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SELBERG'S TRACE FORMULA ON THE k-REGULAR TREE AND APPLICATIONS

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We survey graph theoretic analogues of the Selberg trace and pretrace formulas along with some applications. This paper includes a review of the basic geometry of a k-regular tree Ξ (symmetry group, geodesics, horocycles, and the analogue of the Laplace operator). A detailed discussion of the spherical functions is given. The spherical and horocycle transforms are considered (along with three basic examples, which may be viewed as a short table of these transforms). Two versions of the pretrace formula for a finite connected k-regular graph $X \cong \Gamma\backslash\Xi$ are given along with two applications. The first application is to obtain an asymptotic formula for the number of closed paths of length r in X (without backtracking but possibly with tails). The second application is to deduce the chaotic properties of the induced geodesic flow on X (which is analogous to a result of Wallace for a compact quotient of the Poincaré upper half plane). Finally, the Selberg trace formula is deduced and applied to the Ihara zeta function of X, leading to a graph theoretic analogue of the prime number theorem.

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1. Introduction. The Selberg trace formula has been of great interest to mathematicians for almost 50 years. It was introduced by Selberg [16], who also defined the Selberg zeta function by analogy with the Riemann zeta function, to be a product over prime geodesics in a compact Riemann surface. But the analogue of the Riemann hypothesis is provable for the Selberg zeta function. The trace formula shows that there is a relation between the length spectrum of these prime geodesics and the spectrum of the Laplace operator on the surface.

More recently quantum physicists (specifically those working on quantum chaos theory) have been investigating the Selberg trace formula and its generalizations, because it provides a connection between classical and quantum physics, see Hurt [11]. In fact, of late there has been much communication between mathematicians and physicists on this issue and matters related to the statistics of spectra and zeta zeros. See, for example, Hejhal et al. [10]. Here, our goal is to consider a simpler trace formula and a simpler analogue of Selberg's zeta function. The proofs will require only elementary combinatorics rather than functional analysis. The experimental computations require only a home computer rather than a supercomputer.

The trace formula we discuss here is a graph-theoretic version of Selberg's result. Here, let Ξ be the k-regular tree where $k > 2$ (as the case $k = 2$ is degenerate). This means that Ξ is an infinite connected k-regular graph without cycles. We often write $k = q+1$ as this turns out to be very convenient. In this paper, the Riemann surface, providing a home for the Selberg trace formula, is replaced by a finite k-regular graph X. We can view X as a quotient $\Gamma\backslash\Xi$ where Γ is the fundamental group of the X. There are at least three ways to think about Γ-topological, graph theoretical, and algebraic. We will say more about Γ in Sections 3 and 4. At this point, you can think of Γ as a subgroup of the group of graph isomorphisms of the tree Ξ.

We will normally assume that X is simple (i.e., has no multiple edges or loops), undirected, and connected. We will attempt to keep our discussion of the tree trace formula as elementary as possible and as close to the continuous case in [20] as possible. Thus, you will find in Figures 3.1 and 3.2 a tree analogue of the tessellation of the Poincaré upper half plane by the modular group (see [20, Volume I, page 166]).

This paper is organized as follows. Section 2 concerns the basic geometry of k-regular trees. We consider the geodesics, horocycles, adjacency operator (which gives the tree analogue of the Laplacian of a Riemann surface), and the isomorphism group of the tree. We also investigate the rotation invariant eigenfunctions of the adjacency operator, that is, the spherical functions, in some detail. The horocycle transform of a rotation invariant function f on Ξ is defined in this section.

In Section 3, we obtain two versions of the pretrace formula for a finite k-regular graph X. These formulas involve the spherical transform of a rotation invariant function f on the tree, which is just the tree-inner product of f with a spherical function. The relation between the spherical and horocycle transforms is given in Lemma 3.3. Three examples of horocycle and spherical transforms are computed.

Also, to be found in Section 3, are two applications of pretrace formulas. The first is an asymptotic formula for the number of closed paths of length r in X without backtracking but possibly having tails as r goes to infinity (see (3.28)). Here, "without backtracking but possibly having tails" means that adjacent edges in the path cannot be inverse to each other except possibly for the first and last edges. The second application of a pretrace formula is Theorem 3.8, which says that the intersection of a small rotation invariant set B in X with the image of B propagated forward by the induced geodesic flow on X (when averaged over shells), tends to be what we expect when two random sets intersect in X. This result is a graph-theoretic analogue of a result of Wallace [25], where it is proved that the induced horocycle flow on a compact quotient of the Poincaré upper half plane exhibits chaotic properties in the sense that, in the long term, the area of the intersection of a small rotation invariant set B with the image of B propagated forward by the induced horocycle flow (and averaged over rotations) tends to be what we expect if two random sets intersect. This property is a measure-theoretic analogue of the ergodic "mixing property." Theorem 3.8 gives a graph-theoretic analogue of the Wallace theorem in which the horocycle flow on a compact quotient of the Poincaré upper half plane is replaced with the geodesic flow on a finite k-regular graph. This result also has an interesting combinatorial interpretation, see Example 3.9.

In Section 4, we consider Selberg's trace formula. Here we apply the formula to deduce the basic fact about the Ihara zeta function of a finite regular graph. That is, we show that this zeta function is the reciprocal of a polynomial which is easily computed if one knows the spectrum of the adjacency matrix. And we obtain a graph-theoretic analogue of the prime number theorem (see (4.13)).

Some additional references for the subject are Ahumada [1], Brooks [3], Cartier [5], Figá-Talamanca and Nebbia [7], Hashimoto [9], Quenell [15], Stark and Terras [17, 18], Sunada [19], Terras [21], Venkov and Nikitin [24].

2. Basic facts about k-regular trees

2.1. Geodesics and horocycles. The k-regular tree has a distance function $d(x,y)$ defined for $x,y \in \Xi$ as the number of edges in the unique path connecting x and y. We have a Hilbert space $L^2(\Xi)$ consisting of $f : \Xi \to \mathbb{R}$ such that $(f,f)_\Xi < \infty$, where $(f,g)_\Xi = \sum_{x\in\Xi} f(x)g(x)$.

Our formulas involve the *adjacency operator* A on Ξ which is defined on f in $L^2(\Xi)$ by

$$Af(x) = \sum_{d(x,y)=1} f(y), \tag{2.1}$$

where A is a selfadjoint operator (i.e., $(f,Ag) = (Af,g)$ for $f,g \in L^2(\Xi)$) with continuous spectrum in the interval

$$\left[-2\sqrt{(k-1)}, 2\sqrt{(k-1)}\right]. \tag{2.2}$$

For a proof, see Sunada [19, page 252] and Terras [21, page 410].

However, the adjacency operator is not a compact operator (i.e., there is a bounded sequence f_n such that Af_n does not have a convergent subsequence). Thus, the spectral theorem for A involves a computation of the spectral measure. We summarized the theory for differential operators briefly in Terras [20, Volume I, page 110]. See (3.12) for the spectral measure on the tree. We will not actually need this result here.

A *chain* $c = \{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ in Ξ is a semi-infinite path, that is, vertex x_j is adjacent to vertex x_{j+1}. It is assumed that the chain is without *backtracking*; that is, $x_{n+1} \neq x_{n-1}$. A doubly infinite path is a *geodesic*. It may be viewed as the union of two chains c and c', both of which start at the same vertex x_0.

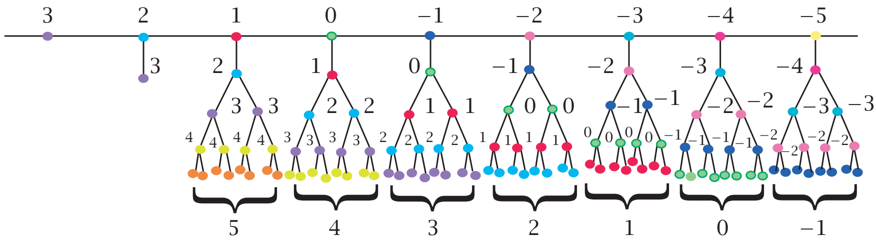

FIGURE 2.1. Horocycles in the 3-regular tree. Points in the nth horocycle are labeled n. We fix a boundary element represented by the chain c and a geodesic $c \cup c'$. Here c consists of the points labeled with $n \le 0$ and c' consists of the points labeled with $n > 0$.

