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ON JORDAN IDEALS AND LEFT (θ,θ)-DERIVATIONS IN PRIME RINGS

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Let R be a ring and S a nonempty subset of R. Suppose that θ and ϕ are endomorphisms of R. An additive mapping $\delta : R \to R$ is called a left (θ,ϕ)-derivation (resp., Jordan left (θ,ϕ)-derivation) on S if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$ (resp., $\delta(x^2) = \theta(x)\delta(x) + \phi(x)\delta(x)$) holds for all $x, y \in S$. Suppose that J is a Jordan ideal and a subring of a 2-torsion-free prime ring R. In the present paper, it is shown that if θ is an automorphism of R such that $\delta(x^2) = 2\theta(x)\delta(x)$ holds for all $x \in J$, then either $J \subseteq Z(R)$ or $\delta(J) = (0)$. Further, a study of left (θ,θ)-derivations of a prime ring R has been made which acts either as a homomorphism or as an antihomomorphism of the ring R.

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1. Introduction. Throughout the present paper, R will denote an associative ring with centre $Z(R)$. We will write for all $x, y \in R$, $[x,y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. A ring R is said to be prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. A ring R is said to be 2-torsion-free if whenever $2a = 0$, with $a \in R$, then $a = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. An additive mapping $d : R \to R$ is called a derivation (resp., Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp., $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Let θ, ϕ be endomorphisms of R. An additive mapping $f : R \to R$ is called a (θ,ϕ)-derivation (resp., Jordan (θ,ϕ)-derivation) if $f(xy) = f(x)\theta(y) + \phi(x)f(y)$ (resp., $f(x^2) = f(x)\theta(x) + \phi(x)f(x)$) holds, for all $x, y \in R$. Of course a $(1,1)$-derivation (resp., a Jordan $(1,1)$-derivation) is a derivation (resp., a Jordan derivation) on R, where 1 is the identity mapping on R. We will make use of the following basic commutator identities without any specific mention:

$$[xy,z] = x[y,z] + [x,z]y, \qquad [x,yz] = y[x,z] + [x,y]z. \tag{1.1}$$

An additive mapping $\delta : R \to R$ is called a left derivation (resp., Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (resp., $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. In view of the definition of a (θ,ϕ)-derivation, the notion of left (θ,ϕ)-derivation can be extended as follows: let θ, ϕ be endomorphisms of R and let S be a nonempty subset of R. An additive mapping $\delta : R \to R$ is called a left (θ,ϕ)-derivation (resp., Jordan left (θ,ϕ)-derivation) on S if $\delta(xy) = \theta(x)\delta(y) + \phi(y)\delta(x)$ (resp., $\delta(x^2) = \theta(x)\delta(x) + \phi(x)\delta(x)$) holds for all $x, y \in S$. Clearly, a left $(1,1)$-derivation (resp., a Jordan left $(1,1)$-derivation) is a left derivation (resp., a Jordan left derivation) on R, where 1 is the identity mapping on R. In [5], Brešar and Vukman have proved that the existence of a nonzero Jordan left derivation on a prime ring R of char $R \neq 2,3$ forces R to be commutative. It should be mentioned that the result obtained in [5] concerning Jordan left derivation has been improved by Deng [7]. Some more related results can be seen in [1, 3, 5, 7, 9]. It is easy to see that every left derivation on a ring R is a Jordan left derivation. However, in general, a Jordan left derivation need not be a left derivation. The following example justifies this statement.

Example 1.1. Let R be a commutative ring and let $a \in R$ such that $xax = 0$ for all $x \in R$ but $xay \neq 0$, for some x and y, $x \neq y$. Define a map $\delta : R \to R$ as follows:

$$\delta(x) = xa + ax. \tag{1.2}$$

Then δ is a Jordan left derivation but not a left derivation.

In the present paper, first it is shown that every Jordan left (θ,θ)-derivation on a Jordan ideal J of a 2-torsion-free prime ring is a left (θ,θ)-derivation on J. Finally, we will study the behaviour of left (θ,θ)-derivation on a prime ring which also acts either as a homomorphism or an antihomomorphism of the underlying ring.

2. Preliminary results.

We begin with the following lemmas which are essential in developing the proof of our main result.

