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# RANDOM SUMS OF RANDOM VECTORS AND MULTITYPE FAMILIES OF PRODUCTIVE INDIVIDUALS

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We prove limit theorems for a family of random vectors whose coordinates are a special form of random sums of Bernoulli random variables. Applying these limit theorems, we study the number of productive individuals in $n$-type indecomposable critical branching stochastic processes with types of individuals $T_1,\dots,T_n$.

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**1. Introduction.** We consider a sequence of random vectors, which is defined as follows. Let $\{\xi_{ij}(k,m),\ j\ge 1\}$, $i=1,2,\dots,n$, for any pair $(k,m)\in\mathbb{N}_0^2$, $\mathbb{N}=\{1,2,\dots\}$, $\mathbb{N}_0=\{0\}\bigcup\mathbb{N}$, be $n$ independent sequences of random variables and let $\{\nu_{ik},\ k\in\mathbb{N}_0\}$, $i=1,2,\dots,n$, be $n$ sequences of (not necessarily independent) random variables taking values $0,1,\dots,$ and independent of family $\{\xi_{ij}(k,m)\}$. We consider the family of random vectors

$$\mathbf{W}(k,m)=\big(W_1(k,m),\dots,W_n(k,m)\big),\quad W_i(k,m)=\sum_{j=1}^{\nu_{ik}}\xi_{ij}(k,m). \tag{1.1}$$

Assume that $\xi_{ij}(k,m)$, $j=1,2,\dots,$ for any fixed $k$, $m$, and $i$, are independent and identically distributed Bernoulli random variables with parameter $P_{km}^{(i)}$ (i.e., have distribution $b(1,P_{km}^{(i)})$).

We will study the asymptotic behavior of $\mathbf{W}(k,m)$ as $k,m\to\infty$ under some assumptions on random variables $\nu_{ik}$ and $\xi_{ij}(k,m)$ in different cases of relationship between parameters $k$ and $m$.

Random sums of independent random variables or random vectors have been considered by many authors. First, it is because of the interest in extending classic limit theorems of the probability theory to a more general situation and to discover new properties of the random sums caused by “randomness” of the number of summands. On the other hand, many problems in different areas of probability can be connected with a sum of a random number of random variables. A rather full list of publications on random sums can be found in a recent monograph by Gnedenko and Korolev [4]. Transfer theorems for the random sum of independent random variables can also be found in [3].

The relationship between random sums and branching stochastic processes is well known. Starting from early studies (see, e.g., [5]) including the recent publications, the fact that the number of particles in a model of branching process can be represented as a random sum has been mentioned. Some investigations show that using this relationship in the study of branching models makes it possible to investigate new variables related to the genealogy of the process to study more general modifications of branching processes and to consider different characteristics of the process from a unique point of view. So, limit distributions for the number of pairs of individuals at time $\tau$ having the same number of descendants at time $t$, $t > \tau$, are found in [7]. A more general variable of this kind, describing the number of individual pairs having a "relatively close" number of descendants, is considered in [8] (see also [9, Chapter IV]). Using this relationship, limit theorems for different models of branching processes with immigration which may depend on the reproduction processes of particles are also proved. This kind of problems is systematically studied in the above-mentioned monograph [9]. Investigations of the maximum family size in a population by Arnold and Villaseñor [1], Rahimov and Yanev [11], and Yanev and Tsokos [16] are also based on this kind of relationship.

Here, we discuss the relationship the random sum of random vectors and multitype branching processes. Although $\mathbf{X}(t)$, the number of individuals of different types at time $t$, is the main object of investigation in the theory of multitype branching processes, there are many other variables related to the population, which are of interest as well. One example of such a variable is the time to the closest common ancestor of the entire population observed at a certain time. For a single-type Galton-Watson process, this variable was considered by Zubkov [17], who proved that, if the process is critical, the time is asymptotically uniformly distributed. Later, it turned out that the time to the closest common ancestor may be treated as a functional of the so-called reduced branching processes. This process was introduced by Fleischmann and Siegmund-Schultze [2] as a process that counts only individuals at a given time $\tau$ having descendants at time $t$, $t > \tau$. They demonstrated that in the critical single-type case, the reduced process can be well approximated by a nonhomogeneous pure birth process. Later, a number of studies extended their results to general single and multitype models of branching processes (see, e.g., [12, 14, 15]).

In this paper, we show that, if one uses theorems proved for the random sums defined in (1.1), one may study a generalized model of multitype reduced processes. Let $\boldsymbol{\theta}(t) = (\theta_1(t),\dots,\theta_n(t))$ be a vector of nonnegative functions, let $\tau$ and $t$, $\tau < t$, be two times of observations. We define the process $\mathbf{X}(\tau,t) = (X_1(\tau,t),\dots,X_n(\tau,t))$, where $X_i(\tau,t)$ is the number of type $T_i$ individuals at time $\tau$ whose number of descendants at time $t$ of at least one type is greater than the corresponding level given by vector $\theta(t-\tau)$. It is clear that $\mathbf{X}(\tau,t)$ counts only "relatively productive" individuals at time $\tau$. We also note that $\mathbf{X}(\tau,t)$ is a usual $n$-type reduced process, if $\boldsymbol{\theta}(t) = \mathbf{0}$ for all $t \in \mathbb{N}_0$. In this paper, we obtain limit distributions for process $\mathbf{X}(\tau,t)$ as $t,\tau \to \infty$ in different cases of relationship between observation times $\tau$ and $t$ for critical processes. It must be noted that the generalized reduced single-type process was introduced and studied by Rahimov [10].

In Section 2, we prove several limit theorems for the random sum of random vectors $\mathbf{W}(k,m)$ under some natural assumptions on parameters. Section 3 is devoted to the construction and definition of the generalized multitype reduced process. Applications of theorems of Section 2 to reduced branching processes are given in Section 4. Section 5 contains proofs of theorems from Section 4. In Section 6, possible applications of theorems on random sum (1.1) in the study of the number of productive ancestors in large populations are discussed.

**2. Convergence of the random sum.** For $n$-dimensional vectors $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_n)$, we denote $\mathbf{x}\oplus\mathbf{y} = (x_1y_1,\dots,x_ny_n)$, $\mathbf{x}^{\mathbf{y}} = (x_1^{y_1},\dots,x_n^{y_n})$, $\mathbf{x}/\mathbf{y} = (x_1/y_1,\dots,x_n/y_n)$, $(\mathbf{x},\mathbf{y}) = x_1y_1+\cdots+x_ny_n$, $\sqrt{\mathbf{x}} = (\sqrt{x_1},\dots,\sqrt{x_n})$, and $\mathbf{x}\ge\mathbf{y}$ or $\mathbf{x}>\mathbf{y}$ if $x_i\ge y_i$ or $x_i>y_i$, respectively.

