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RESTRICTED PARTITIONS

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We prove a known partitions theorem by Bell in an elementary and constructive way. Our proof yields a simple recursive method to compute the corresponding *Sylvester polynomials* for the partition. The previous known methods to obtain these polynomials are in general not elementary.

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1. Proof of Bell's theorem. The main purpose of this section is to prove the following theorem, originally proved by Bell in [1], by elementary methods.

Theorem 1.1. *For a fixed positive integer n, let $A1,\dots,An$ be positive integers and let M' be their least common multiple. For a fixed integer r', the number of nonnegative solutions $Xn,\dots,X1$ of $An\cdot Xn+\cdots+A1\cdot X1 = M'K+r'$, which we indicate by $D_n(M'K+r')$, is given by a polynomial in K, which is either the zero polynomial or a polynomial with rational coefficients of degree $n-1$.*

First, we need the following known result.

Lemma 1.2. *For $N\ge 0$ and $m\ge 1$, $H_m(N) = 0^m+1^m+\cdots+N^m$ is a polynomial in N of degree $m+1$ with rational coefficients. Besides, $H_m(-1) = 0$.*

For example, we have

$$H_1(N) = \frac{1}{2}N^2+\frac{1}{2}N, \qquad H_2(N) = \frac{1}{3}N^3+\frac{1}{2}N^2+\frac{1}{6}N. \tag{1.1}$$

There exist several elementary methods to obtain the polynomials $H_m(N)$.

We will see that $D_n(M'K+r')$ is a polynomial as a direct consequence of Lemma 1.2.

The proof of Theorem 1.1. We are going to prove Bell's theorem by mathematical induction. The theorem is clearly true for $n=1$ since in this case, the number of solutions to the equation $A1\cdot X1 = A1\cdot K+r$' is given by the polynomials

$$\begin{aligned} &D_1(A1\cdot K+r') = 1 \quad \text{if } r' \text{ is multiple of } A1,\\ &D_1(A1\cdot K+r') = 0 \quad \text{if } r' \text{ is not a multiple of } A1. \end{aligned} \tag{1.2}$$

Let $n\ge 1$ be a given, and assume Theorem 1.1 holds for $n-1$; we will prove it is also true for n.

The equation corresponding to n is

$$An \cdot Xn + A(n-1) \cdot X(n-1) + \cdots + A1 \cdot X1 = M' \cdot K + r'. \tag{1.3}$$

From the inductive hypothesis, we know the polynomials $D_{n-1}(MK+r)$ describing the number of solutions to $A(n-1) \cdot X(n-1) + \cdots + A1 \cdot X1 = M \cdot K + r$ for all r, where M is the least common multiple of $A(n-1), A(n-2), \dots, A1$.

We can write

$$M' \cdot K + r' = An(\alpha(K+c)+a)+b, \tag{1.4}$$

where $0 \le b < An$ and $0 \le a < \alpha$.

Note that $M' \cdot K + r' \ge 0$ if and only if $K \ge -c$.

Letting the variable Xn run through all possible values of $n'' \in \{0,1,2,\dots,\alpha(K+c)+a\}$, we obtain

$$D_n(M'K+r') = \sum_{n''=0}^{\alpha(K+c)+a} D_{n-1}(Ann''+b). \tag{1.5}$$

In order to directly use the induction hypothesis, we need to express each of the terms $Ann''+b$ or the form $MK+r$, for suitable K and r.

For that purpose, consider the set partition

$$\begin{aligned}&\{0,1,\dots,\alpha(K+c)+a\}\\ &\quad= \bigcup_{0\le i\le a}\{\alpha S+i : S=0,1,\dots,K+c\} \cup \bigcup_{a+1\le i\le \alpha-1}\{\alpha S+i : S=0,1,\dots,K+c-1\}.\end{aligned} \tag{1.6}$$

Letting $\beta = M'/M$, we have $An \cdot \alpha = M\beta$ and, by (1.5), we obtain

$$\begin{aligned}D_n(M'K+r') &= \sum_{i=0}^{a}\sum_{S=0}^{K+c} D_{n-1}(M\cdot(\beta\cdot S)+Ani+b)\\ &\quad+\sum_{i=a+1}^{\alpha-1}\sum_{S=0}^{K+c-1} D_{n-1}(M\cdot(\beta\cdot S)+Ani+b).\end{aligned} \tag{1.7}$$

Each $D_{n-1}(M\cdot(\beta\cdot S)+Ani+b)$ is, by induction hypothesis, a polynomial in S of degree $n-2$ or the zero polynomial. The proof of Theorem 1.1 now follows directly from Lemma 1.2. □

Note that (1.7) yields a recursive method to obtain D_n from previous $D_{n-1}, D_{n-2}, \dots, D_1$, which we will demonstrate in the next section.

