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ON THE BEURLING ALGEBRAS $A_\alpha^+(\mathbb{D})$—DERIVATIONS AND EXTENSIONS

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Based on a description of the squares of cofinite primary ideals of $A_\alpha^+(\mathbb{D})$, we prove the following results: for $\alpha \geq 1$, there exists a derivation from $A_\alpha^+(\mathbb{D})$ into a finite-dimensional module such that this derivation is unbounded on every dense subalgebra; for $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$, every finite-dimensional extension of $A_\alpha^+(\mathbb{D})$ splits algebraically if and only if $\alpha \geq m + 1/2$.

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1. Introduction. Let α be a positive real number. By $\mathbb{D}$, we denote the open unit disk. The Beurling algebra $A_\alpha^+(\mathbb{D})$ is a subalgebra of the classical disk algebra $A(\mathbb{D})$. For $f \in A(\mathbb{D})$ with power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$, the function f belongs to $A_\alpha^+(\mathbb{D})$ if and only if $\sum_{n=0}^\infty |a_n|(n+1)^\alpha < \infty$. In this case, we define $\|f\|_\alpha := \sum_{n=0}^\infty |a_n|(n+1)^\alpha$. Clearly, $A_\alpha^+(\mathbb{D})$ is a Banach algebra with respect to this norm.

These algebras have been considered in [13] where results on primary ideals were applied to operator theory. More recently, the algebras have appeared in the examination of finite-dimensional extensions of a whole range of commutative Banach algebras [4]. The present paper deals with continuity problems of derivations from $A_\alpha^+(\mathbb{D})$ and with finite-dimensional extensions of this special type of Beurling algebras. Some of the results of the first paper will be the starting point for our investigation.

This paper is organized as follows: as a preparation, Section 2 describes the squares of cofinite primary ideals and exhibits an approximate identity for a special ideal in $A_\alpha^+(\mathbb{D})$. The results of Section 2 will be applied to the questions considered in Sections 3 and 4. These sections are investigating derivations and extensions, respectively, and are rather independent of each other.

Section 3, which is on derivations from $A_\alpha^+(\mathbb{D})$ into finite-dimensional Banach modules, follows the approach used by [2] for Banach algebras of differentiable functions on the unit interval. In our case, we are interested in derivations from $A_\alpha^+(\mathbb{D})$ which are discontinuous on the subalgebra of polynomials. For $\alpha > 1$, we give an example of a derivation which is unbounded on every dense subalgebra.

Section 4 then turns to the problem of finite-dimensional extensions guided by the ideas of [4] which makes a comprehensive approach on extensions of Banach algebras in general. We solve a problem raised there: for $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$, every finite-dimensional extension of $A_\alpha^+(\mathbb{D})$ splits algebraically if and only if $\alpha \geq m + 1/2$.

2. Primary ideals of $A^+_\alpha(\mathbb{D})$. Suppose that $m \in \mathbb{N}$ and that $\alpha \in [m, m+1)$. Let $f \in A^+_\alpha(\mathbb{D})$. Then f is m-times continuously differentiable, and $f^{(m)}$, the mth derivative of f, belongs to $A^+_0(\mathbb{D})$. In fact,

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} a_n z^{n-m} \quad (z \in \overline{\mathbb{D}}). \tag{2.1}$$

Therefore, we expect the algebras $A^+_\alpha(\mathbb{D})$ to resemble the Banach algebras of m-times continuously differentiable functions on the unit interval $C^m[0,1]$ in more than one aspect. It will be of some use for us to turn this observation into a precise statement.

Lemma 2.1. *Let $g \in A(\mathbb{D})$, $m \in \mathbb{N}$, and $\alpha \in [m, m+1)$. Then $g \in A^+_\alpha(\mathbb{D})$ if and only if g is m-times continuously differentiable and $g^{(m)} \in A^+_{\alpha-m}(\mathbb{D})$. Furthermore, there exists a constant $C > 0$ such that*

$$\|f^{(m)}\|_{\alpha-m} \le \|f\|_\alpha \le C \cdot \left(\|f^{(m)}\|_{\alpha-m} + \sum_{n=0}^{m-1} |f^{(n)}(0)| \right) \quad (f \in A^+_\alpha(\mathbb{D})). \tag{2.2}$$

Primary ideals of $A^+_\alpha(\mathbb{D})$ have been investigated before in [13]. This paper gives a full description of all closed ideals which are contained solely in one maximal ideal, the ideal of all functions vanishing at 1. Let $m \in \mathbb{Z}^+$, and let $\alpha \in [m, m+1)$. For $n \le m$, define

$$I_{\alpha,n} = \{ f \in A^+_\alpha(\mathbb{D}) \mid f(1) = f^{(1)}(1) = \cdots = f^{(n)}(1) = 0 \}. \tag{2.3}$$

Then $I_{\alpha,n}$ is a closed ideal in $A^+_\alpha(\mathbb{D})$. For formal reasons, set $I_{\alpha,-1} = A^+_\alpha(\mathbb{D})$. These are the only cofinite primary ideals corresponding to 1 (cf. Section 3). As in [13, Lemma 2.4], it is straightforward to prove that $I_{\alpha,n} = \overline{(Z-1)^{n+1} A^+_\alpha(\mathbb{D})}$ for $n = 0, \dots, m$. Here Z denotes the function $z \mapsto z$ and 1 the constant function with value 1 on $\overline{\mathbb{D}}$.

Of course, all results on the ideals $I_{\alpha,n}$ hold for the corresponding primary ideals for any other point of $\mathbb{T}$.

For an ideal I in a Banach algebra A, we define I^2 to be the linear span of the set $I^{[2]} := \{ a \cdot b \mid a, b \in I \}$. We refer to I^2 as the *square of I*.

According to [13], $I_{\alpha,0}$ has a bounded approximate identity if and only if $\alpha = 0$, and it has an approximate identity if and only if $\alpha < 1$. For $\alpha \ge 1$, the ideal $I_{\alpha,0}$ does not have an approximate identity since $\overline{I^2_{\alpha,0}} = \overline{\overline{(Z-1)A^+_\alpha(\mathbb{D})} \cdot \overline{(Z-1)A^+_\alpha(\mathbb{D})}} = I_{\alpha,1}$. In fact, $I^2_{\alpha,0}$ is not even closed in this case. This can be verified by the following lemma which is essentially [1, Example 3]. Clearly, the lemma implies that $I^2_{\alpha,0} \subsetneq I_{\alpha,1}$.

Lemma 2.2. *Let $m \in \mathbb{N}$. Suppose that $\alpha \in [m, m+1)$. For $g, h \in I_{\alpha,0}$, the function gh is $(m+1)$-times differentiable at 1, and*

$$(gh)^{(m+1)}(1) = \sum_{i=1}^{m} \binom{m+1}{i} g^{(i)}(1) \cdot f^{(m+1-i)}(1). \tag{2.4}$$

Hence, $I^2_{\alpha,0} \subseteq \{ f \in A^+_\alpha(\mathbb{D}) \mid f^{(1)}(1) = f(1) = 0,\ f^{(m+1)}(1)$ exists$\}$.

In order to describe the squares of these ideals, we use the same approach as used in [2] where the Banach algebras $C^m[0,1]$ are investigated. For these algebras, the ideals

$$M_{m,n} = \{f \in C^m[0,1] \mid f(0) = \cdots = f^{(n)}(0) = 0\} \tag{2.5}$$

are defined for $n = 0,\dots,m$. Let $m \in \mathbb{N}$ and let T represent the function which is given by $[0,1] \mapsto [0,1]$, $t \mapsto t$. Then [2, Theorem 2.1] gives the following description:

(i) $M_{m,0}^2 = T \cdot M_{m,0} = \{f \in C^m[0,1] \mid f^{(1)}(0) = f(0) = 0,\ f^{(m+1)}(0) \text{ exists}\}$;
(ii) $M_{m,n}^2 = T^{n+1}M_{m,n}$ for $n = 0,\dots,m-1$;
(iii) $M_{m,m}^2 = T^m M_{m,m}$.

We expect similar results to hold for $A_\alpha^+(\mathbb{D})$. Of course, we will require different arguments due to the different norm structure of $A_\alpha^+(\mathbb{D})$.

The next result is [13, Lemma 2.1]. We give a version which is a bit more precise.

Lemma 2.3. *Suppose that $0 < \alpha < 1$. Let $g \in I_{\alpha,0}$. There exists $h \in A(\mathbb{D})$ with $h(1) = 0$ such that $\|h\|_\infty \le 2\|g\|_\alpha$ and $g = (Z-1)^\alpha h$.*

As in [13, Lemma 2.2], it is easy to check that, for real numbers $\alpha, \beta > 0$, where β is not an integer, $(Z-1)^\beta \in A_\alpha^+(\mathbb{D})$ if and only if $\beta > \alpha$.

In [13, Proposition 2.6], a sequence of polynomials $(e_{n,m})_{n\in\mathbb{N}}$ is defined by

$$e_{n,m} = 1 - \frac{1}{\binom{m+n+1}{m+1}} \sum_{j=0}^{n} \binom{m+n-j}{m} Z^j \tag{2.6}$$

for every $m \in \mathbb{Z}^+$. It is shown that $\lim_{n\to\infty}(e_{n,m}f) = f$ for each $f \in I_{\alpha,m}$ and a given $\alpha \in [m, m+1)$. Note that, for $n, m \in \mathbb{N}$,

$$(Z-1)(e_{n,m} - 1) = \frac{m+1}{n+1} \cdot e_{n+1,m-1}. \tag{2.7}$$

Surprisingly, these polynomials will turn out to define an approximate identity for some other Banach algebra. The next lemma is our key observation.

Lemma 2.4. *Suppose that $\alpha \ge 1$, and let $n \in \mathbb{Z}^+$ such that $n \le \alpha$. Let $g \in I_{\alpha,n}$. Then there exists $f \in I_{\alpha-1,n-1}$ such that $(Z-1)f = g$ and $\|f\|_{\alpha-1} \le \|g\|_\alpha$.*

Proof. Since $g(1) = 0$ and $\alpha \ge 1$, there exists $f \in A(\mathbb{D})$ with $(Z-1)f = g$. Now suppose that $(a_n)_{n\in\mathbb{Z}^+}$ and $(b_n)_{n\in\mathbb{Z}^+}$ are the sequences of the Fourier coefficients for g and f, respectively. Then $b_n = -\sum_{i=0}^n a_i$ $(n \in \mathbb{Z}^+)$. Since $\sum_{i=0}^\infty a_i = g(1) = 0$, it follows that

$$\begin{aligned}\sum_{n=0}^\infty (n+1)^{\alpha-1}|b_n| &= \sum_{n=0}^\infty (n+1)^{\alpha-1}\left|\sum_{i=n+1}^\infty a_i\right| \\ &\le \sum_{i=1}^\infty \sum_{n=0}^{i-1} (n+1)^{\alpha-1}|a_i| \\ &\le \sum_{i=1}^\infty (i+1)^\alpha |a_i| \le \|g\|_\alpha.\end{aligned} \tag{2.8}$$

Hence $f \in A_{\alpha-1}^+(\mathbb{D})$. It is immediate that $f \in I_{(\alpha-1),n-1}$. ∎

By an induction using Lemmas 2.3 and 2.4, we now obtain some useful estimates for the growth of functions in $A_\alpha^+(\mathbb{D})$.