The first theorem concerning the vector (1.1) covers the case when the normalized vector $\boldsymbol{\nu}_k = (\nu_{ik},\ i=1,\dots,n)$ has a limit distribution. Namely, we assume that there exists a sequence of positive vectors $\mathbf{A}_k = (A_{ik},\ i=1,\dots,n)$ such that $A_{ik}\to\infty$, $k\to\infty$,

$$\left\{\frac{\boldsymbol{\nu}_k}{\mathbf{A}_k}\mid \boldsymbol{\nu}_k\neq\mathbf{0}\right\}\longrightarrow \mathbf{Y} = (Y_1,\dots,Y_n) \tag{2.1}$$

in distribution, and for the vector $\mathbf{P}(k,m) = (P_{km}^{(i)},\ i=1,\dots,n)$,

$$\mathbf{P}(k,m)\oplus\mathbf{A}_k\longrightarrow\mathbf{a} = (a_1,\dots,a_n), \tag{2.2}$$

where the components of the vector $\mathbf{a}$ may be $+\infty$.

**Theorem 2.1.** *If conditions (2.1) and (2.2) are satisfied, then*

$$\left\{\frac{\mathbf{W}(k,m)}{\mathbf{P}(k,m)\oplus\mathbf{A}_k}\mid \boldsymbol{\nu}_k\neq\mathbf{0}\right\}\longrightarrow\mathbf{W} \tag{2.3}$$

*in distribution and $Ee^{(\lambda,\mathbf{W})} = \varphi(\lambda^*)$, where $\varphi(\lambda)$ is the Laplace transform of the vector $\mathbf{Y}$, $\lambda^* = (\lambda_1^*,\dots,\lambda_n^*)$ and $\lambda_i^* = \lambda_i$ if $a_i=\infty$, and $\lambda_i^* = a_i(1-e^{-\lambda_i/a_i})$ if $a_i<\infty$.*

**Proof.** First we consider the case when $a_i<\infty$, $i=1,\dots,n$. Since variables $\xi_{ij}(k,m)$, $j=1,2,\dots$, are independent and identically distributed, by total probability arguments, we find for any $\mathbf{S} = (S_1,\dots,S_n)$, $\mathbf{0}<\mathbf{S}<\mathbf{1}$,

$$E\left[\prod_{i=1}^{n} S_i^{W_i(k,m)}\right] = E\left[E\left[\prod_{i=1}^{n}\prod_{j=1}^{\nu_{ik}} S_i^{\xi_{ij}(k,m)}\mid\boldsymbol{\nu}_k\right]\right] = F(k,\mathbf{G}(k,m,\mathbf{S})), \tag{2.4}$$

where

$$\mathbf{G}(k,m,\mathbf{S}) = (G_i(k,m,S_i),\ i=1,\dots,n),\quad G_i(k,m,S_i) = ES_i^{\xi_{ij}(k,m)}, \tag{2.5}$$

and $F(k,\mathbf{S})$ is the probability generating function of the vector $\boldsymbol{\nu}_k$. Note here that, since $\xi_{ij}(k,m)$ are Bernoulli random variables with parameter $P_{km}^{(i)}$,

$$G_i(k,m,S_i) = 1-P_{km}^{(i)}(1-S_i). \tag{2.6}$$

It follows from condition (2.1) that for any $\mathbf{0} < \mathbf{S} < \mathbf{1}$,

$$\frac{1 - F(k, \mathbf{e}^{-\boldsymbol{\lambda}_0/\mathbf{A}_k})}{P\{\boldsymbol{\nu}_k \neq \mathbf{0}\}} \longrightarrow 1 - \varphi(\lambda_0), \tag{2.7}$$

where $\lambda_0 = \mathbf{a} \oplus (\mathbf{1} - \mathbf{S})$.

Now we consider

$$\varepsilon(k, m, \mathbf{S}) = \frac{F(k, \mathbf{G}(k, m, \mathbf{S})) - F(k, \mathbf{e}^{-\lambda_0/\mathbf{A}_k})}{P\{\boldsymbol{\nu}_k \neq \mathbf{0}\}} = E[B(k, m, \boldsymbol{\nu}_k) \mid \boldsymbol{\nu}_k \neq \mathbf{0}], \tag{2.8}$$

where

$$B(k, m, \boldsymbol{\nu}_k) = \prod_{i=1}^{n} G_i^{\nu_{ik}}(k, m, S_i) - \prod_{i=1}^{n} e^{-\nu_{ik} a_i (1 - S_i)/A_{ik}}. \tag{2.9}$$

Let $\Delta$ be a positive number. Introducing the event

$$C(\Delta, \boldsymbol{\nu}_k) = \left\{ \frac{\nu_{ik}}{A_{ik}} < \Delta, \ i = 1, \ldots, n \right\}, \tag{2.10}$$

we write $\varepsilon(k, m, \mathbf{S})$ as follows:

$$\varepsilon(k, m, \mathbf{S}) = E[B(k, m, \boldsymbol{\nu}_k)\chi \mid \boldsymbol{\nu}_k \neq \mathbf{0}] + E[B(k, m, \boldsymbol{\nu}_k)(1 - \chi) \mid \boldsymbol{\nu}_k \neq \mathbf{0}] \tag{2.11}$$

with $\chi = \chi\{C(\Delta, \boldsymbol{\nu}_k)\}$. If we use the inequality

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right| \leq \sum_{i=1}^{n} |a_i - b_i|, \tag{2.12}$$

which holds for any sets of numbers $a_i$, $b_i$ such that $|a_i| \leq 1$, $|b_i| \leq 1$, $i = 1, \ldots, n$, we obtain that the absolute value of the first expectation on the right-hand side of (2.11) is not greater than

$$\begin{aligned} &\sum_{i=1}^{n} E\Big[ \big| G_i^{\nu_{ik}} - e^{-\nu_{ik} a_i (1 - S_i)/A_{ik}} \big| \chi \mid \boldsymbol{\nu}_k \neq \mathbf{0} \Big] \\ &\quad = \sum_{i=1}^{n} E\left[ \left| \exp\left\{ \frac{\nu_{ik} \delta_i(k, m)}{A_{ik}} \right\} - 1 \right| \chi \mid \boldsymbol{\nu}_k \neq \mathbf{0} \right], \end{aligned} \tag{2.13}$$

where

$$\delta_i(k, m) = A_{ik} \ln G_i(k, m, S_i) + a_i(1 - S_i). \tag{2.14}$$

Taking into account the definition of event $C(\Delta, \boldsymbol{\nu}_k)$, we obtain that the last sum can be estimated by

$$\sum_{i=1}^{n} \max_{l \in D} \left| \exp\left\{ \left( \frac{l}{A_{ik}} \right) \delta_i(k, m) \right\} - 1 \right|, \tag{2.15}$$

where $D = \{l \in \mathbb{N}_0 : l/A_{ik} < \Delta\}$. It follows from (2.6) that $1 - G_i(k,m,\mathbf{S_i}) \to 0$ (since $A_{ik} \to \infty$). Therefore, $\ln G_i \sim -(1-G_i)$ and we conclude from condition (2.2) that $\delta_i(k,m) \to 0$, $i = 1,\ldots,n$. Hence, the first expectation on the right-hand side of (2.11) tends to zero.