2. An example of building method. We are going to determine the polynomial $D_3(12K+8)$ that corresponds to the equation

$$4X3+3X2+2X1 = 12\cdot K+8. \tag{2.1}$$

Since $A3 = 4$ and $\alpha = 3$, (1.4) becomes $12 \cdot K + 8 = 4 \cdot (3K + 2)$; therefore $a = 2$, $b = 0$, and $c = 0$. Besides, $M = 6$ and $\beta = 2$. Therefore, (1.7) becomes

$$\begin{aligned} D_3(12K+8) &= \sum_{i=0}^{2} \sum_{S=0}^{K} D_2(6(2S)+4i) \\ &= \sum_{S=0}^{K} D_2(6(2S)) + \sum_{S=0}^{K} D_2(6(2S)+4) + \sum_{S=0}^{K} D_2(6(2S)+8). \end{aligned} \tag{2.2}$$

Computation of $D_2(6K)$ that corresponds to the equation $3X2 + 2 \cdot X1 = 6K$. Since $A2 = 3$ and $\alpha = 2$, (1.4) becomes $6K = 3(2K)$; therefore $a = 0$, $b = 0$, and $c = 0$. Besides, $M = 2$ and $\beta = 3$. Therefore, (1.7) becomes

$$\begin{aligned} D_2(6K) &= \sum_{i=0}^{0} \sum_{S=0}^{K} D_1(2(3S)+3i) + \sum_{i=1}^{1} \sum_{S=0}^{K-1} D_1(2 \cdot (3S)+3i) \\ &= \sum_{S=0}^{K} D_1(2(3S)) + \sum_{S=0}^{K-1} D_1(2(3S)+3). \end{aligned} \tag{2.3}$$

Since the polynomial that corresponds to the equation $2X1 = 2K$ is $D_1(2K) = 1$ and the polynomial that corresponds to the equation $2X1 = 2K+3$ is $D_1(2K+3) = 0$, we obtain $D_2(6K) = K+1$, and hence

$$D_2(6(2S)) = 2S+1. \tag{2.4}$$

Computation of $D_2(6K+4)$ that corresponds to the equation $3 \cdot X2 + 2 \cdot X1 = 6K+4$. Since $A2 = 3$ and $\alpha = 2$, (1.4) becomes $6K+4 = 3(2K+1)+1$; therefore $a = 1$, $b = 1$, and $c = 0$. Besides, $M = 2$ and $\beta = 3$. Therefore, (1.7) becomes

$$\begin{aligned} D_2(6K+4) &= \sum_{i=0}^{1} \sum_{S=0}^{K} D_1(2(3S)+3i+1) \\ &= \sum_{S=0}^{K} D_1(2(3S)+1) + \sum_{S=0}^{K} D_1(2(3S)+4). \end{aligned} \tag{2.5}$$

Since the polynomial that corresponds to the equation $2X1 = 2K+1$ is $D_1(2K+1) = 0$ and the polynomial that corresponds to the equation $2X1 = 2K+4$ is $D_1(2K+4) = 1$, we obtain $D_2(6K+4) = K+1$, and hence

$$D_2(6(2S)+4) = 2S+1. \tag{2.6}$$

Computation of $D_2(6K+8)$ that corresponds to the equation $3X2+2 \cdot X1 = 6K+8$. Since $A2 = 3$ and $\alpha = 2$, (1.4) becomes $6K+8 = 3(2(K+1))+2$; therefore $a = 0$, $b = 2$, and $c = 1$. Besides, $M = 2$ and $\beta = 3$. Therefore, (1.7) becomes

$$\begin{aligned} D_2(6K+8) &= \sum_{i=0}^{0} \sum_{S=0}^{K+1} D_1(2(3S)+3i+2) + \sum_{i=1}^{1} \sum_{S=0}^{K} D_1(2 \cdot (3S)+3i+2) \\ &= \sum_{S=0}^{K+1} D_1(2(3S)+2) + \sum_{S=0}^{K} D_1(2(3S)+5). \end{aligned} \tag{2.7}$$

Since the polynomial that corresponds to the equation $2X1 = 2K+2$ is $D_1(2K+2) = 1$ and the polynomial that corresponds to the equation $2X1 = 2K+5$ is $D_1(2K+5) = 0$, we obtain $D_2(6K+8) = K+2$, and hence

$$D_2(6(2S)+8) = 2S+2. \tag{2.8}$$

From (2.2), (2.4), (2.6), and (2.8), we have

$$\begin{aligned} D_3(12K+8) &= \sum_{S=0}^{K}(2S+1) + \sum_{S=0}^{K}(2S+1) + \sum_{S=0}^{K}(2S+2) \\ &= \sum_{S=0}^{K}(6S+4) = 6H_1(K) + 4(K+1) \\ &= 6\left(\frac{1}{2}K^2 + \frac{1}{2}K\right) + 4(K+1) = 3K^2 + 7K + 4. \end{aligned} \tag{2.9}$$

Remarks 2.1. (i) The recursive method to compute $D_n(M'K+r')$, given in this article, works well for relatively small values of n, but the computations get progressively worse as n grows. (ii) A similar argument shows that there exist polynomials that count the number of solutions $X1,\dots,Xn$, where all $Xi > 0$.

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