Proposition 2.5. *Let $m \in \mathbb{Z}^+$, $n \in \{0,\dots,m\}$, and $\alpha \in [m,m+1)$.*

(i) *Suppose that $m \ge 1$, and that $n < m$. Let $f \in I_{\alpha,n}$. Then there exists $g \in A^+_{\alpha-(n+1)}(\mathbb{D})$ with $\|g\|_{\alpha-(n+1)} \le \|f\|_\alpha$ such that $f = (Z-1)^{n+1}g$.*

(ii) *Let $f \in I_{\alpha,m}$. Then there exists $h \in A(\mathbb{D})$ with $h(1) = 0$, $\|h\|_\infty \le 2\|f\|_\alpha$, such that $f = (Z-1)^\alpha h$.*

Throughout the paper, we will make frequent use of the following corollary of Proposition 2.5. For completeness, we also include the above-mentioned variation of [13, Lemma 2.2] (cf. the remark after Lemma 2.3).

Corollary 2.6. *Let $m \in \mathbb{Z}^+$, $\alpha \in [m,m+1)$, and $\beta > 0$.*

(i) *Suppose that β is not an integer. Then $(Z-1)^\beta \in A_\alpha^+(\mathbb{D})$ if and only if $\beta > \alpha$.*

(ii) *$(Z-1)^\beta \in I_{\alpha,m}$ if and only if $\beta > \alpha$.*

Next we apply the approach of [2, Theorem 2.1] to our situation. We are following the idea that, for the investigation of functions in $I_{\alpha,m}$, the common divisor $(Z-1)^m$ is redundant and therefore division by $(Z-1)^m$ establishes a linear isomorphism. Naturally, the image is a Banach space with respect to the norm induced by this isomorphism. Since the image is also a subspace of $A(\mathbb{D})$ and since $I_{\alpha,m}^2 \subseteq I_{\alpha,m}$, we assume that, with respect to some equivalent norm, this Banach space is in fact a Banach algebra.

Proposition 2.7. *Let $m \in \mathbb{Z}^+$, suppose that $\alpha \in [m,m+1)$, and let $\epsilon = \alpha - m$. Define a linear space B_α via $B_\alpha := \{f \in I_{\epsilon,0} \mid (Z-1)^m f \in I_{\alpha,m}\}$ and a norm $\|\cdot\|_{B_\alpha}$ on B_α by*

$$\|f\|_{B_\alpha} := \sum_{j=0}^{m} \big\|(Z-1)^j f\big\|_{j+\epsilon} \quad (f \in B_\alpha). \tag{2.9}$$

Then the following hold:

(i) *with respect to an equivalent norm, B_α is a Banach algebra;*

(ii) *the Banach algebra B_α has a sequential approximate identity; this approximate identity is bounded if and only if $\alpha = m$;*

(iii) *the map $B_\alpha \mapsto I_{\alpha,m}$, $f \mapsto (Z-1)^m f$, is a linear homeomorphism.*

Proof. Note that all assertions hold for the case $m = 0$ (cf. [13] to verify (ii)).

Now suppose that $m \ge 1$. It follows by a simple induction from Lemma 2.4 that B_α is a Banach space and that $\|f\|_{B_\alpha} \le (m+1)\|(Z-1)^m f\|_\alpha$ for $f \in B_\alpha$. Hence, (iii) holds.

In order to show that B_α is a Banach algebra, we need a different characterization. Using Lemmas 2.1 and 2.4, it can be shown that

$$B_\alpha = \{f \in I_{\epsilon,0} \mid f \in C^m(\overline{\mathbb{D}} - \{1\}),\ f^{(j)}(Z-1)^j \in A_\epsilon^+(\mathbb{D})\ (j = 0,\dots,m)\}. \tag{2.10}$$

Here $C^m(\overline{\mathbb{D}} - \{1\})$ is the algebra of functions on $\overline{\mathbb{D}} - \{1\}$ which are m-times continuously differentiable, and the term "$\in A_\epsilon^+(\mathbb{D})$" implies that the function in question can be extended to all of $\mathbb{D}$. Therefore, we can now introduce a different norm $\|\cdot\|_{B'_\alpha}$ on B_α by

$$\|f\|'_{B_\alpha} := \sum_{j=0}^{m} \left\|(Z-1)^j f^{(j)}\right\|_\epsilon. \tag{2.11}$$

By a straightforward, albeit lengthy induction, this norm is equivalent to the original norm. Here we have to use Lemma 2.1 again.

Now let $f,h \in B_\alpha$. Then $fh \in C^m(\overline{\mathbb{D}}-\{1\})$ and, for $j = 0,\dots,m$,

$$(fh)^{(j)}(Z-1)^j = \sum_{l=0}^{j} \binom{j}{l} f^{(l)}h^{(j-l)}(Z-1)^{j-l}(Z-1)^l \in A_\epsilon^+(\mathbb{D}). \tag{2.12}$$

Thus, $fh \in B_\alpha$, that is, B_α is an algebra. In fact, the multiplication is jointly continuous with respect to $\|\cdot\|'_{B_\alpha}$ since

$$\left\|(fh)^{(j)}(Z-1)^j\right\|_\epsilon \le \sum_{l=0}^{j} \binom{j}{l} \left\|f^{(l)}(Z-1)^l\right\|_\epsilon \left\|h^{(j-l)}(Z-1)^{j-l}\right\|_\epsilon \tag{2.13}$$

for $j = 0,\dots,m$. Hence, (i) has been proved.

To show (ii), consider the sequence $(e_{n,m})_{n\in\mathbb{N}}$ described at the beginning of this section. We have already mentioned that, for $g \in I_{\alpha,m}$, we have $\lim_{n\to\infty}\|ge_{n,m}-g\| = 0$. Now let $f \in B_\alpha$. Then $(Z-1)^m f \in I_{\alpha,m}$ and $\lim_{n\to\infty}\|(Z-1)^m(fe_{n,m}-f)\|_\alpha = 0$. By (iii), $\lim_{n\to\infty}\|fe_{n,m}-f\|_{B_\alpha} = 0$. We have proved that $(e_{n,m})_{n\in\mathbb{N}}$ is an approximate identity for B_α.

Suppose that $\alpha = m$. We show by induction on m that $(\|e_{n,m}\|_{B_m})_{n\in\mathbb{N}}$ is bounded. This is equivalent to $(\|(Z-1)^m e_{n,m}\|_m)_{n\in\mathbb{N}}$ being bounded. The case $m = 0$ is obvious. Now let $m \ge 1$. For $n \in \mathbb{N}$,

$$\left\|(Z-1)^{m-1}e_{n+1,m-1}\right\|_m \le (m+n+1)\left\|(Z-1)^{m-1}e_{n+1,m-1}\right\|_{m-1} \tag{2.14}$$

since this polynomial is of degree $n+m$. Hence,

$$\begin{aligned}\left\|(Z-1)^m e_{n,m}\right\|_m &\le \left\|\frac{m+1}{n+1}(Z-1)^{m-1}\cdot e_{n+1,m-1} + (Z-1)^m\right\|_m \\ &\le \frac{(m+1)(m+n+1)}{n+1}\left\|(Z-1)^{m-1}e_{n+1,m-1}\right\|_{m-1} + \left\|(Z-1)^m\right\|_m\end{aligned} \tag{2.15}$$

which is bounded in n by the induction hypothesis. □

We now obtain a result analogous to [2, Theorem 2.1].

Corollary 2.8. *Let $m,n \in \mathbb{Z}^+$ and $\alpha \in [m,m+1)$.*

(i) *Suppose that $\alpha = m$. Then $I_{m,m}^{[2]} = I_{m,m}^2 = (Z-1)^m I_{\alpha,m}$.*

(ii) *Suppose that $\alpha > m$. Then $I_{\alpha,m}^2 \subsetneq (Z-1)^m I_{\alpha,m}$.*

(iii) *Suppose that $n \le m-1$. Then $I_{\alpha,n}^2 = (Z-1)^{n+1} I_{\alpha,n}$.*

(iv) *$I_{\alpha,m}$ has a multiplier-bounded approximate identity. Thus $I_{\alpha,m} = \overline{I_{\alpha,m}^2}$.*

(v) *Suppose that $0 \le n \le (m-1)/2$. Then $\overline{I_{\alpha,n}^2} = I_{\alpha,2n+1}$.*

(vi) *Suppose that* $(m-1)/2 \le n \le m$*. Then* $\overline{I^2_{\alpha,n}} = I_{\alpha,m}$.

(vii) *Suppose that* $\alpha > 0$*. Then the ideal* $I^2_{\alpha,n}$ *is of infinite codimension for* $0 \le n \le m$.

Proof. First, suppose that $\alpha > m$. Let $\beta \in (\alpha, 2\alpha - m)$. By Corollary 2.6, $(Z-1)^\beta \in I_{\alpha,m}$. Assume towards a contradiction that $(Z-1)^{\beta+m} \in I^2_{\alpha,m}$. By Proposition 2.5,

$$z \longmapsto \frac{(z-1)^{\beta+m}}{(z-1)^{2\alpha}} \quad \text{is bounded on } \mathbb{D}. \tag{2.16}$$

But $\beta + m < 2\alpha$, a contradiction. Now (i) and (ii) follow as in [2, Theorem 2.1].

(iii) Clearly, $(Z-1)^{n+1} I_{\alpha,n} \subseteq I^2_{\alpha,n}$. Let $f, g \in I_{\alpha,n}$. There exist polynomials $p, q \in \mathbb{C}[Z]$ and $\tilde{f}, \tilde{g} \in I_{\alpha,m}$ such that $f = p \cdot (Z-1)^{n+1} + \tilde{f}$ and $g = q \cdot (Z-1)^{n+1} + \tilde{g}$. Therefore,

$$fg = (Z-1)^{2n+2} pq + (Z-1)^{n+1} p\tilde{f} + (Z-1)^{n+1} q\tilde{g} + \tilde{f}\tilde{g}. \tag{2.17}$$

By (i) and (ii), $\tilde{f}\tilde{g} \in (Z-1)^m I_{\alpha,m}$, and $I^2_{\alpha,n} = (Z-1)^{n+1} I_{\alpha,m}$ follows.