In the Poincaré upper half plane, the y-axis is an example of a geodesic and the horizontal lines perpendicular to it are horocycles. We have an analogous concept for the tree.

The *boundary* Ω of Ξ is the set of equivalence classes of chains where two chains $c = \{x_n \mid n \ge 0\}$ and $c' = \{y_n \mid n \ge 0\}$ are equivalent if they have infinite intersection. If we fix an element $\omega \in \Omega$, a horocycle (with respect to ω) is defined as follows. The chain connecting x in Ξ to infinity along ω is $[x,\infty]$. If x and y are any vertices in Ξ, then $[x,\infty] \cap [y,\infty] = [z,\infty]$. We say that x and y are *equivalent* if $d(x,z) = d(y,z)$. *Horocycles*, with respect to ω, are the equivalence classes for this equivalence relation, see Figure 2.1. Note that horocycles are infinite.

The *horocycle transform* of a function $f : \Xi \to \mathbb{C}$ is a sum over a horocycle $\mathfrak{h}$ of our function f,

$$F(\mathfrak{h}) = \sum_{x \in \mathfrak{h}} f(x). \tag{2.3}$$

If we fix a boundary element ω of Ξ represented by $\{x_n \mid n \ge 0\}$ and a geodesic $c \cup c'$ in our tree Ξ, then we can label the horocycles $\mathfrak{h}$ with numbers as in Figure 2.1, where each element of horocycle $\mathfrak{h}_n$ is labeled n. Assume that f is invariant under rotation about the origin and write $f(x) = f(d(x,\mathfrak{o}))$, where $\mathfrak{o}$ is the origin of our tree Ξ and the point $\mathfrak{o}$ is the intersection of c and c'. Then we find that $F(n) = F(\mathfrak{h}_n) = c_n Hf(n)$, where $c_n = q^n$, if $n > 0$; $c_n = 1$, otherwise. Here Hf is defined by

$$Hf(n) = f(|n|) + (q-1) \sum_{j \ge 1} q^{j-1} f(|n| + 2j), \quad \text{for } n \in \mathbb{Z}. \tag{2.4}$$

This transform is invertible,

$$f(|n|) = Hf(|n|) - (q-1) \sum_{j \ge 1} (Hf)(|n| + 2j). \tag{2.5}$$

2.2. Isomorphism group of the tree. The elements of the group G of graph-theoretic isomorphisms of Ξ may be classified into three types of elements as follows.

Type 1. *Rotations* fix a vertex of Ξ.

Type 2. *Inversions* fix an edge of Ξ and exchange endpoints.

Type 3. *Hyperbolic elements* ρ fix a geodesic $\{x_n \mid n \text{ in } \mathbb{Z}\}$ and $\rho(x_n) = x_{n+s}$. That is, ρ shifts along the geodesic by $s = \nu(\rho)$. (See Figá-Talamanca and Nebbia [7] for a proof of this result.)

Notes. A subgroup of G is the free group generated by k elements. The tree is the *Cayley graph* on this group with edge set the set of k generators and their inverses. By a Cayley graph $X = X(G,S)$, we mean that the vertices are the elements of the group G and there are edges from $g \in G$ to gs for all $s \in S$. The Cayley graph will be undirected if $s \in S$ implies $s^{-1} \in S$.

If Γ is the subgroup of G consisting of covering transformations of a finite graph X, then Γ consists only of hyperbolic elements and the identity.

Now consider our finite connected k-regular graph $X = \Gamma\backslash\Xi$, where Γ is the *fundamental group* of X (which may be viewed as covering transformations of the covering of X by Ξ, its universal cover). Then Γ is a strictly hyperbolic group of automorphisms of Ξ; that is, if $\rho \in \Gamma$ and ρ is not the identity, then ρ acts without fixed points. We say that ρ is *primitive* if it generates the centralizer Γ_ρ of ρ in Γ. Note that Γ_ρ must be cyclic since Γ is free.

2.3. Spherical functions on trees. Next we consider tree-analogues of the Laplace spherical harmonics in Euclidean space. These are also analogous to the spherical functions on the Poincaré upper half plane which come from Legendre functions $P_{-s}(\cosh(r))$, r = geodesic radial distance to origin. See [20, Volume I, page 141], where it is noted that these spherical functions are obtained by averaging power functions $(\operatorname{Im}(z))^s$ over the rotation subgroup $K = \mathrm{SO}(2)$ of $G = \mathrm{SL}(2,\mathbb{R})$.

Fix $\mathfrak{o}$ to be the origin of the tree. Define $h : \Xi \to \mathbb{C}$ to be *spherical* if and only if it has the following three properties:

(1) $h(x) = h(d(x,\mathfrak{o}))$; that is, h is invariant under rotation about the origin $\mathfrak{o}$;

(2) $Ah = \lambda h$; that is, h is an eigenfunction of the adjacency operator;

(3) $h(\mathfrak{o}) = 1$.

The spherical function corresponding to the eigenvalue λ is unique and can be written down in a very elementary and explicit manner (see Brooks [3] and Figá-Talamanca and Nebbia [7]).

Start with the *power function* $p_s(x) = q^{-sd(x,\mathfrak{o})}$ for s in $\mathbb{C}$. Here $q+1 = k$ as usual. Then, as long as $d = d(x,\mathfrak{o})$ is nonzero, we have

$$Ap_s = (q^s + q^{1-s})p_s. \tag{2.6}$$

However, if $d(x,\mathfrak{o}) = 0$, we get $Ap_s = (q+1)p_s$. So, the power function p_s just fails to be an eigenfunction of A. Now write

$$h_s(d) = c(s)p_s(d) + c(1-s)p_{1-s}(d). \tag{2.7}$$

This is analogous to a formula for the spherical functions on the Poincaré upper half plane. (See [20, Volume I, page 144].)

You can use what you know about spherical functions to compute the coefficients $c(s)$. This gives

$$c(s) = \frac{q^{s-1} - q^{1-s}}{(q+1)(q^{s-1} - q^{-s})}, \quad \text{if } q^{2s} \neq 1. \tag{2.8}$$

Writing $z = q^{s-1/2}$ yields

$$h_s(d) = \frac{q^{-d/2}}{q+1}\left\{\frac{qz^{-d}(z^{2+2d}-1) - z^d(1-z^{2-2d})}{z^2-1}\right\}. \tag{2.9}$$

We can rewrite this as a polynomial in z divided by z^d,

$$\begin{aligned} h_s(d) &= \frac{1}{z^d q^{d/2}(q+1)}\left\{q\sum_{j=0}^{d} z^{2j} - \sum_{j=1}^{d-2} z^{2+2j}\right\} \\ &= \frac{1}{z^d q^{d/2}(q+1)}\left\{q + qz^{2d} + (q-1)\sum_{j=1}^{d-1} z^{2j}\right\}. \end{aligned} \tag{2.10}$$

Take limits as z^2 goes to 1 to obtain the value if $z^2 = 1$, which is

$$h_s(d) = \frac{1}{z^d q^{d/2}}\left\{\frac{q+1+(q-1)d}{q+1}\right\}, \quad \text{if } z^2 = 1. \tag{2.11}$$

Perhaps the easiest way to understand the spherical functions as a function of the eigenvalue λ of the adjacency operator A is to write $h_s(d) = h_\lambda(d)$, where $\lambda = q^s + q^{1-s}$. Then obtain a recursion from $Ah_\lambda(d) = \lambda h_\lambda(d)$. We obtain

$$\begin{aligned} h_\lambda(d+1) &= \frac{1}{q}(\lambda h_\lambda(d) - h_\lambda(d-1)), \quad \text{for } d = d(x,\mathfrak{o}) > 0, \\ h_\lambda(1) &= \frac{\lambda}{q+1} h_\lambda(0). \end{aligned} \tag{2.12}$$

This allows $h_\lambda(n)$ to be written in terms of the Chebyshev polynomials of the first and second kinds $T_n(x)$ and $U_n(x)$, defined by

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{2.13}$$

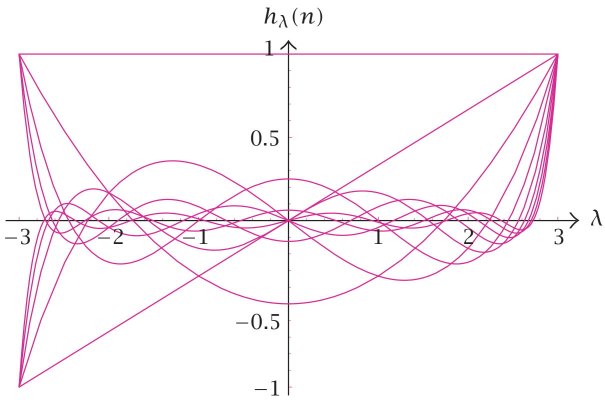

FIGURE 2.2. Ten spherical functions $h_\lambda(n)$, $n = 0,1,\ldots,9$, for the degree 3 tree as a function of the eigenvalue λ of the adjacency matrix.