Now we consider the second term. Since $|B(k,m,\boldsymbol{\nu}_k)| \le 1$ for all sample points, we obtain that the absolute value of the second expectation is not greater than

$$1 - P\{C(\Delta,\boldsymbol{\nu}_k) \mid \boldsymbol{\nu}_k \ne \mathbf{0}\}, \tag{2.16}$$

which, according to condition (2.1) and the definition of $C(\Delta,\boldsymbol{\nu}_k)$, tends to

$$1 - P\{Y_1 \le \Delta,\ Y_2 \le \Delta,\ldots,Y_n \le \Delta\}. \tag{2.17}$$

This estimation shows that the second expectation on the right-hand side of (2.11) can be made arbitrarily small by choosing sufficiently large $\Delta$. Thus, we conclude that, under conditions (2.1) and (2.2), $\varepsilon(k,m,\mathbf{S}) \to 0$. This, along with (2.4), (2.7), and the fact that

$$E\left[\prod_{i=1}^{n} S_i^{W_i(k,m)} \mid \boldsymbol{\nu}_k \ne \mathbf{0}\right] = 1 - \frac{1 - E\left[\prod_{i=1}^{n} S_i^{W_i(k,m)}\right]}{P\{\boldsymbol{\nu}_k \ne \mathbf{0}\}}, \tag{2.18}$$

gives the assertion of the theorem in the case $a_i < \infty$.

Now, we consider the case when the limit in condition (2.2) is not finite, that is, $a_i = \infty$, $i = 1,\ldots,n$. In this case, we use the following notation: $\mathbf{M}(k,m) = \mathbf{A}_k \oplus \mathbf{P}(k,m) = (M_i(k,m),\ i = 1,\ldots,n)$. The proof is the same as of that of the first case and, therefore, we only provide some important points.

We consider relation (2.18) with $\mathbf{S} = (S_1,\ldots,S_n)$ and $S_i = \exp\{-\lambda_i/M_i(k,m)\}$, where $\lambda_i > 0$, $i = 1,\ldots,n$. Using (2.6), we obtain this time that

$$1 - G_i(k,m,S_i) \sim A_{ik}\lambda_i, \quad k,m \longrightarrow \infty,\ i = 1,\ldots,n. \tag{2.19}$$

Therefore, in relation (2.7), we have $\boldsymbol{\lambda}$ in place of $\lambda_0$. Since $A_{ik} \to \infty$, again $\ln G_i \sim -(1 - G_i)$ and

$$\frac{\ln G_i(k,m,S_i)}{A_{ik}} \longrightarrow -\lambda_i \tag{2.20}$$

as $k,m \to \infty$.

We consider again $\varepsilon(k,m,\mathbf{S})$ from (2.8) replacing $\lambda_0$ by $\lambda$ and putting $S_i = \exp\{-\lambda_i/M_i(k,m)\}$. By the same arguments as in the proof of the first case, we obtain that the absolute value of $\varepsilon(k,m,\mathbf{S})$ can be estimated by sum (2.15) with

$$\delta_i(k,m) = \frac{\ln G_i(k,m,S_i)}{A_{ik}} + \lambda_i \tag{2.21}$$

and $\delta_i(k,m) \to 0$, $i = 1,\ldots,n$, due to (2.20). Hence, $\varepsilon(k,m,\mathbf{S}) \to 0$ as $k,m \to \infty$. Again, appealing to relation (2.18), we obtain the assertion of the theorem when $a_i = \infty$. It is now clear that when some of $a_i$ are finite and the others are infinite, the limit random variable has the Laplace transform $\varphi(\lambda^*)$. Theorem 2.1 is proved. □

The family of vectors (1.1) is eventually a sum of independent vectors if vectors $\boldsymbol{\nu}_k = (\nu_{ik},\ i = 1,\dots,n)$ have degenerate distributions. Therefore, one may expect to obtain a normal limit distribution under some natural assumptions. The next theorem obtains the conditions under which the limit of vector $\mathbf{W}(k,m)$ is a mixture of the normal and a given distribution. Assume the following condition:

(C1) for a given sequence of positive vectors $\mathbf{A}_k$, there exists a sequence $\mathbf{l}_k = (l_{ik},\ i = 1,\dots,n)$, $k \ge 1$, such that $A_{ik}/l_{ik} \to \infty$, $k \to \infty$, $i = 1,\dots,n$, and

$$\mathbf{l}_k \oplus \mathbf{P}(k,m) \oplus (1-\mathbf{P}(k,m)) \longrightarrow \mathbf{C}, \quad i = 1,\dots,n, \tag{2.22}$$

as $k,m \to \infty$, where $\mathbf{C} = (C_i,\ i = 1,\dots,n)$ is a positive vector of constants.

**Theorem 2.2.** *If conditions (2.1) and (C1) are satisfied, then*

$$\left\{\frac{\mathbf{W}(k,m) - \nu_{ik} \oplus \mathbf{P}(k,m)}{\sqrt{\mathbf{A}_k \oplus \mathbf{C}/\mathbf{l}_k}} \mid \boldsymbol{\nu}_k \ne \mathbf{0}\right\} \longrightarrow \mathbf{W} \tag{2.23}$$

*as $k,m \to \infty$, where*

$$P\{\mathbf{W} \le \mathbf{x}\} = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n \Phi\left(\frac{x_i}{\sqrt{y_i}}\right) dT(y_1,\dots,y_n), \tag{2.24}$$

*$\Phi(x)$ is the standard normal distribution, and $T(x_1,\dots,x_n)$ is the distribution of the vector $\mathbf{Y}$ in (2.1).*

**Proof.** Let $\mathbf{W}^*(k,m) = (W_i^*(k,m),\ i = 1,\dots,n)$ with

$$W_i^*(k,m) = \sum_{j=1}^{A_{ik}} \xi_{ij}, \quad i = 1,\dots,n. \tag{2.25}$$

In the proof, we use the following proposition.

**Proposition 2.3.** *Assume that there exist sequences $\{l_{ik},\ k \ge 1\}$, $i = 1,\dots,n$, for which condition (C1) is satisfied. Then, for each $i = 1,\dots,n$, the variable $(W_i^*(k,m) - A_{ik}P_{km}^i)/\sqrt{A_{ik}C_i/l_{ik}}$ is asymptotically normal as $k,m \to \infty$.*

**Proof.** The assertion follows directly from central limit theorem and from trivial identities:

$$EW_i^*(k,m) = A_{ik}P_{km}^{(i)}, \qquad \operatorname{var} W_i^*(k,m) = A_{ik}P_{km}^{(i)}(1-P_{km}^{(i)}). \tag{2.26}$$