(iv) By the principle of uniform boundedness, the sequence $(e_{n,m})_n$ is bounded as a sequence of multipliers on $I_{\alpha,m}$. Now consider the net $(b_\alpha) := (e_{n_1} \cdots e_{n_{m+1}})_{n_1,\dots,n_{m+1}}$ in $I_{\alpha,m}$, where the index set $\mathbb{N}^{m+1}$ is directed by the product order. Using the multiplier boundedness of $(e_{n,m})_n$, it is straightforward that (b_λ) is an approximate identity.

(v) and (vi) are now immediate.

(vii) It suffices to show that $I^2_{\alpha,0}$ is of infinite codimension. As shown earlier, the ideal $I^2_{\alpha,0}$ is not closed for $\alpha \ge 1$. For $m = 0$, let $\beta \in (\alpha, \min(2\alpha, 1))$. Then $(Z-1)^\beta \notin I^2_{\alpha,0}$ as we have shown in the proof of (ii). In any case, $I^2_{\alpha,0}$ is not closed. Since $A^+_\alpha(\mathbb{D})$ is separable, $I^2_{\alpha,0}$ is of infinite codimension by [5]. $\square$

Example 2.9. There is one question which remains to be checked in order to obtain a thorough comparison to the $C^m[0,1]$ case; suppose that $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$: does $I^2_{\alpha,0} = \{f \in I_{\alpha,1} \mid f^{(m+1)}(1) \text{ exists}\}$ hold? For $\alpha > m$, it is easy to see that the answer is in the negative since, for $\beta \in (m, \alpha)$, the fact that $(Z-1)^{\beta+1} \in I_{\alpha,1}$ provides a counterexample.

In order to decide the question for $\alpha = m$, we may suppose for simplicity that $\alpha = 1$. We know that $I_{1,1} \subseteq (Z-1) I_{0,0}$. Define $g \in A(\mathbb{D})$ by $\sum_{n=1}^\infty (\exp(i\sqrt{n})/n) \cdot Z^n$. By [15, Theorem 5.2], this series converges uniformly on $\mathbb{T}$. Hence, it converges to an element of $A(\mathbb{D})$. Clearly, $g \notin A^+_0(\mathbb{D})$. Now consider $g \cdot (Z-1)$. The coefficients $(b_n)_n$ of the corresponding Taylor series are $b_0 = 0$, $b_1 = -e^i$, and

$$b_n = \frac{e^{i\sqrt{n-1}}}{n-1} - \frac{e^{i\sqrt{n}}}{n} \quad (n \in \mathbb{N},\ n \ge 2). \tag{2.18}$$

We can now easily estimate the growth of the Fourier coefficients. In fact,

$$|b_n| = \left| \frac{n - (n-1)e^{i(\sqrt{n} - \sqrt{n-1})}}{(n-1)n} \right| \le \frac{1 + C \cdot n/\sqrt{n}}{(n-1)n} \tag{2.19}$$

for $n \in \mathbb{N}$, $n \ge 2$, and some constant $C > 0$. Hence, $g \cdot (Z-1) \in A^+_0(\mathbb{D})$. By Lemma 2.1, there exists $f \in A^+_1(\mathbb{D})$ such that $f(1) = 0$ and $f' = g \cdot (Z-1)$. Clearly, f is two-times differentiable at 1 and $f \in I_{1,1}$. Now assume towards a contradiction that $f \in I^2_{1,0}$. It follows that $f = (Z-1)h$ for some $h \in I_{1,0}$. Hence, $f' = (Z-1)h' + h$.

Recall that $h' \in A_0^+(\mathbb{D})$ by Lemma 2.1. By Proposition 2.5, there exists $\tilde{h} \in A_0^+(\mathbb{D})$ such that $h = (Z-1)\tilde{h}$. Since $g = h' + \tilde{h}$, we see that $g \in A_0^+(\mathbb{D})$, a contradiction. Hence, the answer to the above question is again in the negative.

In order to complete the picture, we give the following obvious results on primary ideals at points in $\mathbb{D}$.

Proposition 2.10. *Let $\lambda \in \mathbb{D}$ and let $\alpha \geq 0$. For $n \in \mathbb{Z}^+$, define*

$$I_{\alpha,n}^{\lambda} = \{f \in A_\alpha^+(\mathbb{D}) \mid f(\lambda) = f^{(1)}(\lambda) = \cdots = f^{(n)}(\lambda) = 0\}. \tag{2.20}$$

Then $I_{\alpha,n}^{\lambda}$ is a principal ideal, that is, $I_{\alpha,n}^{\lambda} = (Z-\lambda)^{n+1}A_\alpha^+(\mathbb{D})$. Every closed primary ideal corresponding to λ is of the form $I_{\alpha,n}^{\lambda}$ for some $n \in \mathbb{Z}^+$. Furthermore, $(I_{\alpha,n}^{\lambda})^2 = I_{\alpha,2n+2}^{\lambda}$.

3. Derivations from $A_\alpha^+(\mathbb{D})$. Let A be a Banach algebra, and E a Banach A-bimodule. A derivation from A to E is a linear map $D : A \to E$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A). \tag{3.1}$$

Suppose that $\dim E = 1$. Then there exist characters φ, ψ on A such that $D(ab) = \varphi(a)D(b) + \psi(b)D(a)$. In this case, D is called a *point derivation*.

Derivations have been investigated for many years. In particular, there are numerous articles concerned with continuity questions for derivations. Whereas there are many results which automatically ensure continuity for a specific class of derivations [7, Section 10], some questions still remain open due to the few established methods for the construction of discontinuous derivations. One of these questions is [8, Question 2.3] which focuses on the disk algebra $A(\mathbb{D})$ and on $l^1(\mathbb{Z}^+)$ (which is $A_0^+(\mathbb{D})$, of course). In a more general setting, we may put forward the following questions.

A Banach algebra A is a *Banach algebra of power series* if it can be embedded continuously into the algebra of formal power series $\mathbb{C}[[X]]$ such that $\mathbb{C}[X]$ is contained in the image of this embedding. Here $\mathbb{C}[[X]]$ is given the topology of coordinatewise convergence (cf. [7, Section 5]). A Banach A-bimodule E is *symmetric* if the right and the left actions of A on E coincide. Then we may simply call E a *Banach A-module*.

(i) Let A be a Banach algebra of power series. Do there exist a Banach A-module E and a derivation $D : A \to E$ such that D is unbounded on every dense subalgebra?

If an answer to this question does not seem to be achievable, we may weaken the question.

(ii) Let A be a Banach algebra of power series with $\overline{\mathbb{C}[Z]} = A$. Do there exist a Banach A-module E and a derivation $D : A \to E$ such that D is unbounded on the polynomials?

Here we identify those elements of A which are mapped to $\mathbb{C}[X]$ by the embedding into $\mathbb{C}[[X]]$ as the *polynomials* of A, and we denote the subalgebra of these polynomials by $\mathbb{C}[Z]$ to obtain a formal distinction.

Although discontinuous derivations on $A(\mathbb{D})$ (or arbitrary Banach algebras of power series) have been constructed [3, 6, 10], all these examples consist of derivations vanishing on the polynomials and thus do not even answer question (ii). There have been attempts to modify these constructions in order to obtain a positive answer for $l^1(\mathbb{Z}^+)$ (see [12]). However, the problem is still open (for $A(\mathbb{D})$ and $l^1(\mathbb{Z}^+)$).

In this context, it is of some interest to consider other related Banach algebras of power series as there are the algebras $A_\alpha^+(\mathbb{D})$, subalgebras of $A(\mathbb{D})$ and $A_0^+(\mathbb{D})$, or weighted discrete convolution algebras $l^1(\mathbb{Z}^+,\omega)$, where ω is a radical weight. In the latter case, where the algebra contains $A(\mathbb{D})$ and $A_0^+(\mathbb{D})$, the author was able to find a positive solution for question (ii) (see [14]).

Surprisingly, it is not too difficult to give a positive answer to question (ii) for $A_\alpha^+(\mathbb{D})$ with $\alpha \geq 1/2$. Using the results of Section 2 on the ideal structure, we are even able to describe all derivations having finite-dimensional image. Again, we follow the approach of [2]. However, it is not always possible to transfer their arguments in a straightforward way and we will make some observations differing from their results. In particular, we obtain an affirmative answer to the first question if $\alpha \geq 1$.

It is easy to see that, for a derivation D from a unital algebra A and for a polynomial $p \in \mathbb{C}[X]$, we have $p'(a)\cdot D(a) = D(p(a))$ for each $a \in A$. In particular, $D(1) = 0$. Note that this implies that, for a Banach algebra of power series in which the polynomials are dense, the set $\overline{D(A)}$ is a submodule of E for every derivation $D : A \to E$.

First, we use arguments similar to [2, page 239] in order to show that a restriction of our investigations to a simple type of modules is justified. Let $m \in \mathbb{Z}^+$ and $\alpha \in [m,m+1)$. Suppose that E is a finite-dimensional Banach $A_\alpha^+(\mathbb{D})$-module. Choosing a basis $\eta_1,\dots,\eta_n$ such that the matrix corresponding to the action of $Z \in A_\alpha^+(\mathbb{D})$ obtains its canonical Jordan form, we see that E can be decomposed into the direct sum of finite-dimensional submodules which correspond to the different Jordan blocks. If there exists only one summand of this type, E is called *indecomposable*. In this case, we see that the module multiplication by $f \in A_\alpha^+(\mathbb{D})$ corresponds to the matrix

$$\begin{pmatrix} f(\lambda) & f^{(1)}(\lambda) & \frac{1}{2}f^{(2)}(\lambda) & \cdots & \frac{1}{(n-1)!}f^{(n-1)}(\lambda) \\ 0 & f(\lambda) & f^{(1)}(\lambda) & & \frac{1}{(n-2)!}f^{(n-2)}(\lambda) \\ & & & \ddots & \\ \vdots & & & & f^{(1)}(\lambda) \\ 0 & \cdots & & 0 & f(\lambda) \end{pmatrix}, \tag{3.2}$$

where $\lambda \in \sigma(Z) = \overline{\mathbb{D}}$. This also shows that the ideals $I_{\alpha,n}$ $(n = 0,\dots,m)$ are the only cofinite closed primary ideals for the character 1.