(See Erdélyi et al. [6, pages 183–187] for more information on these polynomials.) The final result is

$$h_\lambda(n) = q^{-n/2}\left(\frac{2}{q+1}T_n\left(\frac{\lambda}{2\sqrt{q}}\right)+\frac{q-1}{q+1}U_n\left(\frac{\lambda}{2\sqrt{q}}\right)\right). \tag{2.14}$$

Note that since λ is real, so is $h_\lambda(n)$. Figure 2.2 shows graphs of $h_\lambda(d)$ as a function of λ when $q = 2$ and $d = 0,1,\ldots,9$.

We will need to know what happens to the spherical function as d goes to infinity. Suppose the eigenvalue λ of the adjacency operator A on the tree acting on the spherical function $h_s(d)$ is given by $\lambda = q^s + q^{1-s}$. And suppose λ is actually an eigenvalue of the adjacency operator on a connected, k-regular, finite graph X. In this situation, we need to know the locations of the complex numbers s in the complex plane. The answer is given in the following lemma.

Before stating the lemma, we note that a *bipartite graph* is one in which the vertices can be partitioned into two disjoint sets V_1 and V_2 such that every edge has one vertex in V_1 and the other in V_2. For such graphs, if λ is an eigenvalue of the adjacency operator, so is $-\lambda$ (and conversely).

LEMMA 2.1. *Suppose that X is a connected k-regular finite graph. Here $k = q+1$.*

(a) *Assume that X is not bipartite. Then any eigenvalue λ of the adjacency operator on X, with λ not equal to the degree $k = q+1$, satisfies $\lambda = q^s + q^{1-s}$ where $0 < \mathrm{Re}(s) < 1$. We may assume $\mathrm{Re}(s) \geq 1/2$.*

(b) *If X is not bipartite and $k = q+1 = q^s + q^{1-s}$, then take $s = 0$ or 1.*

(c) *If X is bipartite and $\lambda = q^s + q^{1-s}$ is an eigenvalue of A, so is $-\lambda = q^{s'} + q^{1-s'}$, with $s' = s + i\pi/\log q$.*

Proof. (a) Let $s = a+ib$ with a and b real. Then

$$\frac{\lambda}{\sqrt{q}} = 2\cosh\left\{\left(a-\frac{1}{2}\right)\log q\right\}\cos(b\log q) + 2i\sinh\left\{\left(a-\frac{1}{2}\right)\log q\right\}\sin(b\log q). \tag{2.15}$$

Since λ is real, the imaginary part of this expression vanishes. This can happen in two ways:

(1) $\sin(b\log q) = 0$ and $\lambda q^{-1/2} = \pm 2\cosh\{(a-1/2)\log q\}$;

(2) $a = 1/2$ and $\lambda q^{-1/2} = 2\cos(b\log q)$.

In part (a), our hypothesis is $|\lambda| < q+1$ which implies that in case (1), we have

$$\cosh\left\{\left(a-\frac{1}{2}\right)\log q\right\} < \cosh\left\{\left(\frac{1}{2}\right)\log q\right\}. \tag{2.16}$$

Thus, in case (1), $|a-1/2| < 1/2$ and $0 < a < 1$. In case (2), $a = 1/2$, and we are done. In case (2), the eigenvalues satisfy $|\lambda| \le 2\sqrt{q}$, which is the Ramanujan bound (see Terras [21] for more information on this subject).

We leave parts (b) and (c) to the reader. □

Note. In order for $\lambda = q^s + q^{1-s}$, in Lemma 2.1, to be actual eigenvalues of the adjacency operator corresponding to the spherical function h_s, it is necessary for the spherical function to be in $L^2(\Xi)$. This will not be the case, as you can see by noting that for the power function $p_s(x) = q^{-sd(x,o)}$ to be in $L^2(\Xi)$, we need $\operatorname{Re} s > 1/2$. But then, we cannot find any s for which both p_s and p_{1-s} are square summable. This is similar to the situation in the Poincaré upper half plane when the continuous spectrum of the non-Euclidean Laplacian on the fundamental domain of the modular group is considered. See Remark 4.4 and Terras [20, Volume I, pages 206–207].

Corollary 2.2 Asymptotics of Spherical Functions. *Suppose that X is a finite connected k-regular graph which is not bipartite. Let λ be an eigenvalue of the adjacency operator on X, with λ not equal to the degree $k = q+1$. Write $\lambda = q^s + q^{1-s}$ where $1/2 \le \operatorname{Re} s < 1$. Then the corresponding spherical function $h_s(d)$ goes to 0 as d goes to infinity.*

Proof. If s is not $1/2$, note that $0 < \operatorname{Re} s < 1$ implies

$$h_s(d) = c(s)q^{-sd} + c(1-s)q^{-(1-s)d} \tag{2.17}$$

approaches 0 as d goes to infinity. If $\operatorname{Re} s = 1/2$ and $\operatorname{Im} s = n\pi/\log q$, $n \in \mathbb{Z}$, then

$$|h_s(d)| = \frac{1}{q^{d/2}}\left|1 - \frac{q-1}{q+1}d\right| \tag{2.18}$$

approaches 0 as d goes to infinity. □

Note. For any finite connected k-regular graph X, the degree k is an eigenvalue of the adjacency operator corresponding to the constant spherical function $h_0(d) = 1$, $d = d(x,\mathfrak{o})$, for $\mathfrak{o}$ the origin of the tree. If the graph X is finite connected k-regular and bipartite, then $-k$ is also an eigenvalue of the adjacency operator and the corresponding spherical function is $(-1)^d$ where $d = d(x,\mathfrak{o})$.

3. The pretrace formula. Again we assume that $X = \Gamma\backslash\Xi$ is a finite connected k-regular graph. To obtain the pretrace formula, we follow the elementary discussion of Brooks [3]. Consider any rotation-invariant function on the tree $f(y) = f(d(y,x))$, where $x = \mathfrak{o} =$ the origin of the tree. Suppose that f has finite support. Define the Γ-*invariant kernel* associated with f to be

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(d(x,\gamma y)). \tag{3.1}$$

So $K_f(x,\gamma y) = K_f(\gamma x,y) = K_f(x,y)$, for all $\gamma \in \Gamma$ and all x,y in the tree. We may as well take

$$f_r(x,y) = \begin{cases} 1, & \text{if } d(x,y) = r, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

We say this because any finitely supported and rotation-invariant function about x will be a linear combination of the functions f_r.

We need more lemmas.

Lemma 3.1. *Suppose that ϕ is any eigenfunction of the adjacency operator A on Ξ. That is, suppose $A\phi = \lambda\phi$. Then if $r > 0$,*

$$\sum_{y\in\Xi} f_r(x,y)\phi(y) = k(k-1)^{r-1} h_{s_\lambda}(r)\phi(x), \tag{3.3}$$

and the sum is equal to $\phi(x)$, if $r = 0$. Here h_{s_λ} is the spherical function (expressed as a function of the distance to the origin x) corresponding to the eigenvalue

$$\lambda = q^{s_\lambda} + q^{1-s_\lambda} \tag{3.4}$$

for the adjacency operator.

Proof. The case $r = 0$ is clear. When $r > 0$, fix y_0 in a shell of radius r about x. Let $\phi^\#(y_0)$ denote the average of ϕ over the shell of radius $r = d(x,y_0)$ about x. To get $\phi^\#(y_0)$, you must sum $\phi(u)$ over all u with $d(u,x) = d(y_0,x) = r$ and divide by the number of such points u which is $k(k-1)^{r-1}$. So, $\phi^\#(y_0)$ is invariant under rotation about x and it is an eigenfunction of A (as A commutes with the action of the rotation subgroup of G or any element of G).