□

Now we continue the proof of Theorem 2.2. Let $L(k,m,\mathbf{x})$ be the conditional distribution in Theorem 2.2 and for $i = 1,\dots,n$,

$$F_i(k,m,t_i,x_i) = P\left\{\frac{V_i(k,m) - t_iP_{km}^{(i)}}{\sqrt{A_{ik}C_i/l_{ik}}} \le x_i\right\}, \tag{2.27}$$

where $t_i \in \mathbb{N}_0$ and $V_i(k,m) = \xi_{i1}(k,m) + \cdots + \xi_{it_i}(k,m)$. Using independence of sequences $\{\xi_{ij}(k,m),\ j \geq 1\}$ by total probability arguments, we obtain for any $q > 0$,

$$L(k,m,\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{N}_0^n} \prod_{i=1}^{n} \sum_{t_i\in\Delta} F_i(k,m,t_i,x_i) P_k(t), \tag{2.28}$$

where $\mathbf{h} = (h_1,\ldots,h_n)$, $\mathbf{t} = (t_1,\ldots,t_n)$, $P_k(\mathbf{t}) = P\{\boldsymbol{\nu}_k = \mathbf{t} \mid \boldsymbol{\nu}_k \neq \mathbf{0}\}$, and

$$\Delta_i = \left\{ t_i \in \mathbb{N}_0 : \frac{h_i}{q} \leq \frac{t_i}{A_{ik}} < \frac{h_i+1}{q} \right\}. \tag{2.29}$$

Let now $p > 0$ be such that $pq$ is an integer. We partition the sum on the right-hand side of (2.28) as follows:

$$L(k,m,\mathbf{x}) = \Sigma' + \Sigma'' = I_1 + I_2, \tag{2.30}$$

where $\Sigma'$ is the sum over all vectors $\mathbf{h} \in \mathbb{N}_0^n$ such that $h_i \leq pq$, $i = 1,\ldots,n$, and $\Sigma''$ is the sum over all such vectors that at least one of the coordinates is greater than $pq$.

First we consider $I_1$. Using the monotonicity of the distribution function, we obtain the following estimate for $I_1$:

$$I_1 \leq \Sigma' \prod_{i=1}^{n} \sum_{t_i\in\Delta_i} P\left\{ \frac{V_i(k,m) - t_i P_{km}^{(i)}}{\sqrt{t_i C_i / l_{ik}}} \leq x_i \alpha(i,q) \right\} P_k(\mathbf{t}), \tag{2.31}$$

where $\alpha(i,q) = \sqrt{q/h_i}$ if $x_i > 0$, and it is equal to $\sqrt{q/(h_i+1)}$ if $x_i < 0$. We denote by $\mathbf{Y} = (Y_1,\ldots,Y_n)$ a random vector having distribution $T(x_1,\ldots,x_n)$. Since $h_i A_{ik}/q \leq t_i < (h_i+1)A_{ik}/q$ for $t_i \in \Delta_i$, if $A_{ik} \to \infty$, then so does $t_i$ and $t_i/l_{ik} \to \infty$. Consequently, if we use Proposition 2.3 and condition (2.1), we get

$$\limsup_{k,m\to\infty} I_1 \leq \Sigma' \prod_{i=1}^{n} \Phi(x_i \alpha(i,q)) P\left\{ \frac{h_i}{q} \leq Y_i < \frac{h_i+1}{q},\ i = 1,\ldots,n \right\}. \tag{2.32}$$

Repeating similar arguments, we obtain that

$$\liminf_{k,m\to\infty} I_1 \geq \Sigma' \prod_{i=1}^{n} \Phi(x_i \beta(i,q)) P\left\{ \frac{h_i}{q} \leq Y_i < \frac{h_i+1}{q},\ i = 1,\ldots,n \right\}, \tag{2.33}$$

where $\beta(i,q) = \sqrt{q/(h_i+1)}$ if $x_i > 0$, and it is equal to $\sqrt{q/h_i}$ otherwise. Since for each fixed $p$ and $q \to \infty$ right-hand sides of (2.32) and (2.33) have the same limit, we conclude that

$$\lim_{k,m\to\infty} I_1 = \int_0^p \cdots \int_0^p \prod_{i=1}^{n} \Phi\left( \frac{x_i}{\sqrt{y_i}} \right) dT(y_1,\ldots,y_n). \tag{2.34}$$

Now we consider $I_2$. Recall that $\Sigma''$ is the sum over all vectors $\mathbf{h} \in \mathbb{N}_0^n$ such that at least one of the coordinates is greater than $pq$. Let $h_j$ be the coordinate of $\mathbf{h}$, which is greater than $pq$. Then it is not difficult to see that

$$I_2 \le \sum_{h_j=pq+1}^{\infty} P\left\{\frac{h_j}{q} \le \frac{\nu_{jk}}{A_{jk}} < \frac{h_j+1}{q} \mid \boldsymbol{\nu}_k \ne \mathbf{0}\right\} \le P\left\{\frac{\nu_{jk}}{A_{jk}} > p \mid \boldsymbol{\nu}_k \ne \mathbf{0}\right\}. \tag{2.35}$$

From here, due to condition (2.1), we obtain that

$$\limsup_{k,m\to\infty} I_2 \le 1 - T(p), \tag{2.36}$$

where $T(p) = P\{Y_i < \infty,\ i \ne j,\ Y_j \le p\}$. It is clear that the difference on the right-hand side of (2.36) can be made arbitrarily small by choosing $p$ sufficiently large. Therefore, $I_2 \to 0$ as $k,m \to \infty$. Theorem 2.2 is proved. □

**3. Generalized reduced processes.** Now we give a rigorous definition of the generalized reduced process $\mathbf{X}(\tau,t)$. We use the following notation for individuals participating in the process. Let the process start with a single ancestor at time $t = 0$ of type $T_i$, $i = 1,\dots,n$. We denote it by $T_i$ and consider it as the zeroth generation. We denote the direct offspring of the initial ancestor as $(T_i, T_j, m_j)$, where $T_j$, $j = 1,\dots,n$, is the type of the direct descendant and $m_j \in \mathbb{N}$, $\mathbb{N} = \{1,2,\dots\}$, is the label (the number) of the descendant in the set of all immediate descendants of $T_i$. Thus, the $m_{k+1}$th direct descendant of the type $T_{i_{k+1}}$ of the individual $\boldsymbol{\alpha} = (T_i, T_{i_1}, m_1, \dots, T_{i_k}, m_k)$ will be denoted as $\boldsymbol{\alpha}' = (\alpha, T_{i_{k+1}}, m_{k+1})$. Here and later on, for any two vectors $\boldsymbol{\alpha} = (i_1,\dots,i_k)$ and $\boldsymbol{\beta} = (j_1,\dots,j_m)$, we will understand the ordered pair $(\alpha,\beta)$ as a $(k+m)$-dimensional vector $(i_1,\dots,i_k,\ j_1,\dots,j_m)$.