Therefore, in the case where $\lambda \in \mathbb{T}$, it follows that $\dim E \leq m+1$. If $\lambda \in \mathbb{D}$, then no restriction occurs. An indecomposable module of this type is referred to as a *cyclic module at* λ. This term implies that the module is of finite dimension. The basis $\eta_1,\dots,\eta_k$ is called *the standard basis* (which is unique if we demand that $\|\eta_1\| = 1$). Note that, for

$\lambda \in \mathbb{D}$, we obtain a continuous linear map $\rho : A \to \mathcal{B}(E)$,

$$\rho(f)\xi = f' \cdot \xi := \begin{pmatrix} f^{(1)}(\lambda) & f^{(2)}(\lambda) & \cdots & \frac{1}{(n-1)!} f^{(n)}(\lambda) \\ 0 & f^{(1)}(\lambda) & & \frac{1}{(n-2)!} f^{(n-1)}(\lambda) \\ & & \ddots & \\ \vdots & & & f^{(2)}(\lambda) \\ 0 & \cdots & 0 & f^{(1)}(\lambda) \end{pmatrix} \cdot \xi. \tag{3.3}$$

Here we identify ξ and its coordinate vector, and $\mathcal{B}(E)$ denotes the Banach algebra of bounded linear operators on E. This notation is consistent with our earlier definition of the mapping $\mathbb{C}[Z] \mapsto \mathcal{B}(E)$, $p \mapsto p' \cdot (\cdot)$. The same holds in the case where $\lambda \in \mathbb{T}$, provided that $\dim E \le m$.

Next, suppose that $E = \oplus_{i=1}^n E_i$, where $E_1, \ldots, E_n$ are indecomposable submodules of E. Then there exist pairwise orthogonal projections $P_1, \ldots, P_n$ onto $E_1, \ldots, E_n$, respectively, such that each projection commutes with the module action. Now let $D : A_\alpha^+(\mathbb{D}) \to E$ be a derivation. Then $D_i := P_i D$ is a derivation into E_i for each $i = 1, \ldots, n$, and $\sum_{i=1}^n D_i = D$. Obviously, D is continuous (on $\mathbb{C}[Z]$) if and only if $D_1, \ldots, D_n$ are continuous (on $\mathbb{C}[Z]$).

Now consider the case where E is infinite dimensional and the image of D is of finite dimension. Then $D(A)$ is closed and hence, as mentioned above, a submodule. Thus, when considering the continuity of derivations with finite-dimensional image (as a map from $A_\alpha^+(\mathbb{D})$ or from $\mathbb{C}[Z]$), we may always suppose that the module E is finite-dimensional and indecomposable, or, equivalently, that E is cyclic at some $\lambda \in \mathbb{D}$. In this situation, a derivation $D : A_\alpha^+(\mathbb{D}) \to E$ is called a *cyclic* derivation at λ. The dimension of the submodule $D(A)$ is called the height of D.

In order to describe all cyclic derivations for $A_\alpha^+(\mathbb{D})$, we need the notion of a singular derivation. For a Banach algebra of power series A, a derivation D is called *singular* if D vanishes on the polynomials. Thus, a derivation which is bounded on the polynomials can be written as the sum of a continuous and a singular derivation. Such a derivation is called *decomposable*. Our main interest is to find derivations for which this decomposition is not possible.

First, note that, for $\lambda \in \mathbb{D}$, every derivation into a cyclic Banach $A_\alpha^+(\mathbb{D})$-module at λ is continuous. This follows from Proposition 2.10 and the fact that the elements of $A_\alpha^+(\mathbb{D})$ are infinitely differentiable at λ. In this situation, every derivation is given by $f \mapsto f' \cdot \xi$ for some $\xi \in E$.

The last observation implies that, when looking for derivations unbounded on $\mathbb{C}[Z]$, we have to consider cyclic modules at points of $\mathbb{T}$. At the beginning of this section, we have seen that their dimension is necessarily less than $m+1$. Hence, we are dealing merely with point derivations if $m = 0$. We have to consider this case separately. Recall that there is a one-to-one correspondence between point derivations at 1 and those linear functionals on $I_{\alpha,0}$ which vanish on $I_{\alpha,0}^2$ (cf. [9, Proposition 1.8.8]).

Proposition 3.1. *Let $\alpha \in (0,1)$.*

(i) *For $A^+_\alpha(\mathbb{D})$, there exists a singular point derivation at 1.*

(ii) *Suppose that $0 \le \alpha < 1/2$. Then every nontrivial point derivation on $A^+_\alpha(\mathbb{D})$ is singular and hence decomposable.*

(iii) *Suppose that $1/2 \le \alpha < 1$. Then there exists a point derivation on $A^+_\alpha(\mathbb{D})$ which is unbounded on $\mathbb{C}[Z]$.*

Proof. (i) Let $\beta \in (\alpha,1)$ such that $\beta < 2\alpha$. Then $(Z-1)^\beta \in A^+_\alpha(\mathbb{D})$. By Corollary 2.6, we know that $(Z-1)^\beta \notin (I_{\alpha,0})^2 + \mathbb{C}[Z]$. Now the claim follows.

(ii) We may consider point derivations at a point of $\mathbb{T}$, say at 1. Since $\alpha < 1/2$, $(Z-1)^{1/2} \in A^+_\alpha(\mathbb{D})$. Thus $(Z-1)^{1/2}$ belongs to $I_{\alpha,0}$. Hence, $(Z-1) \in (I_{\alpha,0})^2$, and $0 = D(Z-1) = D(Z)$ for every point derivation D at 1.

(iii) We will construct the required derivation at 1. By Proposition 2.5, $(Z-1) \notin (I_{\alpha,0})^2$. Now define a linear functional D on $A^+_\alpha(\mathbb{D})$ such that

$$D(1) = 0, \qquad D\Big(\big(I_{\alpha,0}\big)^2\Big) = \{0\}, \qquad D(Z-1) = 1. \tag{3.4}$$

That D is a point derivation at 1 can be easily verified since, for $f \in A^+_\alpha(\mathbb{D})$, we have $f = f(1)1 + (f - f(1)1)$ and hence, for $f,g \in A^+_\alpha(\mathbb{D})$, we see that

$$D(fg) = f(1)D\big(g - g(1)1\big) + g(1)D\big(f - f(1)1\big) = f(1)D(g) + g(1)D(f). \tag{3.5}$$

In particular, $D(p) = p'(1)$ for every $p \in \mathbb{C}[Z]$. Hence, D is unbounded on the polynomials. □

Note that, for $\alpha = 0$, every point derivation at 1 is zero (and hence continuous) since $I^2_{\alpha,0} = I_{\alpha,0}$. Note further that the last proposition does not provide a positive answer to our initial question (i) in the case $1/2 \le \alpha < 1$: every point derivation at 1 vanishes on $\operatorname{span}\{1, I^2_{\alpha,0}\}$ which is a dense subalgebra.

The first implication of the following result can be proved in exactly the same way as [2, Theorem 5.2]. The second implication is immediate if we recall the definition of the map $f \mapsto f' \cdot \xi$ above.

Proposition 3.2. *Let $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$. Suppose that E is a cyclic Banach $A^+_\alpha(\mathbb{D})$-module at 1.*

(i) *Let $D : A^+_\alpha(\mathbb{D}) \to E$ be a continuous derivation. Then the height of D is at most m, and, for some $\xi \in E$, we have $D(f) = f'(1) \cdot \xi$ ($f \in A^+_\alpha(\mathbb{D})$).*

(ii) *Suppose that $\dim E \le m$ and let $D : A^+_\alpha(\mathbb{D}) \to E$ be a derivation. Then D is continuous and hence decomposable.*

As a next step, we describe singular derivations from $A^+_\alpha(\mathbb{D})$ into cyclic modules at 1.

Proposition 3.3. *Let $m,n \in \mathbb{N}$ with $n \le m+1$, and $\alpha \in [m, m+1)$. Let E be a cyclic, n-dimensional Banach $A^+_\alpha(\mathbb{D})$-module at 1. Suppose that $\eta_1, \dots, \eta_n$ form the standard basis.*

(i) *Let $D : A_\alpha^+(\mathbb{D}) \to E$ be a derivation. Then D is singular if and only if a functional μ on $A_\alpha^+(\mathbb{D})$ with $\mu|_{\mathbb{C}[Z]} = 0$ and $\mu((Z-1)^n A_\alpha^+(\mathbb{D})) = \{0\}$ exists such that*

$$D(f) = \sum_{l=0}^{n-1} \mu((Z-1)^l f)\eta_{l+1} \quad (f \in A_\alpha^+(\mathbb{D})). \tag{3.6}$$

(ii) *Suppose that $\alpha = m$. Then there exists a singular derivation of height n into E if and only if $n \le m$.*

(iii) *Suppose that $\alpha > m$. Then there exists a singular derivation of height n into E.*

Proof. Again, the result may be shown almost completely as the corresponding result in [2, Theorem 5.3], taking into account that this theorem actually deals with height $k+1$. Note that (ii) follows since $I_{\alpha,m}^2 = (Z-1)^m I_{\alpha,m}$ for $\alpha = m$. The only implication we still have to prove is the following: for $\alpha > m$, there exists a singular derivation of height $n = m+1$ into E. In fact,

$$(Z-1)^{m+1} I_{\alpha,m} \subsetneq I_{\alpha,m}^2 \subsetneq (Z-1)^m I_{\alpha,m} \tag{3.7}$$

by Corollary 2.8. Therefore, we may define a linear functional μ on $A_\alpha^+(\mathbb{D})$ such that, for $l = 0,\dots,2m+1$,

$$\mu(I_{\alpha,m}^2) = \{0\}, \qquad \mu((Z-1)^m I_{\alpha,m}) \neq \{0\}, \qquad \mu((Z-1)^l) = 0. \tag{3.8}$$

Here we have used Corollary 2.8(vii). Now define $D : A_\alpha^+(\mathbb{D}) \to E$ and let $D(f) = \sum_{l=0}^m \mu((Z-1)^l f)\eta_{l+1}$. Proceeding as in the proof of [2, Theorem 5.3], we see that D is a derivation and $D(A_\alpha^+(\mathbb{D})) = E$. □

We are now constructing a derivation on $A_\alpha^+(\mathbb{D})$ which is unbounded on the polynomials. The result should be compared with [2, Theorem 5.4]. Recall that the dual Banach space of $A_\alpha^+(\mathbb{D})$ can be identified naturally and isometrically with the space

$$l^\infty(\mathbb{Z}^+, (n+1)^{-\alpha}) := \left\{ (c_n)_{n\in\mathbb{Z}^+} \,\middle|\, \|(c_n)\| := \sup_n \frac{|c_n|}{(n+1)^\alpha} < \infty \right\}. \tag{3.9}$$

Theorem 3.4. *Suppose that $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$. Let E be an $(m+1)$-dimensional, cyclic Banach $A_\alpha^+(\mathbb{D})$-module at 1 with standard basis $\eta_1,\dots,\eta_{m+1}$. There exists a linear functional μ on $A_\alpha^+(\mathbb{D})$ such that $\mu(1) = 0$, $\mu(Z-1) = 0$, and $\mu(f) = (1/m!) f^{(m+1)}(1)$ $(f \in I_{\alpha,0}^2)$. The map $D : A_\alpha^+(\mathbb{D}) \to E$,*

$$D(f) = \mu(f)\eta_1 + \sum_{i=1}^m \frac{1}{(m-i)!} f^{(m+1-i)}(1) \cdot \eta_{i+1}, \tag{3.10}$$

is a derivation which is unbounded on $\mathbb{C}[Z]$. Furthermore, D is discontinuous on every dense subalgebra.