By the uniqueness of the spherical function associated with the eigenvalue λ of A,

$$\phi^{\#}(y_0) = h_{s_\lambda}(d(x,y_0))\phi(x), \tag{3.5}$$

where s_λ is defined by $\lambda = q^{s_\lambda} + q^{1-s_\lambda}$.

Note. We have to think a bit about what happens if $\phi^{\#}$ is zero. (See Quenell [15].)

So, now we find that, with y_0 fixed such that $d(y_0,x) = r$,

$$\sum_{y\in\Xi} f_r(x,y)\phi(y) = \sum_{y\in\Xi, d(x,y)=r} \phi(y) = \phi^{\#}(y_0)k(k-1)^{r-1}. \tag{3.6}$$

This completes the proof of the lemma. □

Note. The operator on the left-hand side of (3.3) (with x = the origin) is the rth *Hecke operator* which we denote $T_r\phi$. See Cartier [5]. The algebra generated by the T_r is called the Hecke algebra. We see that T_0 = Identity, $T_1 = A$, $(T_1)^2 = T_2 + (q+1)T_0$, and $T_1T_m = T_{m+1} + qT_{m-1} = T_mT_1$, for $m > 1$. Thus, the Hecke algebra is a polynomial algebra in $A = T_1$. Moreover,

$$\sum_{m\geq 0} T_m u^m = (1-u^2)(1-uT_1+qu^2)^{-1}. \tag{3.7}$$

Corollary 3.2 (Selberg's lemma)**.** *If f is a finitely supported rotation invariant (real-valued) function on Ξ and ϕ is an eigenfunction of the adjacency operator A on Ξ with $A\phi = \lambda\phi$, then, writing $\mathfrak{o}$ for the origin of Ξ,*

$$(f,\phi)_\Xi = \phi(\mathfrak{o})(f,h_s)_\Xi, \tag{3.8}$$

where $\lambda = q^s + q^{1-s}$ and h_s is the spherical function in (2.7). Here $(f,g)_\Xi$ denotes the inner product defined by

$$(f,g)_\Xi = \sum_{x\in\Xi} f(x)g(x). \tag{3.9}$$

Proof. Set $x = \mathfrak{o}$ in Lemma 3.1. If $r = 0$, we get $(f_0,\phi)_\Xi = \phi(\mathfrak{o}) = \phi(\mathfrak{o})(f_0,h_s)_\Xi$ since $h_s(\mathfrak{o}) = 1$. If $r > 0$, then Lemma 3.1 implies that

$$(f_r,\phi)_\Xi = k(k-1)^{r-1}h_s(r)\phi(\mathfrak{o}) = \phi(\mathfrak{o})(f_r,h_s)_\Xi. \tag{3.10}$$

Since the f_r form a vector space basis of the space of finitely supported rotation invariant functions on Ξ, the proof is complete. □

The inner product on the right-hand side of the formula in Selberg's lemma has a name—the spherical transform of f. The *spherical transform* of any rotationally invariant function f on the tree is defined to be

$$\hat{f}(s) = (f, h_s)_{\Xi}. \tag{3.11}$$

This is an invertible transform. The inversion formula is part of the spectral theorem for the adjacency operator on the tree. It can be obtained by making use of the resolvent $R_\mu = (A - \mu I)^{-1}$. See Figà-Talamanca and Nebbia [7, page 61]. The Plancherel theorem for rotation-invariant functions f on the tree with finite support says

$$f(o) = \int_0^{\pi/\log q} \hat{f}\left(\frac{1}{2} + it\right) \frac{q \log q}{2\pi(q+1)\,|c(1/2+it)|^2}\, dt, \quad \text{for } c(s) \text{ as in (2.8)}. \tag{3.12}$$

We will not need this result here.

The following lemma proves useful when applying the Selberg trace formula in Section 4.

Lemma 3.3 (relation between spherical and horocycle transforms). *Suppose that f is a rotation-invariant function on the tree and that $z = q^{s-1/2}$. If $\hat{f}(s)$ denotes the spherical transform in (3.11) and Hf denotes the horocycle transform in (2.4), then*

$$\hat{f}(s) = \sum_{n \in \mathbb{Z}} Hf(n) q^{|n|/2} z^n. \tag{3.13}$$

Proof. Using (2.10), we have

$$\begin{aligned} (f, h_s) &= f(0) + \sum_{n=1}^{\infty} (q+1) q^{n-1} f(n) h_s(n) \\ &= f(0) + \sum_{n=1}^{\infty} f(n) q^{n/2} z^{-n} \left(1 + z^{2n} + \frac{q-1}{q} \sum_{j=1}^{n-1} z^{2j}\right). \end{aligned} \tag{3.14}$$

Rearranging the sums finishes the proof after a bit of work. □

In the following examples, we essentially find a short table of horocycle and spherical transforms.

Example 3.4. By Lemma 3.3, we find that if the horocycle transform is defined to be

$$H\alpha(n) = \begin{cases} u^{|n|-1}, & \text{for } n \neq 0, \\ 0, & \text{for } n = 0, \end{cases} \tag{3.15}$$

then the spherical transform $(\alpha, h_s) = q^s/(1-uq^s) + q^{1-s}/(1-uq^{1-s})$ when $|u| < 1/q$. Setting $\lambda = q^s + q^{1-s}$ as usual, this means

$$(\alpha, h_s) = \frac{d}{du} \log \frac{1}{1-\lambda u + qu^2}. \tag{3.16}$$

Then using the inversion formula for the horocycle transform with $|u| < 1$, we have

$$\alpha(n) = \begin{cases} \dfrac{(1-q)u}{1-u^2}, & \text{for } n = 0, \\ \dfrac{u^{|n|-1}(1-qu^2)}{1-u^2}, & \text{for } n > 0. \end{cases} \tag{3.17}$$

Example 3.5. Set $f = f_r$ as in (3.2). Then

$$(f, h_s) = \begin{cases} 1, & \text{for } r = 0, \\ (q+1)q^{r-1}h_s(r), & \text{for } r > 0, \end{cases}$$

$$Hf_r = \begin{cases} 1, & \text{if } |n| = r, \\ (q-1)q^{j-1}, & \text{if } |n| + 2j = r, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

So

$$Hf_r = \delta_r(|n|) + (q-1) \sum_{j=1}^{[r/2]} q^{j-1} \delta_{r-2j}(|n|). \tag{3.19}$$

Example 3.6. Suppose that $Hg_n(m) = \delta_n(|m|)$. Then using the inversion formula for the horocycle transform in (2.5), we have

$$g_n = \delta_n - (q-1) \sum_{j=1}^{[n/2]} \delta_{n-2j}. \tag{3.20}$$

By Lemma 3.3, with $z = q^{s-1/2}$, the spherical transform is

$$(g_n, h_s) = q^{n/2}(z^n + z^{-n}) = q^{ns} + q^{n(1-s)}. \tag{3.21}$$

Let A be the adjacency operator on the finite graph $X = \Gamma \backslash \Xi$. It is essentially the same as that on the tree covering X by the local isomorphism. Suppose that $\{\phi_i\}_{i=1,\dots,|X|}$ is a complete orthonormal set of eigenfunctions of A on X. We can assume that the ϕ_i are real valued. Let Φ_i be the lift of ϕ_i to the tree Ξ; that is, $\Phi_i(x) = \phi_i(\pi(x))$ for $x \in \Xi$, where $\pi : \Xi \to X$ is the natural projection map. Then, $A\Phi_i = \lambda\Phi_i$ since $\pi : \Xi \to X$ is a local graph isomorphism. Usually we will omit π.