If we use the above notation, the set $\Re_n \in E$, where $E$ is the space of all finite subsets of

$$\bigcup_{k=1}^{\infty} \mathbb{N}_1^k, \quad \mathbb{N}_1^k = \mathbb{N}_1^{k-1} \times \mathbb{N}_1,\ \mathbb{N}_1 = \{T_i\} \times \{T_1,\dots,T_n\} \times \mathbb{N}, \tag{3.1}$$

corresponds to the population of the $n$th generation. It is clear that $\Re_n$ can be decomposed as $\Re_n = \cup_{i=1}^n \Re_n^{(i)}$, where $\Re_n^{(i)}$ is the population of the type $T_i$ individuals of the $n$th generation. Consequently, components of the process $\mathbf{X}(t)$ are found as $X_i(t) = \operatorname{card}\{\Re_t^{(i)}\}$, $t \in \mathbb{N}_0$, and for any $\tau$ and $t$ such that $\tau < t$, we have

$$\mathbf{X}(t) = \sum_{i=1}^{n} \sum_{\alpha \in \Re_\tau^{(i)}} \mathbf{X}^{(\alpha)}(t-\tau), \tag{3.2}$$

where $\mathbf{X}^{(\alpha)}(t) = (X_1^{(\alpha)}(t),\dots,X_n^{(\alpha)}(t))$ is the $n$-type branching process generated by individual $\alpha$.

Let $\Im_i([\boldsymbol{\theta}],\tau,t)$ be the set of individuals in $\Re_\tau^{(i)}$ having at least one type of descendants at time $t$ more than the corresponding component $\boldsymbol{\theta}(t-\tau)$. It is not difficult to see that it can be described as follows:

$$\begin{aligned}\mathfrak{I}_i([\boldsymbol{\theta}],\tau,t) = \{\alpha \in \mathfrak{R}_\tau^{(i)} : &\text{for at least one } j \ \exists \text{ more than } \theta_j(t-\tau) \\ &\beta\text{-sets such that } (\alpha,\beta) \in \mathfrak{R}_t^{(i)}\},\end{aligned} \tag{3.3}$$

where $\alpha \in \mathbb{N}_1^\tau$, $\beta \in \mathbb{N}_1^{t-\tau}$. Thus, the generalized reduced process is defined as $\mathbf{X}(\tau,t) = (X_i(\tau,t),\ i = 1,\dots,n)$ with $X_i(\tau,t) = \operatorname{card}\{\mathfrak{I}_i([\boldsymbol{\theta}],\tau,t)\}$.

In particular, if $\theta(t) = \mathbf{0}$ for all $t$, then $\mathfrak{I}_i([\mathbf{0}],\tau,t)$ contains all individuals of type $T_i$ only living in the $\tau$th generation and having descendants (at least of one type) in generations $\tau+1, \tau+2,\dots,t$. Consequently, in this case, $\mathbf{X}(\tau,t)$, $0<\tau<t$, is the $n$-type usual reduced branching process.

**4. Limit behavior of the reduced process.** We denote by $P_\alpha^i$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, the offspring distribution of the process $\mathbf{X}(t)$, that is,

$$P_\alpha^i = P\{\mathbf{X}(1) = \alpha \mid \mathbf{X}(0) = \boldsymbol{\delta}_i\} \tag{4.1}$$

is the probability that an individual of type $T_i$ generates the total number $\boldsymbol{\alpha}$ of new individuals. Here, $\boldsymbol{\delta}_i = (\delta_{ij},\ j = 1,\dots,n)$, $\delta_{ij} = 0$, if $i \neq j$ and $\delta_{ii} = 1$. We also denote

$$\begin{aligned}F^i(\mathbf{S}) &= \sum_{\alpha\in\mathbb{N}_0^n} P_\alpha^i S_1^{\alpha_1}\cdots S_n^{\alpha_n}, \qquad \mathbf{F}(\mathbf{S}) = \left(F^1(\mathbf{S}),\dots,F^n(\mathbf{S})\right), \\ Q^i(t) &= P\{\mathbf{X}(t) \neq \mathbf{0} \mid \mathbf{X}(0) = \delta_i\}, \qquad \mathbf{Q}(t) = \left(Q^1(t),\dots,Q^n(t)\right).\end{aligned} \tag{4.2}$$

Let for $i,j,k = 1,2,\dots,n$,

$$a_i^j = \frac{\partial F^j(\mathbf{S})}{\partial S_i}\bigg|_{\mathbf{S}=\mathbf{1}}, \qquad b_{ik}^j = \frac{\partial^2 F^j(\mathbf{S})}{\partial S_i \partial S_k}\bigg|_{\mathbf{S}=\mathbf{1}}, \tag{4.3}$$

let $\mathbf{A} = \|a_i^j\|$ be the matrix of expectations, let $\rho$ be its Peron root, and let the right and the left eigenvectors $\mathbf{U} = (u_1,u_2,\dots,u_n)$ and $\mathbf{V} = (v_1,v_2,\dots,v_n)$ corresponding to the Peron root be such that

$$\mathbf{AU} = \rho\mathbf{U}, \qquad \mathbf{VA} = \rho\mathbf{V}, \qquad (\mathbf{U},\mathbf{V}) = 1, \qquad (\mathbf{U},\mathbf{1}) = 1. \tag{4.4}$$

If $\mathbf{A}$ is indecomposable, aperiodic, and $\rho = 1$, the process $\mathbf{X}(t)$ is called a critical indecomposable multitype branching process. We assume that the generating function $\mathbf{F}(\mathbf{S})$ satisfies the following representation:

$$x - \sum_{j=1}^n v_j\left(\mathbf{1} - F^j(\mathbf{1} - \mathbf{U}x)\right) = x^{1+\alpha}L(x), \tag{4.5}$$

where $0 < x \le 1$, $\alpha \in (0,1]$, and $L(x)$ is a slowly varying function as $x \downarrow 0$. Note that in this case, $\rho = 1$, that is, the process is critical and the second moments of the offspring distribution $b_{ik}^j$, $i,j,k = 1,\dots,n$, may not be finite. Under this assumption, the following limit theorem for the process $\mathbf{X}(t)$ holds (see [13]).

**Proposition 4.1.** *If the offspring generating function* $\mathbf{F}(\mathbf{S})$ *satisfies representation (4.5), then*

(a) $Q^i(t) \sim u_j t^{-1/\alpha} L_1(t)$ *as* $t \to \infty$, *where* $L_1(t)$ *is a slowly varying function as* $t \to \infty$;

(b) $\lim_{t\to\infty} P\{\mathbf{X}(t)q(t) \le \mathbf{x} \oplus \mathbf{V} \mid \mathbf{X}(t) \ne \mathbf{0},\ \mathbf{X}(0) = \boldsymbol{\delta}_i\} = \pi(\mathbf{x})$, *where* $q(t) = \sum_{j=1}^n v_j Q^j(t)$ *and* $\pi(\mathbf{x}) = \pi(x_1, x_2, \dots, x_n)$, *a distribution having the Laplace transform*

$$\phi(\boldsymbol{\lambda}) = \int_{\mathbb{R}_+^n} e^{-(\mathbf{x},\lambda)} d\pi(\mathbf{x}) = 1 - \left(1 + \bar{\lambda}^{-\alpha}\right)^{-1/\alpha}, \quad \bar{\lambda} = (\boldsymbol{\lambda}, \mathbf{1}). \tag{4.6}$$

Now, we are in a position to state our first result about $\mathbf{X}(\tau, t)$. Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n$, $\mathbb{R}_+ = [0, \infty)$, $\mathbf{C} = (C_1, \dots, C_n) \in \mathbb{R}_+^n$ be some nonnegative vectors.