Proof. By Lemma 2.2, μ is well defined since $(Z-1) \notin I_{\alpha,0}^2$ and $1 \notin I_{\alpha,0}$. It is easily checked that D is indeed a derivation. Clearly $D(\mathbb{C}[Z]) = E$. Since $\dim E = m+1$, it follows that D is unbounded on the polynomials by Proposition 3.2.

Now let $\mathcal{A}$ be a dense subalgebra of $A_\alpha^+(\mathbb{D})$. Assume towards a contradiction that D is continuous on $\mathcal{A}$. We may suppose that $1 \in \mathcal{A}$. Now μ is continuous on $\mathcal{A}$. Therefore, we can define a continuous linear functional $\gamma = (c_n)_{n\in\mathbb{Z}^+} \in l^\infty(\mathbb{Z}^+, (n+1)^{-\alpha})$ such that $\gamma|_{\mathcal{A}} = \mu|_{\mathcal{A}}$. Note that the algebra $\mathcal{B}$, where $\mathcal{B} = \mathcal{A} \cap I_{\alpha,0}$, is dense in $I_{\alpha,0}$.

Let $p \in \mathbb{C}[Z]$. We may find sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ with $\lim_{n\to\infty} f_n = (Z-1)p$, $\lim_{n\to\infty} g_n = (Z-1)^m$, and $g_n \in I_{\alpha,m-1}$ $(n \in \mathbb{N})$. Then

$$\gamma(f_n \cdot g_n) = \mu(f_n \cdot g_n) = \frac{1}{m!}(f_n \cdot g_n)^{(m+1)}(1) = \frac{m+1}{m!} f_n^{(1)}(1) g_n^{(m)}(1) \tag{3.11}$$

by Lemma 2.2. Hence

$$\begin{aligned}\gamma\big((Z-1)^{m+1}p\big) &= \lim_{n\to\infty} \gamma(f_n g_n) = \lim_{n\to\infty} \frac{m+1}{(m!)} f_n^{(1)}(1) g_n^{(m)}(1) \\ &= (m+1)p(1) = \frac{1}{m!}\big(p\cdot(Z-1)^{m+1}\big)^{(m+1)}(1).\end{aligned} \tag{3.12}$$

In other words, γ coincides with μ on $(Z-1)^{m+1}\mathbb{C}[Z]$.

Now define a sequence $(a_n)_{n\in\mathbb{Z}^+}$ by setting $a_n = \mu(Z^n)$. We claim that $c_n - a_n = O(n^m)$. Let $b_n = a_n - c_n$ $(n \in \mathbb{Z}^+)$. Then, for $n \in \mathbb{Z}^+$,

$$0 = (\mu-\gamma)\big((Z-1)^{m+1}Z^n\big) = \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^{m+1-i} b_{i+n}. \tag{3.13}$$

For $n \in \mathbb{Z}^+$, define $\xi_{m+n} \in \mathbb{C}^{m+1}$ by $\xi_{m+n} = (b_{m+n}, \ldots, b_n)$, and define further $M \in M_{m+1}(\mathbb{C})$ by

$$\begin{pmatrix} +\binom{m+1}{m} & (-1)\binom{m+1}{m-1} & \cdots & & (-1)^m\binom{m+1}{0} \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}. \tag{3.14}$$

Then $\xi_{n+1} = M \cdot \xi_n$.

The characteristic polynomial of M is $(Z-1)^{m+1}$ again. On the other hand, the minimal polynomial of M is of degree $m+1$ since $\{M^k e_{n+1}\}_{k=1}^m$ is a linearly independent set. Here e_{n+1} is the $(n+1)$th canonical basis vector. Therefore, the Jordan form N of M is given by

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 1 \\ 0 & \cdots & & 0 & 1 \end{pmatrix} \tag{3.15}$$

and, for $n \geq m$, we have

$$N^n = \begin{pmatrix} 1 & \binom{n}{1} & \cdots & \cdots & \binom{n}{m} \\ 0 & 1 & \binom{n}{1} & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & \ddots & \binom{n}{1} \\ 0 & \cdots & & 0 & 1 \end{pmatrix}. \tag{3.16}$$

If we define the norm on $M_{m+1}(\mathbb{C})$ to be the maximum of the moduli of the matrix coefficients, then $\|M^n\| = \binom{n}{m}$ for large n. Thus $\|M^n\| = O(n^m)$.

Hence, $(b_n)_{n\in\mathbb{Z}^+} \in l^\infty(\mathbb{Z}^+,(n+1)^{-\alpha})$. But now this implies that $(a_n)_{n\in\mathbb{Z}^+} \in l^\infty(\mathbb{Z}^+,(n+1)^{-\alpha})$, a contradiction to D being unbounded on $\mathbb{C}[Z]$. □

The theorem is somewhat surprising when we take into account that the derivation maps into a finite-dimensional module and that $\ker D$ is a cofinite subalgebra. However, for $m \in \mathbb{N}$, we are able to extend this result to the algebra $C^m[0,1]$. In fact, using the embedding

$$\iota : A_m^+(\mathbb{D}) \longrightarrow C^m[0,1], \qquad \iota(f)(t) = f(e^{2\pi i t}), \tag{3.17}$$

the proof of the theorem can be easily modified to obtain the following.

Corollary 3.5. *Let $m \in \mathbb{N}$. There exists a derivation from $C^m[0,1]$ which is unbounded on every dense subalgebra and maps into a finite-dimensional module.*

This establishes a simpler example of a derivation of this special type than that given by [2, Proposition 6.2].

For $\alpha \in [m,m+1)$ and $\lambda \in \mathbb{T}$, we will call a functional μ on $A_\alpha^+(\mathbb{D})$ with $\mu(1) = 0$, $\mu((Z-1)^n) = 0$ $(n = 1,\ldots,m)$, and $\mu(f) = (1/m!)f^{(m+1)}(1)$ for $f \in I_{\alpha,0}^2$ a *generalized derivative of order* $(m+1)$ *at* λ. The corresponding derivation into a cyclic, $m+1$-dimensional module will be denoted by D_μ.

We would like to conclude that every derivation into an $(m+1)$-dimensional, cyclic module at 1 can be decomposed into the sum of a continuous derivation, a singular derivation, and the scalar multiple of a fixed derivation which is unbounded on the polynomials. Obviously, we cannot use the same argument as [2] since $A_\alpha^+(\mathbb{D})$ is not a regular Banach function algebra. Nevertheless, we have the following lemma.

Lemma 3.6. *Let $m \geq 1$, $\alpha \in [m,m+1)$, and let E be a cyclic, $(m+1)$-dimensional $A_\alpha^+(\mathbb{D})$-module at 1. Suppose that D is a cyclic derivation of height $m+1$ at 1 such that, with respect to the standard basis $\eta_1,\ldots,\eta_{m+1}$,*

$$D : A_\alpha^+(\mathbb{D}) \longrightarrow E, \qquad D(f) = \sum_{i=1}^{m+1} \mu_i(f)\eta_i \tag{3.18}$$

for linear functionals $\mu_1, \ldots, \mu_{m+1}$ on $A_\alpha^+(\mathbb{D})$. Suppose further that μ_{m+1} vanishes on the polynomials. Then D is decomposable.

Proof. Let j be the maximum integer with $\mu_j|_{\mathbb{C}[Z]} \neq 0$. It follows that $(Z-1)^j D(Z) = 0$. Define

$$D_1 : \mathbb{C}[Z] \longrightarrow E, \qquad D_1(p) = \sum_{i=1}^{j} \mu_j(p)\eta_j. \tag{3.19}$$

Hence, $D(p) = D_1(p)$ for every $p \in \mathbb{C}[Z]$. We see that

$$D_1(p) = \sum_{i=0}^{j-1} \frac{p^{(i+1)}(1)}{i!}(Z-1)^i \cdot D(Z) \quad (p \in \mathbb{C}[Z]), \tag{3.20}$$

D_1 can be extended to a continuous derivation into E. Then $D = D_1 + D_2$, where D_2 is a singular derivation. □

Theorem 3.7. *Let $m \in \mathbb{N}$ and let $\alpha \in [m, m+1)$. Suppose that E is a cyclic, $(m+1)$-dimensional $A_\alpha^+(\mathbb{D})$-module at 1 and μ is a generalized derivative of order $m+1$ at 1. Let $D : A_\alpha^+(\mathbb{D}) \to E$ be a derivation. Then there exist $\kappa \in \mathbb{C}$, a continuous derivation D_c, and a singular derivation D_s such that $D = D_c + D_s + \kappa D_\mu$.*

Proof. If D is of height less than $m+1$, then the claim holds for $\kappa = 0$ by Proposition 3.2. Thus we may suppose that D is of height $m+1$. With respect to the standard basis,

$$D(f) = \sum_{i=1}^{m+1} \mu_i(f)\eta_i \quad (f \in A_\alpha^+(\mathbb{D})) \tag{3.21}$$

for linear functionals $\mu_1, \ldots, \mu_{m+1}$. It follows that μ_{m+1} is a point derivation. By Proposition 3.2, μ_{m+1} is decomposable, and there exist $\kappa \in \mathbb{C}$ and a singular point derivation λ on $A_\alpha^+(\mathbb{D})$ such that

$$\mu_{m+1}(f) = \kappa f'(1) + \lambda(f) \quad (f \in A_\alpha^+(\mathbb{D})). \tag{3.22}$$

Now set $\tilde{D} = D - \kappa F$. Then the derivation $\tilde{D}$ maps the polynomials into $(Z-1)E$. By Lemma 3.6, $\tilde{D}$ is decomposable, and the claim follows. □

Corollary 3.8. *Let $m \in \mathbb{N}$. Suppose that E is a finite-dimensional $A_m^+(\mathbb{D})$-module. Then a derivation $D : A_m^+(\mathbb{D}) \to E$ is unbounded on the polynomials if and only if D is unbounded on every dense subalgebra.*

We doubt that the last corollary remains true in the case where $\alpha > m$. In this situation, there exists a singular derivation of height $m+1$ into a cyclic module at 1. It might happen that this singular derivation coincides with $(-D_\mu)$ on a dense subalgebra $\mathcal{A}$. However, $\mathcal{A}$ has to satisfy additional properties, that is, $\mathcal{A} \cap \mathbb{C}[Z] = \varnothing$ and $\mathcal{A} \cap (Z-1)^m A_\alpha^+(\mathbb{D}) \subseteq I_{\alpha,m}$.