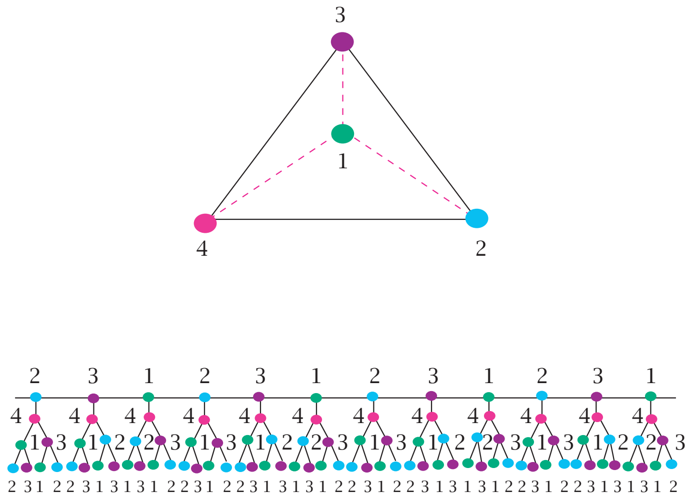

FIGURE 3.1. Tessellation of the 3-regular tree from the complete graph on 4 vertices (K_4 or the tetrahedron). Obtain the tessellation by choosing a spanning tree in K_4—the dashed edges. Repeat the spanning tree for each of the infinitely many sheets of the cover. Connections between sheets are forced by the ordering of points on the geodesic labelled with 1, 2, 3 repeated infinitely many times.

Now, using our favorite functions $f_r(x,y)$, defined in (3.2), we have for $\pi(x),\pi(y)\in X$,

$$K_{f_r}(x,y)=K_{f_r}(\pi(x),\pi(y))=\sum_{\gamma\in\Gamma}f_r(d(x,\gamma y))=(A_r)_{\pi(x),\pi(y)}. \tag{3.22}$$

Here, the right-hand side is the number of paths of length r without backtracking from $\pi(x)$ to $\pi(y)$ in X. A path C in X consists of a set of adjacent vertices $C=(v_0,\dots,v_n)$, $v_j\in X$. As before, we say C has backtracking if $v_{j-1}=v_{j+1}$ for some j, $2\le j\le r-1$. To see that we are counting paths without backtracking, consider Figure 3.1.

So, there must be constants $c_{i,j}$ such that K_f is a sum over i,j from 1 to $|X|$ of $c_{i,j}\phi_i(x)\phi_j(y)$. We find these constants in Lemma 3.7.

LEMMA 3.7. *Suppose that the notation is as in (3.7) and $\{\phi_i,\ i=1,\dots,|X|\}$ forms a complete orthonormal set of eigenfunctions of the adjacency operator on X. Let h_{s_i} be the spherical function associated with the same eigenvalue for the adjacency operator on Ξ*

$$\lambda_i=q^{s_i}+q^{1-s_i} \tag{3.23}$$

as ϕ_i. Then for $r > 0$,

$$K_{f_r}(x,y) = \sum_{i=1}^{|X|} k(k-1)^{r-1} h_{s_i}(r)\phi_i(x)\phi_i(y). \tag{3.24}$$

If $r = 0$, the right-hand side of (3.24) becomes the sum of $\phi_i(x)\phi_i(y)$.

Proof. Look at $r > 0$. Then, as above, Φ_i is the lift of ϕ_i to Ξ. So, by Lemma 3.1, we have

$$\begin{aligned}\sum_{y\in X} K_{f_r}(x,y)\phi_m(y) &= \sum_{y\in\Gamma\backslash\Xi}\sum_{\gamma\in\Gamma} f_r(x,\gamma y)\phi_m(y)\\ &= \sum_{y\in\Xi} f_r(x,y)\Phi_m(y)\\ &= k(k-1)^r h_{s_m}(r)\phi_m(x).\end{aligned} \tag{3.25}$$

□

Now, by Lemma 3.7, *pretrace formula I* says

$$\sum_{x\in X} K_{f_r}(x,x) = \sum_{i=1}^{|X|} k(k-1)^{r-1} h_{s_i}(r) = \sum_{i=1}^{|X|} (f_r, h_{s_i}). \tag{3.26}$$

This can be viewed as the trace of the operator K_{f_r}, acting on functions $g \in L^2(X)$ via $K_{f_r}(g) = \sum_{y\in X} K_{f_r}(x,y)g(y)$ and it is the left-hand side of the trace formula in Theorem 4.2 when $f = f_r$.

Note that using (3.22), the trace of K_{f_r} is $a_X(r)$ = the number of closed paths without backtracking of length r in X. Here, we count paths as different if they start at a different vertex. Note also that the paths being counted can have tails. A *tail* in a path means that the starting edge is the inverse of the terminal edge. More explicitly, we have, for $r > 0$,

$$\begin{aligned} a_X(r) &= \#\text{ closed paths without backtracking, length } r \text{ in } X\\ &= \sum_{i=1}^{|X|} (q+1)q^{r-1} h_{s_i}(r),\end{aligned} \tag{3.27}$$

where s_i corresponds to the eigenvalue $\lambda_i = q^{s_i} + q^{1-s_i}$ of the adjacency operator A of X.

Brooks [3] uses (3.27) to obtain bounds on the second largest (in absolute value) eigenvalue of A on X. We will use it to obtain an asymptotic formula for $a_X(r)$.

Suppose that X is a nonbipartite k-regular finite graph. Then

$$a_X(r) \sim \left(1 + \frac{1}{q}\right)q^r, \quad \text{as } r \longrightarrow \infty. \tag{3.28}$$

To prove this, we can use (3.27) and the method of generating functions. For set

$$w(\lambda,x) = \sum_{n=1}^{\infty} x^{n-1} h_\lambda(n). \tag{3.29}$$

From the recursion (2.12), we see that

$$w(\lambda,x) = \frac{\lambda/(q+1) - x/q}{x^2/q - \lambda x/q + 1}. \tag{3.30}$$

It follows that

$$\sum_{n=1}^{\infty} a_X(n) x^{n-1} = \sum_{i=1}^{|X|} \frac{\lambda_i - x(q+1)}{qx^2 - \lambda_i x + 1}. \tag{3.31}$$

The closest pole to the origin of the right-hand side is $x = 1/q$. It comes from the largest eigenvalue (the degree of the graph). Then, a standard method from generating function theory, using the formula for the radius of convergence of a power series, leads to (3.28). See Wilf [26, page 171].

Now, we need to discuss the right-hand side of the trace formula.

Define Γ_γ = the *centralizer* of γ in Γ; that is,

$$\Gamma_\gamma = \{\sigma \in \Gamma \mid \gamma\sigma = \sigma\gamma\}. \tag{3.32}$$

And let $\{\gamma\}$ be the *conjugacy class* of γ in Γ; that is,

$$\{\gamma\} = \{\sigma\gamma\sigma^{-1} \mid \sigma \in \Gamma\}. \tag{3.33}$$

Then we break up Γ into the disjoint union of its conjugacy classes and note that the conjugacy class $\{\gamma\}$ is the image of $\Gamma_\gamma\backslash\Gamma$ under the map that send σ to $\sigma\gamma\sigma^{-1}$. And $\Gamma_\gamma\backslash\Xi$ is a union of images of $\Gamma\backslash\Xi$ under elements of $\Gamma_\gamma\backslash\Gamma$. So we obtain, writing $f(d(x,y)) = f(x,y)$,

$$\sum_{x\in X} K_{f_r}(x,x) = \sum_{x\in\Gamma\backslash\Xi} \sum_{\gamma\in\Gamma} f_r(x,\gamma x) = \sum_{\{\gamma\}} \sum_{x\in\Gamma_\gamma\backslash\Xi} f_r(x,\gamma x), \tag{3.34}$$

where the sum over $\{\gamma\}$ is a sum over all conjugacy classes in Γ.

Thus, we have the *pretrace formula II* which says that for any f which is rotation invariant and of finite support on Ξ,

$$\sum_{i=1}^{|X|} \hat{f}(s_i) = \sum_{\{\gamma\}} I_\gamma(f), \tag{3.35}$$

where $\{\gamma\}$ is summed over all conjugacy classes in Γ. Here the *orbital sum* is

$$I_\gamma(f) = \sum_{x\in\Gamma_\gamma\backslash\Xi} f(x,\gamma x). \tag{3.36}$$

Note that $\gamma \in \Gamma$, $\gamma \neq$ identity, implies that γ is hyperbolic. For Γ is the group of covering transformations of X. See Figure 3.1 for a tessellation of the 3-regular tree covering the tetrahedron or K_4, the complete graph with 4 vertices. See Stark and Terras [17, 18] for examples of finite covers of the tetrahedron.