**Theorem 4.2.** *If condition (4.5) is satisfied,* $\theta(t) = \theta \oplus \mathbf{V}/q(t)$, *and* $t, \tau \to \infty$, $t - \tau \to \infty$ *such that* $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$, *then*

$$P\{\mathbf{X}(\tau, t) = \mathbf{k} \mid \mathbf{X}(\tau) \ne \mathbf{0},\ \mathbf{X}(0) = \delta_i\} \longrightarrow P_{\mathbf{k}}^*, \tag{4.7}$$

*where* $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ *and the probability distribution* $\{P_k^*, \mathbf{k} \in \mathbb{N}_0^n\}$ *has the generating function* $\phi^*(\mathbf{S}) = \phi(\mathbf{a})$ *with* $\mathbf{a} = b\mathbf{C} \oplus \mathbf{U} \oplus \mathbf{V} \oplus (\mathbf{1} - \mathbf{S})$, $b = 1 - \pi(\theta)$, $\mathbf{S} = (S_1, \dots, S_n)$, *and* $\phi(\boldsymbol{\lambda})$ *is the Laplace transform defined in (4.6).*

**Remark 4.3.** It is clear that vector $\mathbf{C}$ in the condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau) \to \mathbf{C}$ necessarily has the form $\mathbf{C} = C\mathbf{1}$, where $C \ge 0$ is some constant.

**Example 4.4.** Let $\mathbf{F}(\mathbf{S})$ satisfy condition (4.5) with $\alpha = 1$. We note here that in this case, the second moments of the offspring distribution may be still infinite. For this kind of a process, the limit distribution $\pi(\theta)$ is exponential and the generating function $\phi^*(\mathbf{S})$ has the form $\phi^*(\mathbf{S}) = (1+d)^{-1}$, where $d = bC\sum_{j=1}^n u_j v_j (1 - S_j)$, $b = e^{-\theta^*}$, $\theta^* = \min\{\theta_1, \dots, \theta_n\}$. We represent it as follows:

$$\phi^*(\mathbf{S}) = \frac{1}{1 + Ce^{-\theta^*}} \left(1 - \frac{Ce^{-\theta^*}}{1 + Ce^{-\theta^*}} \sum_{i=1}^n u_i v_i S_i\right)^{-1}. \tag{4.8}$$

What is the distribution having the last probability generating function? To answer this question we consider a sequence of independent random variables $X_1, X_2, \dots$ such that $P\{X_i = j\} = p_j$, $j = 0, 1, 2, \dots, n$, $\sum_{j=0}^n p_j = 1$, where $p_0 = (1 + Ce^{-\theta^*})^{-1}$, $p_j = Ce^{-\theta^*} u_j v_j / (1 + Ce^{-\theta^*})$, $j = 1, 2, \dots, n$. Let $\Delta_1$ be the number of 1's, let $\Delta_2$ be the number of 2's, and so on, let $\Delta_n$ be the number of $n$'s observed in the sequence $X_1, X_2, \dots$ before the first zero is obtained. Then it follows from the formula of the generating function of generalized multivariate geometric distribution in [6, Chapter 36.9] that the vector $(\Delta_1, \dots, \Delta_n)$ has the probability generating function given by (4.8), that is,

$$E\left(S_1^{\Delta_1} S_2^{\Delta_2} \cdots S_n^{\Delta_n}\right) = \phi^*(\mathbf{S}). \tag{4.9}$$

Hence, we have the following result.

**Corollary 4.5.** *If the assumptions of Theorem 4.2 are satisfied with $\boldsymbol{\alpha} = 1$, then the probability distribution $\{P_k^*, \mathbf{k} \in \mathbb{N}_0^n\}$ is a multivariate geometric distribution defined by the generating function (4.8) such that*

$$P_k^* = P\{\Delta_i = k_i,\ i = 1,\dots,n\}. \tag{4.10}$$

*It is clear that, if $n = 1$, the distribution is geometric, that is, $P_k^* = pq^k$, $k = 0,1,\dots$ with $p = (1+Ce^{-\theta_1})^{-1}$, $q = Ce^{-\theta_1}(1+Ce^{-\theta_1})^{-1}$.*

**Example 4.6.** Let the assumptions of Theorem 4.2 be satisfied and $\tau = [\varepsilon t]$, $0 < \varepsilon < 1$. Using the asymptotic behavior of $\mathbf{Q}(t)$ and the uniform convergence theorem for the slowly varying functions, we obtain that as $t \to \infty$,

$$\frac{\mathbf{Q}(t-\tau)}{\mathbf{Q}(\tau)} \longrightarrow \left(\frac{\varepsilon}{1+\varepsilon}\right)^{1/\alpha} \mathbf{1}. \tag{4.11}$$

Consequently, in this case, the limit distribution has the generating function $\phi^*(\mathbf{S})$ with $C = (\varepsilon/(1+\varepsilon))^{1/\alpha}$. In particular, we have the following result.

**Corollary 4.7.** *If the assumptions of Theorem 4.2 are satisfied and $\tau = o(t)$, then*

$$\lim_{t\to\infty} P\{\mathbf{X}(\tau,t) = \mathbf{k} \mid \mathbf{X}(\tau) \neq \mathbf{0},\ \mathbf{X}(0) = \delta_i\} = 0 \tag{4.12}$$

*for all $\mathbf{k} \in \mathbb{N}_0^n$ and $\mathbf{k} \neq \mathbf{0}$.*

It is known that in the critical case, the process $\mathbf{X}(t)$ goes to extinction with probability 1. Corollary 4.7 shows that, if $\tau = o(t)$, even conditioned process $\mathbf{X}(\tau,t)$ given $\mathbf{X}(\tau) \neq \mathbf{0}$ vanishes with a probability approaching 1.

Theorem 4.2 gives a limit distribution for $\mathbf{X}(\tau,t)$ when the times of observation $\tau, t \to \infty$ such that $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau)$ has a finite limit. Now, we consider the case when this limit is not finite. Let $T_i(\tau,t) = Q^i(t-\tau)/Q^i(\tau)$ and $\mathbf{T}(\tau,t) = (T_1(\tau,t),\dots,T_n(\tau,t))$.

**Theorem 4.8.** *If condition (4.5) holds, $\theta(t) = \theta \oplus \mathbf{V}/q(t)$, and $t,\tau \to \infty$, $t-\tau \to \infty$, such that $T_i(\tau,t) \to \infty$, $i = 1,2,\dots,n$, then*

$$P\left\{\frac{\mathbf{X}(\tau,t)}{\mathbf{T}(\boldsymbol{\tau},\mathbf{t})} \leq \mathbf{x} \mid \mathbf{X}(\tau) \neq \mathbf{0},\ \mathbf{X}(0) = \delta_i\right\} \longrightarrow \pi\left(\frac{1}{b}\mathbf{x}\right), \tag{4.13}$$

*where $\pi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}_+^n$, is the distribution from Proposition 4.1 and $b = 1-\pi(\boldsymbol{\theta})$.*

**Remark 4.9.** It follows from the asymptotic behavior of $Q^i(t)$ that, if $T_i(\tau,t) \to \infty$ for at least one $i$, then it holds for each $i = 1,2,\dots,n$.