Lemma 2.1 [6, Lemma 4]. *Let G and H be additive groups and let R be a 2-torsion-free ring. Let $f : G\times G \to H$ and $g : G\times G \to R$ be biadditive mappings. Suppose that for each pair $a,b \in G$ either $f(a,b) = 0$ or $g(a,b)^2 = 0$. In this case, either $f = 0$ or $g(a,b)^2 = 0$ for all $a,b \in G$.*

If J is assumed to be a Jordan ideal and a subring of a ring R, then using similar techniques as used in the proofs of Lemmas 2.2 and 2.3 of [1], one can easily obtain the following lemma.

Lemma 2.2. *Let R be a 2-torsion-free ring, let J be a Jordan ideal and a subring of R. If θ is an endomorphism of R and $\delta : R \to R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in J$, then*

(i) $\delta(uv+vu) = 2\theta(u)\delta(v)+2\theta(v)\delta(u)$, *for all* $u,v \in J$,

(ii) $\delta(uvu) = \theta(u^2)\delta(v)+3\theta(u)\theta(v)\delta(u)-\theta(v)\theta(u)\delta(u)$, *for all* $u,v \in J$,

(iii) $\delta(uvw + wvu) = (\theta(u)\theta(w) + \theta(w)\theta(u))\delta(v) + 3\theta(u)\theta(v)\delta(w) + 3\theta(w)\theta(v)\delta(u)-\theta(v)\theta(u)\delta(w)-\theta(v)\theta(w)\delta(u)$, *for all* $u,v,w \in J$,

(iv) $[\theta(u),\theta(v)]\theta(u)\delta(u) = \theta(u)[\theta(u),\theta(v)]\delta(u)$, *for all* $u,v \in J$,

(v) $[\theta(u),\theta(v)](\delta(uv)-\theta(u)\delta(v)-\theta(v)\delta(u)) = 0$, *for all* $u,v \in J$.

Lemma 2.3. *Let R be a 2-torsion-free ring, J a Jordan ideal and a subring of R. If θ is an endomorphism of R and $\delta : R \to R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$ for all $u \in J$, then*

(i) $[\theta(u),\theta(v)]\delta([u,v]) = 0$, *for all* $u,v \in J$;

(ii) $(\theta(u^2)\theta(v)-2\theta(u)\theta(v)\theta(u)+\theta(v)\theta(u^2))\delta(v) = 0$, *for all* $u,v \in J$.

We begin with the following lemma.

Lemma 2.4. *If R is a ring and J a nonzero Jordan ideal of R, then $2[R,R]J \subseteq J$ and $2J[R,R] \subseteq J$.*

Proof. Let $x,y \in R$ and $u \in J$. Then $u \circ [x,y] - (u \circ x) \circ y + (u \circ y) \circ x \in J$. This implies that $uxy - uyx + xyu - yxu - uxy - xuy - yux - yxu + uyx + yux + xuy + xyu \in J$ and hence $2[x,y]u \in J$, for all $x,y \in R$, and $u \in J$, that is, $2[R,R]J \subseteq J$.

Similarly, it is easy to see that $2u[x,y] = (u \circ y) \circ x - u \circ [x,y] - (u \circ x) \circ y \in J$, for all $x,y \in R$ and $u \in J$, and hence $2J[R,R] \subseteq J$. □

Lemma 2.5. *Let R be a prime ring and J a nonzero Jordan ideal of R. If $a \in R$ and $aJ = (0)$ (or $Ja = (0)$), then $a = 0$.*

Proof. Since J is a Jordan ideal of R, $u \circ x \in J$, for all $x \in R$ and $u \in J$. By hypotheses, we have $a(u \circ x) = 0$, for all $x \in R$, $u \in J$, and hence we get $axu = 0$, for all $x \in R$, $u \in J$, that is, $aRJ = (0)$. Since J is a nonzero Jordan ideal and R is prime, the above relation yields that $a = 0$.

If $Ja = (0)$, then using similar arguments with necessary variations, we get the required result. □

Lemma 2.6. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If $aJb = (0)$, then $a = 0$ or $b = 0$.*

Proof. By Lemma 2.4, we find that $2[R,R]J \subseteq J$. Thus, for any $x,y \in R$ and $u \in J$, we have $2a[x,y]ub = 0$. This implies that

$$a[x,y]ub = 0, \quad \forall x,y \in R,\ u \in J. \tag{2.1}$$

Replacing y by ya in the above expression, we get $a[x,ya]ub = 0$, for all $x,y \in R$ and $u \in J$ or $ay[x,a]ub + a[x,y]aub = 0$. Now, using the fact that $aJb = (0)$, we find that $ay[x,a]ub = 0$, for all $x,y \in R$ and $u \in J$ and hence $aR[x,a]ub = (0)$. Thus, primeness of R forces that either $a = 0$ or $[x,a]ub = 0$. If $[x,a]ub = 0$, for all $x \in R$, $u \in J$, then by our hypotheses we have $axub = 0$, for all $x \in R$, $u \in J$, that is, $aRub = (0)$. Again, primeness of R gives that either $a = 0$ or $ub = 0$. If $ub = 0$, for all $u \in J$, then by Lemma 2.5, we get $b = 0$. □