We say that $\rho \in \Gamma$ is a *primitive hyperbolic* element if $\rho \neq$ the identity and ρ generates its centralizer in Γ. As in the case of discrete groups Γ acting on the Poincaré upper half plane, primitive hyperbolic conjugacy classes $\{\gamma\}$ in Γ correspond to closed paths in X, which are the graph theoretic analogues of prime geodesics or curves minimizing distance which are not traversed more than once. We call the equivalence classes of such paths *primes* in X. Here, the equivalence relation on paths simply identifies closed paths with different initial vertices. We will make use of this fact when considering the Ihara zeta function (4.7). See Terras [20, Volume I, page 277] for a discussion of the hyperbolic plane case.

Now, we proceed to a discussion of another application of a pretrace formula. Take some geodesic $\{x_m\}_{m\in\mathbb{Z}}$ which is a union of two chains c and c', on Ξ. We assume that c and c' intersect in the origin $\mathfrak{o}$ of Ξ. The *geodesic flow* is defined by $\tau_n(x_m) = x_{m+n}$ on the geodesic itself and by moving points off the geodesic by just shifting the whole picture n units to the left if the geodesic is, for example, the top horizontal line in Figure 2.1.

We could also consider the horocycle flow, but this seems somewhat less natural since it does not travel along edges of the tree. By a small set B in Ξ, we mean a set contained in a fundamental region for $\Gamma\backslash\Xi$. If B is a small rotation-invariant set in Ξ such as the shell of radius r defined by $S(r) = \{y \in \Xi \mid d(y,\mathfrak{o}) = r\}$, for r sufficiently small, then define $B_n = \tau_n(B)$. For example, consider Figure 2.1 and let $B = \{x \mid d(x,\mathfrak{o}) = 1\}$. Then, B_n consists of points having distance $\geq n-1$ from the origin.

This flow induces trajectories on X which are not true flows, as they have many self-intersections. If the radius of B is less than or equal to 1, it is locally mapped by the projection $\pi : \Xi \to X = \Gamma\backslash\Xi$, one-to-one, onto the corresponding set in X. It is natural to ask, to what extent does the *induced flow* on X mix up the vertices of B on X. Theorem 3.8 gives an answer to this question.

Let f denote the characteristic function of the set B and define $F(x)$ to be the Γ-*periodization* of f,

$$F(x) = \sum_{\gamma\in\Gamma} f(\gamma x). \tag{3.37}$$

Similarly, define the *rotationally averaged* Γ-*periodization of the shift of* f,

$$F_n^{\#}(x) = \sum_{\gamma\in\Gamma} \frac{1}{k(k-1)^{d-1}} \sum_{\substack{y\in\Xi \\ d(y,\mathfrak{o})=d(\gamma x,\mathfrak{o})=d}} f(\tau_{-n}y). \tag{3.38}$$

Theorem 3.8. *Assume that the k-regular graph X is connected and non-bipartite. Using definitions (3.37) and (3.38), with f equal to the characteristic* *function of the rotation-invariant set B, consider the sum*

$$U(n,B) = \frac{1}{|X|}\sum_{x\in X} F(x)F_n^{\#}(x). \tag{3.39}$$

Then $U(n,B)$ approaches $|B|^2|X|^{-2}$, as n goes to infinity.

Proof. Suppose ϕ_i, $i = 1,\dots,|X|$, denotes a complete orthonormal set of eigenfunctions of the adjacency operator on X. Now

$$F(x) = \sum_{i=1}^{|X|} c_i\phi_i(x), \tag{3.40}$$

where $c_i = (F,\phi_i)_X$. By Selberg's lemma, we have

$$c_i = \sum_{x\in X} F(x)\phi_i(x) = \sum_{y\in\Xi} f(x)\phi_i(x) = \hat{f}(s_i)\phi_i(\mathbf{o}) \tag{3.41}$$

with $\hat{f}(s_i)$ defined by (3.11).

Then the sum $U(n,B)$ is

$$U(n,B) = \frac{1}{|X|}\sum_{x\in X}\sum_{i=1}^{|X|}\hat{f}(s_i)\phi_i(\mathbf{o})\phi_i(x)F_n^{\#}(x). \tag{3.42}$$

We look at the term corresponding to $i = 1$, where we have chosen $\phi_1 = |X|^{-1/2}$ to be the constant eigenfunction of the adjacency operator on X. This term is

$$S = \frac{1}{|X|^2}\sum_{x\in X}\hat{f}(s_1)F_n^{\#}(x). \tag{3.43}$$

Since f is the characteristic function of the set B and the spherical function corresponding to $s_1 = 0$ is the function which is identically 1, we have

$$\hat{f}(s_1) = (f,1)_{\Xi} = |B|. \tag{3.44}$$

Therefore, we find, using definition (3.38), that the sum S in (3.43) is $|B|^2|X|^{-2}$.

It remains to estimate the terms given by

$$R = \frac{1}{|X|}\sum_{i=2}^{|X|} c_i\sum_{x\in X}\sum_{\gamma\in\Gamma}\frac{1}{(q+1)q^{d-1}}\sum_{\substack{y\in\Xi\\ d(y,\mathbf{o})=d(\gamma x,\mathbf{o})=d}} f(\tau_{-n}y)\phi_i(x). \tag{3.45}$$

We can combine the sums over X and Γ to get a sum over the tree Ξ. Then by Selberg's lemma, we have

$$R = \frac{1}{|X|}\sum_{i=2}^{|X|} c_i\sum_{x\in\Xi}\frac{1}{(q+1)q^{d-1}}\sum_{\substack{y\in\Xi;\ d(y,\mathbf{o})\\ =d(x,\mathbf{o})=d}} f(\tau_{-n}y)h_{s_i}(x)\phi_i(\mathbf{o}). \tag{3.46}$$

Here $y = kx$ where k is a rotation about o. Make the change of variables from x to y and use the fact that the spherical function is rotation invariant to see that

$$R = \frac{1}{|X|}\sum_{i=2}^{|X|} c_i \sum_{y\in\Xi} f(\tau_{-n}y)h_{s_i}(y)\phi_i(o). \tag{3.47}$$

Since $f(y)$ is the characteristic function of the rotation-invariant set B, $f(\tau_{-n}y)$ is the characteristic function of $B_n = \tau_n(B)$ whose distance from o approaches infinity as n goes to infinity.

So, now we must make use of the asymptotics of the spherical function as d approaches infinity. Corollary 2.2 says that our spherical function does approach 0 as the distance from the origin increases. And thus, the sum R of the remaining terms approaches 0 and the theorem is proved. □

Theorem 3.8 says that the intersection of a small rotation-invariant set B with the image of B, propagated forward by the geodesic flow induced on X (when averaged over shells), tends to be what we expect when two random sets are intersected in X.

What happens if X is bipartite? Then you must look at the term corresponding to the eigenvalue $-k$ separately and this is harder to compute.

Example 3.9. Consider Figure 3.1. Let X be the tetrahedron. Take $f = f_0$ in Theorem 3.8. Then the function $F_n^{\#}$ in (3.38) has support consisting of Γ-translates of rotations of $\tau_n o$. So the sum in Theorem 3.8 is $U(n,\{o\}) = u_n/12(2^{n-1})$, where u_n is the number of points in the tree of Figure 3.1 which are labeled 1 and are at a distance n from the origin o.

It might help to have a more detailed version of Figure 3.1. So see Figure 3.2. From Figure 3.2, we see that $u_3 = u_4 = u_5 = 6$ and $u_6 = 30$, $u_7 = 54$. Theorem 3.8 says that $U(n,o)$ approaches $1/16$ as n approaches infinity. We see in the example that $16U(n,o)$ has the values $2,1,1/2,5/4,9/8$ for $n = 3,4,5,6,7$. When $n \geq 35$, we can check, using a computer, that $16U(n,o)$ is very close to 1.

There is another interpretation of u_n. As in (3.22), let A_n be the 4×4 matrix whose i,j entry is the number of paths in X of length n *with no backtracking* starting at vertex i in X and ending at j in X. Then u_n is the $1,1$ entry of A_n. In Stark and Terras [17, Lemma 1] says that $A_0 = I$, $A_1 = A =$ the adjacency matrix of the tetrahedron, $A_2 = A^2 - 3I$, $A_n = A_{n-1}A - 2A_{n-2}$ for $n \geq 3$. We can use this recursion to prove Theorem 3.8 in the case that $B = \{o\}$. These recursions for A_n are the same as those defining the Hecke operator T_n in (3.7). Using the method of generating functions (see Wilf [26, page 171]) plus the fact that $(A_n)_{1,1} = (1/|X|)Tr(A_n)$, we easily prove $u_n \sim 3\cdot 2^{n-3}$, as $n \to \infty$.