4. Finite-dimensional extensions. This section now turns to finite-dimensional extensions of Beurling algebras. As it is shown here, the splitting problem for (finite-dimensional) extensions is closely connected to the structure of (cofinite) ideals. Thus, our results are mainly consequences of Section 2.

An *extension* $\Sigma(\mathcal{A},I)$ of a Banach algebra A is a short exact sequence of Banach algebras

$$\Sigma(\mathcal{A},I): 0 \longrightarrow I \xrightarrow{i} \mathcal{A} \xrightarrow{p} A \longrightarrow 0. \tag{4.1}$$

The Banach algebra I is usually considered as an ideal of $\mathcal{A}$. The extension is called *radical* (*nilpotent, finite-dimensional*) if I is radical (nilpotent, finite-dimensional). The extension is *commutative* if $\mathcal{A}$ is commutative, and *singular* if $I^2 = \{0\}$. In the latter case, we can regard I as a Banach A-bimodule, and there is a corresponding concept of a singular extension of a Banach algebra A by a Banach A-bimodule E.

An extension is *admissible* if the sequence splits as a sequence of Banach spaces, that is, there exists a continuous linear map $\Phi: A \to \mathcal{A}$ with $p \circ \Phi = \mathrm{Id}_A$. Thus, every finite-dimensional extension is admissible. An extension *splits algebraically* if the sequence splits as a sequence of complex algebras, that is, if there exists a homomorphism $\rho: A \to \mathcal{A}$ such that $p \circ \rho = \mathrm{Id}_A$. It *splits strongly* if it splits algebraically and if the splitting homomorphism ρ can be chosen to be continuous, or, equivalently, if the sequence splits as a sequence of Banach algebras. For a detailed discussion of extensions of Banach algebras in a more general context, see [4].

As usual, the principal tool for the investigation of a singular extension $\Sigma(\mathcal{A},E)$ of A by a Banach A-bimodule E is the continuous Hochschild cohomology groups $\mathcal{H}^n(A,E)$, where $n \in \mathbb{N}$. For a definition, see [11]. All admissible, singular extensions of A by E split strongly if and only if $\mathcal{H}^2(A,E) = \{0\}$. $\mathcal{B}^n(A,E)$ denotes the Banach space of continuous n-linear maps from A into E. For the connecting maps of the Hochschild-Kamowitz complex, we write $\delta^n: \mathcal{B}^n(A,E) \to \mathcal{B}^{n+1}(A,E)$. The Hochschild cohomology groups are given by $\mathcal{H}^n(A,E) = \ker\delta^n / \operatorname{im}\delta^{n-1}$. Further, we set $\mathcal{Z}^n(A,E) = \ker\delta^n$ and $\mathcal{N}^n(A,E) = \operatorname{im}\delta^{n-1}$. Then $\mathcal{Z}^n(A,E)$ is called the set of *n-cocycles*, whereas $\mathcal{N}^n(A,E)$ is called the set of *n-coboundaries*.

For example, $\mu \in \mathcal{B}^2(A,E)$ is a 2-cocycle if

$$0 = a \cdot \mu(b,c) - \mu(ab,c) + \mu(a,bc) - \mu(a,b) \cdot c \quad (a,b,c \in A). \tag{4.2}$$

This equation is called the *cocycle identity*. μ is a (continuous) 2-coboundary if there exists a continuous linear map $\lambda: A \to E$ such that

$$\mu(a,b) = a \cdot \lambda(b) - \lambda(ab) + \lambda(a) \cdot b \quad (a,b \in A). \tag{4.3}$$

μ is *symmetric* if $\mu(a,b) = \mu(b,a)$ for all $a,b \in A$. For A commutative and E symmetric, this is equivalent to the commutativity of the corresponding extension.

In [4], a related class of groups, $\tilde{H}^2(A,E)$, is defined. For this definition, $\tilde{N}^2(A,E)$ is taken to be the set of all continuous cocycles which are coboundaries in the algebraic sense, that is, the set of all $\mu \in \mathscr{Z}^2(A,E)$ such that there exists a (not necessarily continuous) linear map $\lambda : A \to E$ satisfying (4.3). Now we set $\widetilde{H}^2(A,E) := \mathscr{Z}^2(A,E)/\widetilde{N}^2(A,E)$. All singular admissible extensions of A by E split algebraically if and only if $\widetilde{H}^2(A,E) = \{0\}$.

An important observation (for the case $n = 2$, but it is obvious that the proof holds for each $n \in \mathbb{N}$) is made in the remark after [4, Proposition 2.2]: let A be a unital Banach algebra and let M be a maximal ideal in A. Let E be a unital A-module. Then $\mathscr{H}^n(A,E) = \{0\}$ $[\widetilde{H}^2(A,E) = \{0\}]$ if and only if $\mathscr{H}^n(M,E) = \{0\}$ $[\widetilde{H}^2(M,E) = \{0\}]$.

Recall that, for finite-dimensional extensions, the problem of strong splitting can be reduced to singular, one-dimensional extensions. However, for the investigation of possible algebraic splittings, one has to consider all finite-dimensional singular extensions by a certain type of modules [4, pages 63–64].

Extensions of the algebras $A_\alpha^+(\mathbb{D})$ have been considered before in [4]. For the case $\alpha = 0$, we have $A_0^+(\mathbb{D}) = l^1(\mathbb{Z}^+)$, and every finite-dimensional extension splits strongly since every maximal ideal has a bounded approximate identity [4, Proposition 4.4]. For $\alpha > 0$, we have the following result on strong splittings which is [4, Proposition 5.9(i)].

Proposition 4.1. *Let $\alpha > 0$. Then there exists a one-dimensional extension of $A_\alpha^+(\mathbb{D})$ which does not split strongly.*

Thus our objective is to establish algebraic splitting of extensions of $A_\alpha^+(\mathbb{D})$.

Proposition 5.9 in [4] also shows that each one-dimensional extension of $A_\alpha^+(\mathbb{D})$ splits algebraically ($\alpha \geq 0$), and that there exists a two-dimensional extension which does not split algebraically provided that $1 \leq \alpha < 3/2$. The case $\alpha \geq 3/2$ remains unsolved.

In this section, we prove that, for $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$, every finite-dimensional extension splits algebraically if and only if $\alpha \geq m + 1/2$.

Note that there is a simple solution for the case where $\alpha \in (0,1)$. Then each maximal ideal of $A_\alpha^+(\mathbb{D})$ either has an approximate identity or is a principal ideal. Thus, every finite-dimensional extension splits algebraically by [4, Theorem 4.13].

To cover the case $\alpha \geq 1$, we begin with a reduction to the case of singular, commutative extensions. The result may be proved in a way similar to [4, Theorem 5.5]. We think that one should be more careful showing this reduction. However, this does not require any new arguments but simple (albeit tedious) matrix manipulations. Therefore, we omit the proof.

The proof would also contain arguments showing that $\mu \in \mathscr{Z}^2(A_\alpha^+(\mathbb{D}), E)$ is symmetric provided that E is a symmetric module, that is, every extension by a symmetric module is commutative.

Proposition 4.2. *Let $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$. Suppose that Σ is a finite-dimensional extension of $A_\alpha^+(\mathbb{D})$. Then at least one of the following assertions is true:*

(i) *Σ splits algebraically;*

(ii) *Σ is singular and commutative.*

Decomposing symmetric modules as shown in Section 3, it suffices to consider cyclic modules at an arbitrary $\lambda \in \overline{\mathbb{D}}$.

In the case where $\lambda \in \mathbb{D}$, $I_{\alpha,0}^\lambda$ is a principal ideal by Proposition 2.10. Now it follows from a result of Pugach (cf. [4, Theorem 4.8] and the remark preceding it that

$\mathcal{H}^2(A_\alpha^+(\mathbb{D}),\mathbb{C}_\lambda)=\{0\}$. By the basic lemma of homological algebra [11, Proposition 1.7], we conclude that $\mathcal{H}^2(A_\alpha^+(\mathbb{D}),E)=\{0\}$.

Hence, we may suppose that $\lambda\in\mathbb{T}$, say $\lambda=1$. First, we observe that a construction from [4] provides a counterexample for $\alpha<m+1/2$.

Lemma 4.3. *Let $m\in\mathbb{N}$ and $\alpha\in[m,m+1/2)$. Then there exists an $(m+1)$-dimensional, commutative, singular extension of $A_\alpha^+(\mathbb{D})$ which does not split algebraically.*

Proof. Suppose that E is a cyclic, $(m+1)$-dimensional module at 1. Now we define a cocycle $\mu\in\mathcal{Z}^2(I_{\alpha,0},E)$ such that, with respect to the standard basis, $\mu=(\mu_1,\dots,\mu_{m+1})$, where $\mu_1,\dots,\mu_{m+1}\in\mathcal{B}^2(I_{\alpha,0},\mathbb{C})$ are continuous bilinear functionals given by

$$\mu_1(f,g)=0,$$
$$\mu_{m+1-j}(f,g)=\sum_{i=j+1}^{m}\frac{1}{i!(m+1+j-i)!}f^{(i)}(1)\cdot g^{(m+1+j-i)}(1) \tag{4.4}$$

for $f,g\in I_{\alpha,0}$ and $j=0,\dots,m-1$. Clearly μ is bilinear, symmetric, and continuous. Using exactly the same arguments as in the proof of [4, Theorem 5.6], it follows that μ is a continuous 2-cocycle which is not algebraically cobound. Thus, $\tilde{H}^2(I_{\alpha,0},E)\neq\{0\}$ and therefore $\tilde{H}^2(A_\alpha^+(\mathbb{D}),E)\neq\{0\}$. $\square$

It is obvious from [4, Theorem 5.6] that the proof of Lemma 4.3 depends on the fact that, by the hypothesis, $(Z-1)^{2m+1}\in I_{\alpha,m}^2$. By Proposition 2.5, this does not hold for $\alpha\geq m+1/2$. We will show that this observation forces every cocycle to cobound algebraically for $\alpha\geq m+1/2$.