**Example 4.10.** If matrix $\mathbf{A}$ is indecomposable, aperiodic, $\rho = 1$, and $b_{jk}^i < \infty$, $i,j,k = 1,\dots,n$, then (4.5) is satisfied with $\boldsymbol{\alpha} = 1$, $L(x) \to \text{const}$, $x \to 0$. In this case, $Q^i(t) \sim 2u_i/\sigma^2 t$, $i = 1,\dots,n$, as $t \to \infty$, where $\sigma^2 = \sum_{j,m,k=1}^n v_j b_{mk}^j u_m u_k$. Consequently,

$$q(t) = \sum_{j=1}^n Q^j(t) v_j \sim \frac{2}{\sigma^2 t}, \quad t \longrightarrow \infty, \tag{4.14}$$

and $\boldsymbol{\theta}(t) \sim \sigma^2 t \boldsymbol{\theta} \oplus \mathbf{V}/2$. On the other hand, $b = e^{-\theta^*}$, $\theta^* = \min\{\theta_1, \dots, \theta_n\}$, and $T_j(\tau, t) \sim \tau/(t-\tau)$, $j = 1, \dots, n$. Thus, $T_j(\tau, t) \to \infty$ if, for example, $\tau \sim t$ and we obtain the following result from Theorem 4.8.

**Corollary 4.11.** *If $\rho = 1$, $0 < \sigma^2 < \infty$, and $t, \tau \to \infty$, $t - \tau \to \infty$, such that $\tau \sim t$, then*

$$P\left\{\frac{t-\tau}{\tau}\mathbf{X}(\tau, t) \le \mathbf{x} \mid \mathbf{X}(\tau) \ne \mathbf{0},\ \mathbf{X}(0) = \delta_i\right\} \longrightarrow 1 - \exp\left\{-\frac{x^*}{b^*}\right\}, \tag{4.15}$$

*where $\mathbf{x} \in \mathbb{R}_+^n$, $x^* = \min\{x_1, \dots, x_n\}$, and $b^* = \exp\{-\min\{\theta_1, \dots, \theta_n\}\}$.*

The above two theorems describe the asymptotic behavior of $\mathbf{X}(\tau, t)$ when $t - \tau \to \infty$. Now we consider the case $\tau = t - \Delta$, where $\Delta \in (0, \infty)$ is a constant.

**Theorem 4.12.** *If condition (4.5) is satisfied, $t, \tau \to \infty$, such that $t - \tau = \Delta \in (0, \infty)$ and $\boldsymbol{\theta}(t) = \boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n$, then*

$$P\{\mathbf{X}(\tau, t) \oplus \mathbf{Q}(\tau) \le \mathbf{x} \mid \mathbf{X}(\tau) \ne \mathbf{0},\ \mathbf{X}(0) = \delta_i\} \longrightarrow \pi\left(\frac{\mathbf{x}}{\mathbf{R}(\Delta)}\right), \tag{4.16}$$

*where $\mathbf{x} \in \mathbb{R}_+^n$ and*

$$\mathbf{R}(\Delta) = \left(R^1(\Delta), \dots, R^n(\Delta)\right), \qquad R^i(\Delta) = P\left\{\bigcup_{j=1}^n \{X_j(\Delta) > \theta_j\} \mid \mathbf{X}(0) = \delta_i\right\}. \tag{4.17}$$

**Remark 4.13.** It follows from Proposition 4.1 that

$$\frac{Q^i(\tau)}{Q^i(t)} \sim \left(\frac{t}{t-\Delta}\right)^{1/\alpha} \frac{L_1(t-\Delta)}{L_1(t)}, \tag{4.18}$$

which shows that $Q^i(\tau) \sim Q^i(t)$ as $t, \tau \to \infty$, $t - \tau = \Delta$, for each $i = 1, \dots, n$. Therefore, the vector of normalizing functions $\mathbf{Q}(\tau)$ in Theorem 4.12 can be replaced by $\mathbf{Q}(t)$.

## 5. Proofs of the theorems of Section 4

**Proof of Theorem 4.2.** It follows from the definition of $\mathbf{X}(\tau, t)$ in Section 3 that its $i$th component can be written as

$$X_i(\tau, t) = \sum_{\alpha \in R_\tau^{(i)}} \chi\left(\bigcup_{j=1}^n \{X_j^{(\alpha)}(t-\tau) > \theta_j(t-\tau)\}\right). \tag{5.1}$$

Since $\operatorname{card}\{R_\tau^{(i)}\} = X_i(\tau)$, from here we can see that it can be presented in the form (1.1) with $\nu_{i\tau} = X_i(\tau)$ and

$$\xi_{ij}(\tau, t) = \chi\left(\bigcup_{l=1}^n \{X_{il}^j(t-\tau) > \theta_l(t-\tau)\}\right), \tag{5.2}$$

where $X_{il}^{j}(t)$ is the number of individuals of type $T_l$ at time $t$ in the process initiated by the $j$th individual of type $T_i$. Hence, Theorem 2.1 can be applied. It follows from Proposition 4.1 that condition (2.1) is satisfied with $\mathbf{A}_\tau = \mathbf{V}/q(\tau)$ and with

$$\varphi(\lambda) = Ee^{-(\mathbf{Y},\lambda)} = \phi(\lambda), \tag{5.3}$$

where $\phi(\lambda)$ is defined in (4.6). Furthermore, it is not difficult to see that

$$\begin{aligned} E\xi_{ij}(\tau,t) &= P\left(\bigcup_{l=1}^{n}\{X_{il}^{j}(t-\tau) > \theta_l(t-\tau)\}\right) \\ &= P\left(\bigcup_{l=1}^{n}\{X_{il}^{j}(t-\tau)q(t-\tau) > \theta_l v_l\},\ \mathbf{X}(t-\tau)\neq\mathbf{0},\ \mathbf{X}(0)=\delta_i\right). \end{aligned} \tag{5.4}$$

Hence, using the asymptotic behavior of $\mathbf{Q}(t)$ again, we obtain that when $\tau,t\to\infty$, $t-\tau\to\infty$,

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \sim bv_i\frac{Q^i(t-\tau)}{q(\tau)}. \tag{5.5}$$

From here, taking into account the condition $\mathbf{Q}(t-\tau)/\mathbf{Q}(\tau)\to\mathbf{C}$, we conclude that

$$\frac{E\xi_{ij}(\tau,t)}{q(\tau)/v_i} \longrightarrow bC_iu_iv_i, \tag{5.6}$$

which shows that condition (2.2) of Theorem 2.1 is also satisfied with $\mathbf{a} = b\mathbf{C}\oplus\mathbf{U}\oplus\mathbf{V}$. Consequently, the assertion of Theorem 4.2 follows from Theorem 2.1. Theorem 4.2 is proved. □