4. The trace formula.

In order to derive the trace formula, we only need to recall pretrace formula II (3.35) and to evaluate our orbital sums I_γ (3.36). If γ

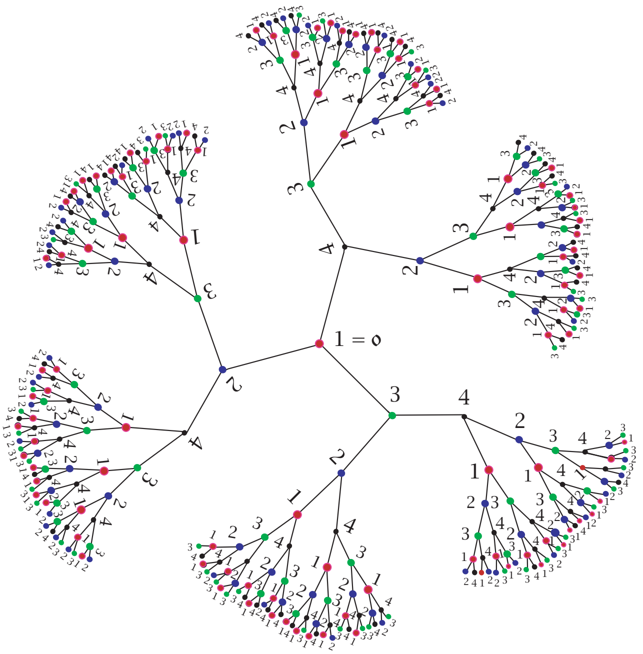

FIGURE 3.2. Another view of the tessellation in Figure 3.1.

is the identity, this is easy. Writing $f(x,y) = f(d(x,y))$, we find $I_\gamma = f(0)|X|$. Otherwise, use Lemma 4.1. Recall that we say $\rho \in \Gamma$ is primitive hyperbolic if $\rho \neq$ the identity and ρ generates its centralizer in Γ.

LEMMA 4.1 (orbital sums for hyperbolic elements are horocycle transforms). *Suppose that ρ is a primitive hyperbolic element of Γ. Then we have the following formula relating the orbital integral defined by (3.36) and the horocycle transform defined by (2.4) for $r \geq 1$,*

$$I_{\rho^r}(f) = \nu(\rho)Hf(r\nu(\rho)). \tag{4.1}$$

Here $\nu(\rho)$ is the integer giving the size of the shift by ρ along its fixed geodesic.

PROOF. The quotient $\Gamma_\rho \backslash \Xi$ is found by looking at the geodesic fixed by the primitive hyperbolic element ρ. Then consider this geodesic modulo ρ. Assume $\nu(\rho) = 3$ and look at Figure 4.1 where there are only three Γ_ρ-inequivalent points on the geodesic fixed by ρ, which is the left line of points in the picture.

We claim that

$$\#\{y \in \Gamma_\rho \backslash \Xi \mid d(y, \rho^r y) = r\nu(\rho) + 2j\} = \nu(\rho)(q-1)q^{j-1}. \tag{4.2}$$

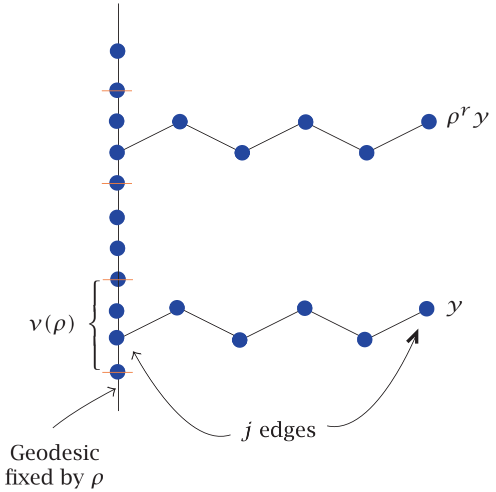

FIGURE 4.1. Finding the fundamental domain $\Gamma_\rho\backslash\Xi$ when $\nu(\rho)=3$, $r=2$, $j=5$.

For Figure 4.1, we have $q=2$, $j=5$, $r=2$, $\nu(\rho)=3$ and the number in (4.2) is $3\cdot 16$. Why does $\nu(\rho)$ appear? Because y can be any of $\nu(\rho)$ things. It follows that the orbital sum of f associated with ρ^r is

$$\begin{aligned} I_{\rho^r}(f) &= \sum_{y\in\Gamma_\rho\backslash\Xi} f(d(y,\rho^r y)) \\ &= \nu(\rho)f(r\nu(\rho))+\nu(\rho)(q-1)\sum_{j=1}^{\infty} q^{j-1}f(r\nu(\rho)+2j). \end{aligned} \tag{4.3}$$

This completes the proof of the lemma. □

So, we have proved the trace formula finally.

THEOREM 4.2 (the trace formula for a k-regular finite graph)**.** *Suppose that $f:\Xi\to\mathbb{R}$ has finite support and is invariant under rotation about the origin $\mathfrak{o}$ of Ξ. Then if $\mathfrak{P}_\Gamma$ denotes the set of primitive hyperbolic conjugacy classes in Γ,*

$$\sum_{i=1}^{|X|}\hat{f}(s_i)=f(\mathfrak{o})|X|+\sum_{\{\rho\}\in\mathfrak{P}_\Gamma}\nu(\rho)\sum_{e\geq 1}Hf(e\nu(\rho)). \tag{4.4}$$

Here, $A\phi_i=\lambda_i\phi_i$ where the ϕ_i form a complete orthonormal set of eigenfunctions of the adjacency operator A on X. Here the sum on the left is of spherical transforms of f at the s_i corresponding to the eigenvalues λ_i as in Lemma 3.7. In the sum on the right, Hf is the horocycle transform defined by (2.4).

Special case. Suppose $f = f_0$, defined by (3.2) with $r = 0$. Then the trace formula is trivial—saying that $|X| = |X|$. If $f = f_r$, defined by (3.2) for $r > 0$, the trace formula says

$$k(k-1)^{r-1}\sum_{i=1}^{|X|} h_{s_i}(r) = f_r(\mathfrak{o})|X| + \sum_{\{\rho\}\in\mathfrak{P}_\Gamma} \nu(\rho)\sum_{e\geq 1} Hf_r(e\nu(\rho)), \tag{4.5}$$

where $Hf_r(m)$ was computed in (3.19).

If $r = 1$ or 2, $f_r(e\nu(\rho)+2j) = 0$ as $e \geq 1$, $\nu(\rho) \geq 1$, and $j \geq 1$. And $f_r(e\nu(\rho)) = Hf_r(e\nu(\rho))$ is only nonzero for $e\nu(\rho) \leq 2$. In particular, when $r = 1$, we see that the trace formula says

$$k\sum_{i=1}^{|X|} h_{s_i}(1) = \#\{\{\rho\}\in\mathfrak{P}_\Gamma \mid \nu(\rho) = 1\} = 0. \tag{4.6}$$

If we plug in our formula for $h_s(1)$, we see that the sum on the left is $Tr(A) = 0$. It is also obvious that there are 0 primitive hyperbolic conjugacy classes $\{\rho\}$ in Γ with $\nu(\rho) = 1$ since there are no closed paths in X of length 1. For our graph, X is assumed to have no loops. It is a simple graph.

Rather than proceeding in this way, to find how many primes there are of various lengths, we put all the prime information together into a zeta function—Ihara's zeta function, which is the graph theoretic analogue of the Selberg zeta function. It can also be viewed as an analogue of the Dedekind zeta function of an algebraic number field. References are Ahumada [1], Bass [2], Hashimoto [9], Ihara [12], Stark and Terras [17, 18], Sunada [19], Venkov and Nikitin [24].