Let $m\in\mathbb{N}$ and $\alpha\in[m,m+1)$. Suppose that E is an $(m+1)$-dimensional, cyclic $A_\alpha^+(\mathbb{D})$-module at 1 and choose a standard basis. Let F be an n-dimensional cyclic $A_\alpha^+(\mathbb{D})$-module at 1 $(n\leq m+1)$. Let $\nu\in\mathcal{Z}^2(A_\alpha^+(\mathbb{D}),F)$. Again, $\nu=(\nu_1,\dots,\nu_n)$ in the standard basis of F. Defining $\tilde{\nu}=(\nu_1,\dots,\nu_n,0,\dots,0)$, we obtain $\tilde{\nu}\in\mathcal{Z}^2(A_\alpha^+(\mathbb{D}),E)$. It is easily seen that $\tilde{\nu}$ is cobound if and only if ν is. Thus, when dealing with extensions by cyclic modules at 1, we may always suppose that they are of dimension $m+1$.

The proof of our following main result is simplified considerably for the case $m=1$.

Proposition 4.4. *Let $m\in\mathbb{N}$ and $\alpha\in[m+1/2,m+1)$. Let E be a cyclic Banach $A_\alpha^+(\mathbb{D})$-module at 1. Then $\tilde{H}^2(A_\alpha^+(\mathbb{D}),E)=\{0\}$.*

Proof. Without loss of generality, we suppose that $\dim E=m+1$. With respect to the standard basis, the module multiplication is given as in Section 3. Again, it suffices to show that $\tilde{H}^2(I_{\alpha,0},E)=\{0\}$. Now let $\mu=(\mu_1,\dots,\mu_{m+1})\in\mathcal{Z}^2(I_{\alpha,0},E)$, where $\mu_1,\dots,\mu_{m+1}$ are continuous bilinear functionals.

For each component $j=1,\dots,m+1$, the cocycle identity has the following form:

$$\mu_j(f,gh)+\sum_{i=1}^{m+1-j}\frac{f^{(i)}(1)}{i!}\mu_{j+i}(g,h)=\mu_j(fg,h)+\sum_{i=1}^{m+1-j}\frac{h^{(i)}(1)}{i!}\mu_{j+i}(f,g) \tag{4.5}$$

for $f,g,h\in I_{\alpha,0}$. In particular, for $f=(Z-1)^k$ $(k=1,\dots,m)$ and $h\in I_{\alpha,m}$,

$$\mu_j((Z-1)^k,gh)+\mu_{j+k}(g,h)=\mu_j((Z-1)^kg,h), \tag{4.6}$$

and, for $f \in I_{\alpha,m}$ and $h = (Z-1)^k$ $(k = 1,\dots,m)$,

$$\mu_j\big(f, g(Z-1)^k\big) = \mu_{j+k}(f,g) + \mu_j\big(fg,(Z-1)^k\big). \tag{4.7}$$

Consider the ideal $J = \{h \in A_\alpha^+(\mathbb{D}) \mid (Z-1)^m h \in I_{\alpha,m}^2\}$. By Corollary 2.8, we have $I_{\alpha,m}^2 = (Z-1)^m J$ and $J \subseteq I_{\alpha,m}$. First, define a linear functional λ_{m+1} on $I_{\alpha,0}$ such that, for $h \in J$ with $(Z-1)^m h = \sum_{i=1}^n f_i g_i$ and $f_1,\dots,f_n,g_1,\dots,g_n \in I_{\alpha,m}$,

$$\begin{gathered}\lambda_{m+1}(h) = \mu_1\big((Z-1)^m,h\big) - \sum_{i=1}^n \mu_1(f_i,g_i),\\ \lambda_{m+1}\big((Z-1)^k\big) = -\mu\big((Z-1)^{k-1},Z-1\big) \quad (k = 2,\dots,m+1),\\ \lambda_{m+1}(Z-1) = 0.\end{gathered} \tag{4.8}$$

Here we have used the fact that $(Z-1)^{2m+1} \notin I_{\alpha,m}^2$ (see Proposition 2.5 and Corollary 2.6). We have to show that λ_m is well defined on J, that is, its value does not depend on the decomposition of $(Z-1)^m h$. To see this, suppose that $(Z-1)^m h = \sum_{i=1}^n f_i g_i = 0$ and let (e_γ) denote the approximate identity of $I_{\alpha,m}$. Recall that (e_γ) is given by polynomials. Then

$$\begin{aligned}\mu_1\big((Z-1)^m,h\big) - \sum_{i=1}^n \mu_1(f_i,g_i) &= \lim_\gamma \Bigg[\mu_1\big((Z-1)^m,he_\gamma\big) - \sum_{i=1}^n \mu_1(f_ie_\gamma,g_i)\Bigg]\\ &= \lim_\gamma \sum_{i=1}^n \big[\mu_1\big((Z-1)^m,p_\gamma f_ig_i\big) - \mu_1\big((Z-1)^m f_ip_\gamma,g_i\big)\big]\\ &= \lim_\gamma \sum_{i=1}^n \mu_{m+1}(p_\gamma\cdot f_i,g_i),\end{aligned} \tag{4.9}$$

by (4.6). Here p_γ is the polynomial which one obtains dividing e_γ by the polynomial $(Z-1)^m$. Now $\lim_\delta \mu_{m+1}(p_\gamma f_i, g_ie_\delta) = \lim_\delta \mu_{m+1}(p_\gamma f_ig_i, e_\delta)$. Hence,

$$\mu_1\big((Z-1)^m,h\big) - \sum_{i=1}^n \mu_1(f_i,g_i) = -\lim_{\gamma,\delta} \sum_{i=1}^n \mu_{m+1}(p_\gamma\cdot f_ig_i,e_\delta) = 0, \tag{4.10}$$

and λ_{m+1} is well defined.

Next, we inductively define linear functionals $\lambda_m,\lambda_{m-1},\dots,\lambda_1$ such that, for $f \in I_{\alpha,m}$,

$$\begin{gathered}\lambda_j\big((Z-1)f\big) = -\mu_j\big((Z-1),f\big) + \lambda_{j+1}(f),\\ \lambda_j\big((Z-1)^k\big) = -\mu_j\big((Z-1),(Z-1)^{k-1}\big) + \lambda_{j+1}\big((Z-1)^{k-1}\big),\\ \lambda_j(Z-1) = 0,\end{gathered} \tag{4.11}$$

where $k = 2,\dots,m+1$ and $j = 1,\dots,m$. Here we have used the fact that $(Z-1)^k \notin (Z-1)I_{\alpha,m}$ for $k = 1,\dots,m+1$. Set $\lambda = (\lambda_1,\dots,\lambda_{m+1})$. Then λ is an $(m+1)$-linear map from $I_{\alpha,0}$ to E.

We will prove that $\delta^1\lambda = \mu$. For each component $j \in \{1,\dots,m+1\}$, we have to show that, for $f,g \in I_{\alpha,0}$,

$$\lambda_j(fg) = -\mu_j(f,g) + \sum_{i=1}^{m+1-j} \frac{f^{(i)}(1)}{i!}\lambda_{j+i}(g) + \sum_{i=1}^{m+1-j} \frac{g^{(i)}(1)}{i!}\lambda_{j+i}(f). \tag{4.12}$$

It suffices to verify this equation for the special cases where f and g belong to $I_{\alpha,m}$ or have the form $(Z-1)^k$ $(k = 1,\dots,m)$. We prove the claim by an induction on j starting with $j = m+1$. Let $g,f \in I_{\alpha,m}$. Then $(Z-1)^m(fg) = ((Z-1)^m f)\cdot g$, and

$$\lambda_{m+1}(fg) = \mu_1((Z-1)^m, fg) - \mu_1((Z-1)^m f, g) = -\mu_{m+1}(f,g) \tag{4.13}$$

by (4.6). For $k = 1,\dots,m$, we see that $(Z-1)^m \cdot (Z-1)^k f = (Z-1)^{m+k}\cdot f$, and using (4.5), we obtain

$$\begin{aligned} \lambda_{m+1}((Z-1)^k f) &= \mu_1((Z-1)^m, (Z-1)^k f) - \mu_1((Z-1)^{m+k}, f) \\ &= -\mu_{m+1}((Z-1)^k, f). \end{aligned} \tag{4.14}$$

It is easily seen that, for $k,l = 1,\dots,m$,

$$\begin{aligned} \lambda_{m+1}((Z-1)^k(Z-1)^l) &= -\mu_{m+1}((Z-1)^{k+l-1}, (Z-1)) \\ &= -\mu_{m+1}((Z-1)^k, (Z-1)^l) \end{aligned} \tag{4.15}$$

if $k+l \le m+1$, and by (4.6),

$$\begin{aligned} \lambda_{m+1}((Z-1)^k(Z-1)^l) &= \mu_1((Z-1)^m, (Z-1)^{k+l}) - \mu_1((Z-1)^{m+1}, (Z-1)^{k+l-1}) \\ &= -\mu_{m+1}((Z-1)^1, (Z-1)^{k+l-1}) \\ &= -\mu_{m+1}((Z-1)^k, (Z-1)^l) \end{aligned} \tag{4.16}$$

if $k+l \ge m+2$. By the remark preceding Proposition 4.2, μ_{m+1} is symmetric, hence, by (4.6),

$$\begin{aligned} \lambda_{m+1}(f(Z-1)^k) &= \mu_1((Z-1)^m, f(Z-1)^k) - \mu_1((Z-1)^{m+k}, f) \\ &= -\mu_{m+1}(f, (Z-1)^k) \quad (k = 1,\dots,m,\ f \in I_{\alpha,m}). \end{aligned} \tag{4.17}$$

So (4.12) has been verified for $j = m+1$.