**Proof of Theorem 4.8.** We again use Theorem 2.1. As it was shown in the proof of Theorem 4.2, condition (2.1) of Theorem 2.1 is satisfied with $\mathbf{A}_\tau = \mathbf{V}/q(\tau)$. Now, we consider

$$M_i(\tau,t) = \frac{E\xi_{ij}(\tau,t)}{q(\tau)}v_i, \tag{5.7}$$

where $\xi_{ij}(\tau,t)$ is the same as in (5.2). Appealing again to the asymptotic behavior of $\mathbf{Q}(t)$, we obtain that

$$M_i(\tau,t) \sim bv_iT_i(\tau,t)\frac{Q^i(\tau)}{q(\tau)} \tag{5.8}$$

as $t,\tau\to\infty$, $t-\tau\to\infty$. It follows from (5.8) that the condition $M_i(\tau,t)\to\infty$ of Theorem 2.1 is also satisfied when $T_i(\tau,t) = Q^i(t-\tau)/Q^i(\tau)\to\infty$. The assertion of Theorem 4.8 follows now from Theorem 2.1. □

**Proof of Theorem 4.12.** We use again Theorem 2.1. As in the proofs of the preceding theorems, condition (2.1) follows from Proposition 4.1. If $t-\tau \in (0,\infty)$, we obtain from (5.2) that

$$M_i(\tau,t) = \frac{R_i(\Delta)}{q(\tau)}v_i. \tag{5.9}$$

Thus, we have that $M_i(\tau,t) \to \infty$ as $t,\tau \to \infty$, $t-\tau \in (0,\infty)$. We obtain from Theorem 2.1 that

$$E\left[\prod_{i=1}^{n} e^{-\lambda_i X_i(\tau,t)Q^i(\tau)} \mid \mathbf{X}(\tau) \neq \mathbf{0}\right] \longrightarrow \varphi(\lambda \oplus \mathbf{R}(\Delta)). \tag{5.10}$$

This yields the assertion of Theorem 4.12. □

**6. The number of productive ancestors.** Now we consider a population containing, at time $t=0$, a random number $\nu_i(t)$, $i=1,\dots,n$, $t \in \mathbb{N}_0$, of individuals (ancestors) of $n$ different types $T_1,\dots,T_n$. Each of these individuals generates a discrete-time indecomposable $n$-type branching stochastic process. Let $\theta(t) = (\theta_1(t),\dots,\theta_n(t))$ be a vector of nonnegative functions. In how many processes generated by these ancestors the number of descendants at time $t$ of at least one type will exceed the corresponding level given by $\theta(t)$? To answer the question, we investigate the process $\mathbf{Y}(t) = \mathbf{Y}([\theta],t) = (Y_1(t),\dots,Y_n(t))$, where $Y_i(t)$ is the number of initial individuals of type $T_i$, whose number of descendants at time $t$ of at least one type is greater than the corresponding component of the vector $\theta(t)$. It is clear that $\mathbf{Y}(t)$ takes into account only "relatively productive" ancestors regulated by the family of levels $\theta(t)$, $t \in \mathbb{N}_0$.

Process $\mathbf{Y}(t)$ may be associated with the following scheme describing the growth of $n$-type trees in a forest. Suppose at time zero we have $\nu_i(t)$, $i=1,\dots,n$, one-branch trees of types $T_i$. Each of these trees will grow and give new branches of types $T_1,\dots,T_n$ according to independent, indecomposable $n$-type branching processes. Then process $\mathbf{Y}(t) = (Y_1(t),\dots,Y_n(t))$ will count the number of "big trees": the variable $Y_i(t)$ is the number of big trees of type $T_i$ having more than $\theta_j(t)$ new branches at time $t$ for at least one $j$, $j=1,\dots,n$.

It is not difficult to see that the components of the process $Y_i(t)$ can be presented as

$$Y_i(t) = \sum_{j=1}^{\nu_i(t)} \xi_{ij}(t), \tag{6.1}$$

where $\xi_{ij}(t) = \chi(\bigcup_{l=1}^{n}\{X_{il}^{j}(t) > \theta_l(t)\})$ and $X_{il}^{j}(t)$ is as before, the number of individuals of type $T_l$ at time $t$ in the process initiated by the $j$th ancestor of type $T_i$. Consequently, theorems proved for random sum (1.1) may be applied to this process.

Let all the assumptions from Section 4 on $n$-type branching process $\mathbf{X}(t)$, $t \in \mathbb{N}_0$, be satisfied and the generating function corresponding to probability distribution $P_\alpha^i$, $\alpha \in \mathbb{N}_0^n$, satisfy (4.5).

**Theorem 6.1.** *Let condition (4.5) be satisfied and $\theta(t) = \theta \oplus \mathbf{V}/q(t)$, $\theta \in \mathbb{R}_+^n$. If condition (2.1) is satisfied and for the normalizing coefficients in (2.1)*

$$A_{it} Q^i(t) \longrightarrow \infty \tag{6.2}$$

*as $t \to \infty$ for $i = 1,\dots,n$, then*

$$P\left\{\frac{Y_i(t) - \nu_{it} a_i(t)}{\sqrt{\nu_{it} a_i(t)}} \le x_i,\ i = 1,\dots,n \mid \nu \ne \mathbf{0}\right\} \longrightarrow L(\mathbf{x}), \tag{6.3}$$

*where $\mathbf{x} \in \mathbb{R}^n$, $a_i(t) = bQ^i(t)$, $b = 1 - \pi(\theta)$, $\theta \in \mathbb{R}_+^n$, and $L(\mathbf{x})$ is defined in (2.24).*

**Proof.** We demonstrate that the conditions of Theorem 2.2 are satisfied. It is clear that we just need to show that condition (C1) holds for the variables defined in (6.1). As in the proof of Theorem 4.2, we easily obtain that

$$\begin{aligned} E\xi_{ij}(t) &= P\left(\bigcup_{l=1}^{n} \{X_{il}^j(t) > \theta_l(t)\}\right) \\ &= P\left(\bigcup_{l=1}^{n} \{X_{il}^j(t) q(t) > \theta_l v_l\},\ \mathbf{X}(t) \ne \mathbf{0},\ \mathbf{X} = \delta_i\right) \sim bQ^i(t). \end{aligned} \tag{6.4}$$

Consequently, if we take $l_{it} = 1/Q^i(t)$, then for $i = 1,\dots,n$ as $t \to \infty$,

$$l_{it} E\xi_{ij}(t)\left(1 - E\xi_{ij}(t)\right) \longrightarrow b. \tag{6.5}$$

On the other hand, $A_{it}/l_{it} \to \infty$, $i = 1,\dots,n$, as $t \to \infty$ due to condition (6.2). Hence, condition (C1) of Theorem 2.2 is satisfied and the assertion of the theorem follows from Theorem 2.2. Theorem 6.1 is proved. □

In conclusion, we note that according to condition (6.2), the assertion of Theorem 6.1 holds when the initial population is large enough. One may obtain limit distributions for $\mathbf{Y}(t)$ when this condition is not satisfied. To do it, one needs to apply Theorem 2.1.

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