Consider a connected finite (not necessarily regular) graph with vertex set X and undirected edge set E. For an example, look at the tetrahedron. We orient the edges of X and label them $e_1, e_2, \ldots, e_{|E|}, e_{|E|+1} = e_1^{-1}, \ldots, e_{2|E|} = e_{|E|}^{-1}$. Here, the inverse of an edge is the edge taken with the opposite orientation. A *prime* $[C]$ in X is an equivalence class of tailless backtrackless primitive paths in X. Here, write $C = a_1a_2\cdots a_s$, where a_j is an oriented edge of X. The *length* $\nu(C) = s$. *Backtrackless* means that $a_{i+1} \neq a_i^{-1}$ for all i. *Tailless* means that $a_s \neq a_1^{-1}$. The *equivalence class* of C is $[C] = \{a_1a_2\cdots a_s, a_sa_1a_2\cdots a_{s-1}, \ldots, a_2\cdots a_sa_1\}$; that is, the same path with all possible starting points. We call the equivalence class $[C]$ a *prime* or *primitive* if $C \neq D^m$, for all integers $m \geq 2$, and all paths D in X. It can be shown that such $[C]$ are in one-to-one correspondence with primitive hyperbolic conjugacy classes $\mathfrak{P}_\Gamma$. See Stark's article in Hejhal et al. [10, pages 601–615].

The Ihara zeta function of X is defined for $u \in \mathbb{C}$ with $|u|$ sufficiently small by

$$\zeta_X(u) = \prod_{[C] \text{ prime in } X} \left(1-u^{\nu(C)}\right)^{-1}. \tag{4.7}$$

Note that the preceding product is infinite since X cannot be a cycle graph as its degree is greater than two.

Theorem 4.3 can be attributed to many people in the case of both regular and irregular finite graphs. Bass [2], Hashimoto [9], and Sunada [19] certainly should be mentioned. The proof we sketch is due to Ahumada [1]. We found the discussion we outline in Venkov and Nikitin [24]. Ihara [12] considers these zeta functions within the framework of p-adic groups.

Theorem 4.3 (Ihara [12]). *If A denotes the adjacency matrix of X and Q, the diagonal matrix with jth entry q_j = (degree of the jth vertex-1), then*

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2). \tag{4.8}$$

Here, r denotes the rank of the fundamental group of X. That is, $r = |E| - |V| + 1$.

Proof. Following Venkov and Nikitin, plug the following function into the trace formula:

$$Hf(d) = g(d) = \begin{cases} u^{|d|-1}, & \text{for } d \neq 0, \\ 0, & \text{for } d = 0. \end{cases} \tag{4.9}$$

After a certain amount of computation, we find the following formulas. The

$$\text{right-hand nonidentity terms} = \frac{d \log \zeta_X(u)}{du}. \tag{4.10}$$

By the inversion formula for the horocycle transform,

$$\text{right-hand identity term} = nf(0) = \frac{d \log (1-u^2)^{n(q-1)/2}}{du}. \tag{4.11}$$

Here, $n = |X|$ and $n(q-1)/2 = r-1$ where r is the rank of the fundamental group of X. See Stark's article in Hejhal et al. [10, pages 604–605] for a proof.

Using (3.16) Example 3.4 of spherical and horocycle transforms, the left-hand side is a sum over the eigenvalues λ of the adjacency matrix of X of terms of the form

$$-\frac{d \log (1 - \lambda u + qu^2)}{du}. \tag{4.12}$$

□

In the general case, when X is not a regular graph, formula (4.8) for the Ihara zeta function still holds. There are many proofs. See Stark and Terras [17, 18] for some elementary ones and more references.

We can use Theorem 4.3 to obtain an analogue of the prime number theorem. First, we define a prime path to be a closed path without backtracking or tails that is not a power of another closed path modulo equivalence. Here, two paths running through the same edges (vertices) but starting at different vertices are called equivalent. Let $\pi_X(r)$ denote the number of prime path equivalence classes $[C]$ in X where the length of C is r. Then, we have for nonbipartite $(q+1)$-regular graphs X,

$$\pi_X(r) \sim \frac{q^r}{r}, \quad \text{as } r \longrightarrow \infty. \tag{4.13}$$

To prove (4.13), look at the generating function

$$u\frac{d}{du}\log\zeta_X(u) = \sum_{m=1}^{\infty} n_X(m)u^m, \tag{4.14}$$

where $n_X(r)$ is the number of closed paths C in X of length r without backtracking or tails. Since the closest singularity of $\zeta_X(u)$ to the origin is at $1/q$, it follows that $n_X(r) \sim q^r$, as $r \to \infty$. Compare this with (3.28) where we were counting paths that could have tails. Then, we easily see that the asymptotic behavior of the number of prime paths C of length r is the same as that of $n_X(r)$. Counting equivalence classes $[C]$ of prime paths of length r divides the result by r.

Note that since the Ihara zeta function is the reciprocal of a polynomial, it has no zeros. Thus, when discussing the Riemann hypothesis, we consider only poles. When X is a finite connected $(q+1)$-regular graph, there are many analogues of the facts about the other zeta functions. For any unramified graph covering (not necessarily normal or even involving regular graphs), it is easy to show that the reciprocal of the zeta function below divides that above (see Stark and Terras [17, 18]). The analogue of this for Dedekind zeta functions of extensions of number fields is still unproved. There are functional equations. Special values give graph theoretic constants such as the number of spanning trees. See the references mentioned at the beginning of this section for more details.

For example, when X is a finite connected $(q+1)$-regular graph, we say that $\zeta_X(q^{-s})$ *satisfies the Riemann hypothesis* if and only if

$$\text{for } 0 < \operatorname{Re} s < 1, \quad \zeta_X(q^{-s})^{-1} = 0 \Longleftrightarrow \operatorname{Re} s = \frac{1}{2}. \tag{4.15}$$

Remark 4.4. It is easy to see that (4.15) is equivalent to saying that X is a *Ramanujan graph* in the sense of Lubotzky et al. [14]. This means that when λ is an eigenvalue of the adjacency matrix of X such that $|\lambda| \neq q+1$, then $|\lambda| \leq 2\sqrt{q}$. Such graphs are optimal expanders and (when X is nonbipartite) the standard random walk on X converges extremely rapidly to uniform. See Terras [21] for more information. The statistics of the zeros of the Ihara zeta function of a regular graph can be viewed as the statistics of the eigenvalues of the adjacency matrix. Such statistics have recently been of interest to number theorists and physicists. See Katz and Sarnak [13]. This has been investigated for various families of Cayley graphs such as the finite upper half plane graphs. See Terras [22, 23] for a discussion of some connections with quantum chaos.

Example 4.5 (the Ihara zeta function of the tetrahedron). It is really easy to compute the eigenvalues of the adjacency operator in this case. They are $\{3,-1,-1,-1\}$. So, we find that the reciprocal of the Ihara zeta function of K_4 is

$$\zeta_{K_4}(u)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3. \tag{4.16}$$

In this case, the generating function for the $n_{K_4}(r)$ in (4.14) is

$$\begin{aligned} x\frac{d}{dx}\log\zeta_{K_4}(x) &= \sum_{m=1}^{\infty} n_{K_4}(m)x^m \\ &= 24x^3+24x^4+96x^6+168x^7+168x^8 \\ &\quad +528x^9+1200x^{10}+1848x^{11}+O(x^{12}). \end{aligned} \tag{4.17}$$

Equation (4.17) says that there are 8 equivalence classes of closed paths of length 3 (without backtracking or tails) on the tetrahedron, for instance. It is easy to check this result, recalling that we distinguish between paths and their inverses, the path traversed in the opposite direction. And there are six classes of closed paths of length four. There are no closed paths of length five. Note that the coefficient of x^9 is not divisible by nine. This happens since a nonprime path such as that which goes around a given triangle three times will have only three equivalent paths in its equivalence class rather than nine.

Many more examples can be found in Stark and Terras [17, 18]—as well as examples of Artin L-functions associated with graph coverings. In [18, Figure 19], we find an example of two nonisomorphic graphs without loops or multiple edges having the same Ihara zeta functions. This is analogous to similar examples of Dedekind zeta functions of two algebraic numbers fields being equal without the number fields being isomorphic. It comes from an example of Buser [4], which ultimately led to the example of two planar drums whose shape cannot be heard since they have the same Laplace spectra but cannot be obtained from one another by rotation or translation. See Gordon et al. [8].

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