is a continuous linear operator. Since $W^{2,p}(B)$ is compactly embedded into $C^1(\overline{B})$, it follows that $K : L^p(B) \to L^q(B)$ is a linear compact operator. Therefore, Ψ turns to be a weakly sequentially continuous functional. Moreover, since K is positive and symmetric, it follows that Ψ is also strictly convex. The Gateaux differentiability of Ψ is straightforward; and it is easy to see that the derivative of Ψ at v can be identified with $Kv+\eta$. From all this we can see that Lemma 2.1 is applicable. So (1.5) is solvable, and if $\hat{\zeta}$ is any solution of (1.5), then

$$\hat{\zeta} = \phi(K\hat{\zeta}+\eta), \tag{3.1}$$

almost everywhere in B, for an increasing function ϕ.

We set $H_0^1(B) \equiv \mathcal{H}$, and the norm on $\mathcal{H}$ is denoted $\|u\| = (\int_B |\nabla u|^2)^{1/2}$. We define a parametrized convex functional $\mathcal{K}$ by

$$\mathcal{K}_c(u) = \begin{cases} \dfrac{1}{2}\|u\| - \displaystyle\int_B hu + c, & u \in \mathcal{H}, \\ \infty, & u \in L^q(B) \setminus \mathcal{H}, \end{cases} \tag{3.2}$$

where c is a real parameter. We now consider the conjugate convex functional $\mathcal{K}_c^*$ of $\mathcal{K}_c$ defined by

$$\mathcal{K}_c^*(v) = \sup_{u \in L^q(B)} \left(\int_B uv - \mathcal{K}_c(u) \right), \quad v \in L^p(B). \tag{3.3}$$

Recalling the variational setup for K, it is easy to obtain

$$\mathcal{K}_c^*(v) = \frac{1}{2}\int_B (v+h)K(v+h) - c, \tag{3.4}$$

from which, by setting $c = 1/2\int_B h\eta$, and from the symmetry property of K we infer that

$$\mathcal{K}_c^* = \Psi. \tag{3.5}$$

We fix a nonnegative function $v \in L^p(B)$. Then the supremum in (3.3) is attained at $u \equiv Kv+\eta$. Therefore, from (3.3), we obtain

$$\Psi(v) + \mathcal{K}_c(u) = \int_B uv. \tag{3.6}$$

Again, from (3.3), we infer that

$$\Psi(v^*) + \mathcal{K}_c(u^*) \geq \int_B u^* v^*. \tag{3.7}$$

So from (3.6), (3.7), and a standard rearrangement inequality, it follows that

$$\Psi(v^*) + \mathcal{K}_c(u^*) \geq \Psi(v) + \mathcal{K}_c(u). \tag{3.8}$$

At this stage, we make another assumption; namely, we suppose that $v \in \Sigma$. Since (1.5) is solvable, Σ is not empty. Thus, from (3.8), we infer that

$$\mathcal{K}_c(u^*) \geq \mathcal{K}_c(u). \tag{3.9}$$

Therefore,

$$\frac{1}{2}||u^*||^2 - \int_B hu^* \geq \frac{1}{2}\|u\|^2 - \int_B hu. \tag{3.10}$$

Since $\int_B hu \leq \int_B hu^*$, it follows from (3.10) that $\|u^*\| \geq \|u\|$. So, in view of (2.3), we deduce that $\|u\| = \|u^*\|$.

Claim. We have $u = u^*$.

Proof of the claim. From the maximum principle and elliptic regularity theory, it follows that u is a positive function in $C^1(\overline{B})$. We fix $x_1 \in B$. The set

$$S \equiv \{x \in B : u(x) \geq u(x_1)\} = u^{-1}([u(x_1), \infty)) \tag{3.11}$$

is a ball according to Lemma 2.1. If $x \in \operatorname{int} S$, the interior of S, then, by the maximum principle, $u(x) > u(x_1)$; thus $x_1 \in \partial S$, the boundary of S. Now we can apply the Hopf boundary point lemma to deduce that $\partial u/\partial \nu(x_1) < 0$, where ν is the unit normal to ∂S at x_1 pointing outward. Therefore, the set

$$\{x \in B : \nabla u(x) = 0,\ 0 < u(x) < M\}, \tag{3.12}$$

where $M = \max_\Omega u$, is empty, so its measure is zero. Hence, from Lemma 2.3, the set $\{x \in B : \nabla u^*(x) = 0,\ 0 < u^*(x) < M\}$ also has zero measure. Therefore, by Lemma 2.2, it follows that u is a translate of u^*. However, since u is a positive function, we infer that $u = u^*$ as desired. This completes the proof of the claim. □

Note that, from (3.1), we have

$$v = \phi(u), \tag{3.13}$$

almost everywhere in B, for some increasing function ϕ. So v is also spherically symmetric and radially decreasing; hence, $v = v^* = \zeta_0^*$. Since $-\Delta u = v + h$, it follows that

$$-\Delta u = \phi(u) + h, \tag{3.14}$$

which is the differential equation in (1.1). It is easy to check that (u, ϕ) satisfies all other conditions in (1.1), so (u, ϕ) is a maximizing energy solution of (1.1) as desired. The function u is obviously unique; in fact,

$$u = K\zeta_0^*. \tag{3.15}$$

Thus the proof of the theorem is completed.

References

[1] J. E. Brothers and W. P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. reine angew. Math. **384** (1988), 153–179.

[2] G. R. Burton, *Rearrangements of functions, maximization of convex functionals, and vortex rings*, Math. Ann. **276** (1987), no. 2, 225–253.

[3] G. R. Burton and B. Emamizadeh, *A constrained variational problem for steady vortices in a shear flow*, Comm. Partial Differential Equations **24** (1999), no. 7-8, 1341–1365.

[4] B. Emamizadeh, *Steady vortex in a uniform shear flow of an ideal fluid*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 4, 801–812.

[5] ———, *Existence of a steady flow with a bounded vortex in an unbounded domain*, J. Sci. Islam. Repub. Iran **12** (2001), no. 1, 57–63.

[6] B. Emamizadeh and F. Bahrami, *Steady vortex flows obtained from an inverse problem*, Bull. Austral. Math. Soc. **66** (2002), no. 2, 213–226.

[7] J. Nycander and B. Emamizadeh, *Variational problem for vortices attached to seamount*, to appear in Nonlinear Analysis Theory, Methods and Applications.

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