Now suppose that, for $j \in \{1,\dots,m\}$, (4.12) has been verified for $l = j+1,\dots,m+1$. Let $f,g \in I_{\alpha,m}$. Then there exists $h \in J$ such that $fg = (Z-1)^m h$. Consider the case $j = m$ first. Then

$$\begin{aligned} \lambda_m(fg) &= -\mu_m(Z-1, (Z-1)^{m-1}h) + \lambda_{m+1}((Z-1)^{m-1}h) \\ &= -\mu_m(Z-1, (Z-1)^{m-1}h) + \mu_1((Z-1)^m, (Z-1)^{m-1}h) \\ &\quad - \mu_1((Z-1)^{m-1}f, g) \end{aligned} \tag{4.18}$$

by the definition of λ_{m+1}. By (4.6),

$$\begin{aligned}\lambda_m(fg) = &-\mu_m(Z-1,(Z-1)^{m-1}h)+\mu_1((Z-1)^m,(Z-1)^{m-1}h)\\ &-\mu_1((Z-1)^{m-1},fg)-\mu_m(f,g).\end{aligned} \tag{4.19}$$

Since $fg = (Z-1)(Z-1)^{m-1}h$, (4.6) gives

$$\lambda_m(fg) = -\mu_m(f,g). \tag{4.20}$$

Now suppose that $j \le m-1$. In this case,

$$\begin{aligned}\lambda_j(fg) &= -\mu_j(Z-1,(Z-1)^{m-1}h)+\lambda_{j+1}((Z-1)^{m-1}h)\\ &= -\mu_j(Z-1,(Z-1)^{m-1}h)-\mu_{j+1}((Z-1)^{m-j},(Z-1)^{j-1}h)\\ &\quad +\lambda_{m+1}((Z-1)^{j-1}h),\end{aligned} \tag{4.21}$$

where we have used the induction hypothesis. The definition of λ_{m+1} and (4.6) yield

$$\begin{aligned}\lambda_j(fg) &= -\mu_j(Z-1,(Z-1)^{m-1}h)-\mu_{j+1}((Z-1)^{m-j},(Z-1)^{j-1}h)\\ &\quad +\mu_1((Z-1)^m,(Z-1)^{j-1}h)-\mu_1((Z-1)^{j-1}f,g)\\ &= -\mu_j(Z-1,(Z-1)^{m-1}h)-\mu_{j+1}((Z-1)^{m-j},(Z-1)^{j-1}h)\\ &\quad +\mu_1((Z-1)^m,(Z-1)^{j-1}h)-\mu_1((Z-1)^{j-1},fg)-\mu_j(f,g).\end{aligned} \tag{4.22}$$

Since $fg = (Z-1)^{j-1}(Z-1)^{m-j+1}h$, a double application of (4.6) yields

$$\begin{aligned}\lambda_j(fg) &= -\mu_j(Z-1,(Z-1)^{m-1}h)-\mu_{j+1}((Z-1)^{m-j},(Z-1)^{j-1}h)\\ &\quad +\mu_j((Z-1)^{m-j+1},(Z-1)^{j-1}h)-\mu_j(f,g)\\ &= -\mu_j(f,g).\end{aligned} \tag{4.23}$$

Hence (4.12) holds for $j \in \{1,\dots,m\}$ and $f,g \in I_{\alpha,m}$. For the following calculation, it is convenient to introduce functionals $\lambda_{m+2},\dots,\lambda_{2m}$ and bilinear functionals $\mu_{m+2},\dots,\mu_{2m}$ which are identically zero. Let $k,l \in \{1,\dots,m\}$. Then

$$\begin{aligned}\lambda_j((Z-1)^k(Z-1)^l) &= \lambda_{j+1}((Z-1)^{k+l-1})-\mu_j(Z-1,(Z-1)^{k+l-1})\\ &= \lambda_{j+k}((Z-1)^l)-\mu_{j+1}((Z-1)^{k-1},(Z-1)^l)\\ &\quad +\lambda_{j+1+l}((Z-1)^{k-1})-\mu_j(Z-1,(Z-1)^{k+l-1})\end{aligned} \tag{4.24}$$

by the definition of λ_j (here we do not have to distinguish the cases where $k+l \le m+1$ and $l+k > m+1$) and by the induction hypothesis. We apply (4.5) again to obtain

$$\begin{aligned}\lambda_j((Z-1)^k(Z-1)^l) &= \lambda_{j+k}((Z-1)^l)+\lambda_{j+1+l}((Z-1)^{k-1})\\ &\quad -\mu_j((Z-1)^k,(Z-1)^l)-\mu_{j+l}(Z-1,(Z-1)^{k-1})\\ &= \lambda_{j+k}((Z-1)^l)+\lambda_{j+l}((Z-1)^k)-\mu_j((Z-1)^k,(Z-1)^l).\end{aligned} \tag{4.25}$$

For the last equality, we have used the definition of λ_{j+l} (which might be zero), or, if $l+j = m+1$, we have used the starting point of our induction. Thus we have verified (4.12) for $j \in \{1,\dots,m\}$, $f = (Z-1)^k$, and $g = (Z-1)^l$, where $k,l \in \{1,\dots,m\}$.

For the last combination, let $k \in \{1,\dots,m\}$ and $f \in I_{\alpha,m}$. The definition of λ_j gives

$$\lambda_j\big((Z-1)^k f\big) = \lambda_{j+1}\big((Z-1)^{k-1} f\big) - \mu_j\big(Z-1,(Z-1)^{k-1} f\big). \tag{4.26}$$

By the induction hypothesis, or by the definition of $\lambda_{m+2},\dots,\lambda_{2m}$, respectively,

$$\begin{aligned}\lambda_j\big((Z-1)^k f\big) &= \lambda_{j+k}(f) - \mu_{j+1}\big((Z-1)^{k-1}, f\big) - \mu_j\big(Z-1,(Z-1)^{k-1} f\big)\\ &= \lambda_{j+k}(f) - \mu_j\big((Z-1)^k, f\big).\end{aligned} \tag{4.27}$$

Since μ_j is symmetric by the remark preceding Proposition 4.2, we have also shown that

$$\lambda_j\big(f(Z-1)^k\big) = \lambda_{j+k}(f) - \mu_j\big(f,(Z-1)^k\big). \tag{4.28}$$

Hence, (4.12) holds for $j \in \{1,\dots,m\}$, $f \in I_{\alpha,m}$, and $g = (Z-1)^k$, where $k \in \{1,\dots,m\}$ and the induction continues. □

Inspecting the proof of the previous result carefully, the hypothesis that $\alpha \geq m+1/2$ is needed only once: we have to ensure that $(Z-1)^{2m+1} \notin I^2_{\alpha,m}$ in order to define λ_{m+1} consistently. Thus, our approach can be modified to obtain two interesting observations.

Proposition 4.5. *Let $m \in \mathbb{Z}^+$ and $\alpha \in [m,m+1)$. Let E be a cyclic $A_\alpha^+(\mathbb{D})$-module at 1.*

(i) *Suppose that* $\dim E \leq m$. *Then* $\widetilde{H}^2(A_\alpha^+(\mathbb{D}),E) = \{0\}$.

(ii) *Suppose that $\alpha < m+1/2$ and* $\dim E = m+1$. *Then* $\widetilde{H}^2(A_\alpha^+(\mathbb{D}),E) = \mathbb{C}$.

Proof. (i) We may suppose that $\alpha < m+1/2$. Now suppose that E is a cyclic $A_\alpha^+(\mathbb{D})$-module, that $\mu \in \mathscr{Z}^2(A_\alpha^+(\mathbb{D}),E)$, and that $k := \dim E \leq m$. As in the earlier remark, we may consider E as a submodule of an $(m+1)$-dimensional, cyclic module F, and $\mu = (\mu_1,\dots,\mu_k,0,\dots,0)$ with respect to the standard basis of F. If we now define λ_{m+1} as we did in the proof, we might obtain an inconsistency since $(Z-1)^{2m+1} \in I^2_{\alpha,m}$. In fact, we have

$$\begin{aligned}\lambda_{m+1}\big((Z-1)^{m+1}\big) &= \mu_1\big((Z-1)^m,(Z-1)^{m+1}\big)\\ &\quad - \mu_1\big((Z-1)^{m+1/2},(Z-1)^{m+1/2}\big)\\ &= \lim_\gamma \mu_{m+1}\big(p_\gamma (Z-1)^{m+1/2},(Z-1)^{m+1/2}\big).\end{aligned} \tag{4.29}$$

On the other hand, our definition requires

$$\lambda_{m+1}\big((Z-1)^{m+1}\big) = -\mu_{m+1}\big((Z-1)^m, Z-1\big). \tag{4.30}$$

But, since $\mu_{m+1} \equiv 0$, this no longer yields a contradiction. Hence, we may proceed with the proof and μ is cobound.

(ii) Suppose that $\alpha < m+1/2$ and $\dim E = m+1$. Let $\mu \in \mathscr{Z}^2(A_\alpha^+(\mathbb{D}),E)$ and let $\nu \in \mathscr{Z}^2(A_\alpha^+(\mathbb{D}),E) \setminus \widetilde{N}^2(A_\alpha^+(\mathbb{D}),E)$ be the 2-cocycle constructed in Lemma 4.3. Again, we may write $\mu = (\mu_1,\dots,\mu_{m+1})$ and $\nu = (\nu_1,\dots,\nu_{m+1})$ with respect to the standard basis. Recall that, by definition, $\nu_1 = 0$ and $\nu_{m+1}((Z-1)^m, Z-1) = 1$. Hence, adding a scalar multiple of ν to μ, we may suppose that

$$\mu_1\big((Z-1)^m,(Z-1)^{m+1}\big)-\mu_1\big((Z-1)^{m+1/2},(Z-1)^{m+1/2}\big)=\mu_{m+1}\big((Z-1)^m,Z-1\big). \tag{4.31}$$

Now there no longer occurs any obstruction for the definition of a functional λ_{m+1} and we may proceed as before. The claim follows. □

This section can be summarized as follows.

Theorem 4.6. *Let $m \in \mathbb{N}$ and $\alpha \in [m, m+1)$.*

(i) *Suppose that $\alpha < m+1/2$. Then every finite-dimensional extension of $A_\alpha^+(\mathbb{D})$ with dimension at most m splits algebraically, and there exists an $(m+1)$-dimensional extension which does not split algebraically.*

(ii) *Suppose that $\alpha \geq m+1/2$. Then every finite-dimensional extension of $A_\alpha^+(\mathbb{D})$ splits algebraically.*

It is remarkable that, for certain α's, $\alpha \geq 1$, all finite-dimensional extensions split, whereas, considering the so closely related algebras $C^m[0,1]$, this does not hold for any $m \in \mathbb{N}$.

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