Lemma 2.7. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R. If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.*

Proof. By Lemma 2.4, we have $2[R,R]J \subseteq J$. Thus, for any $x,y \in R$ and $u,v \in J$, we find that $[2[x,y]u,v] = 0$ and hence $2[[x,y],v]u + 2[x,y][u,v] = 0$, for all $x,y \in R$ and $u,v \in J$. By hypotheses, we obtain $[[x,y],v]u = 0$, for all $x,y \in R$ and $u,v \in J$. Using Lemma 2.5, we get $[[x,y],v] = 0$, for all $x,y \in R$ and $v \in J$. Now, replace y by xy to get $[x,v][x,y] = 0$, for all $x,y \in R$ and $v \in J$. Further replacing y by yv, we have $[x,v]y[x,v] = 0$, for all $x,y \in R$ and $v \in J$. This implies that $[x,v]R[x,v] = (0)$, for all $x \in R$, $v \in J$. Now, primeness of R forces that $v \in Z(R)$, for all $v \in J$. Hence, $J \subseteq Z(R)$. □

The next lemma can be regarded as a generalization of a lemma due to Smiley [8] for Jordan ideals of a prime ring.

Lemma 2.8. *Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of R such that $[u,v]^2 = 0$, for all $u,v \in J$. Then J is commutative and hence central.*

Proof. By hypothesis, we have $[u,v]^2 = 0$, for all $u,v \in J$. On linearizing, we get $[u,v][u,w]+[u,w][u,v] = 0$, for all $u,v,w \in J$. Replacing v by vu in the above expression and using it, we obtain $[u,v][u,[u,w]] = 0$, for all $u,v,w \in J$. Again, replacing v by vv_1 in latter relation, we find that $[u,v]v_1[u,[u,w]] = 0$, that is, $[u,v]J[u,[u,w]] = (0)$, for all $u,v,w \in J$. Thus by Lemma 2.6, we have for each $u \in J$ either $[u,v] = 0$ or $[u,[u,w]] = 0$, for all $u,v,w \in J$. If $[u,[u,w]] = 0$, for all $u,w \in J$, then on replacing w by wv, we get $[u,w][u,v] = 0$, for all $u,v,w \in J$. Again, replacing v by vw, we have $[u,w]v[u,w] = 0$, for all $u,v,w \in J$ and hence $[u,w]J[u,w] = (0)$, for all $w \in J$. Again, by Lemma 2.6, we obtain $[u,w] = 0$. Thus in both cases we find that $[u,w] = 0$, for all $u,w \in J$. Thus, J is commutative, and by Lemma 2.7, J is central, that is, $J \subseteq Z(R)$. □

Lemma 2.9. *Let R be a 2-torsion-free ring, J a Jordan ideal and a subring of R. If $\delta : R \to R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in U$, then*

(i) $\delta(u^2v) = \theta(u^2)\delta(v) + (\theta(u)\theta(v) + \theta(v)\theta(u))\delta(u) + \theta(u)\delta([u,v])$, *for all* $u,v \in J$,

(ii) $\delta(vu^2) = \theta(u^2)\delta(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))\delta(u) - \theta(u)\delta([u,v])$, *for all* $u,v \in J$.

Proof. (i) Replacing v by vu and uv in Lemma 2.2(i), we find that

$$\delta(uvu+vu^2) = 2(\theta(u)\delta(vu)+\theta(v)\theta(u)\delta(u)), \quad \forall u,v \in J, \tag{2.2}$$

$$\delta(u^2v+uvu) = 2(\theta(u)\delta(uv)+\theta(u)\theta(v)\delta(u)), \quad \forall u,v \in J. \tag{2.3}$$

Now, subtracting (2.2) from (2.3), we get

$$\delta(u^2v-vu^2) = 2(\theta(u)\delta([u,v])+[\theta(u),\theta(v)]\delta(u)), \quad \forall u,v \in J. \tag{2.4}$$

Replacing u by u^2 in Lemma 2.2(i), we have

$$\begin{aligned}\delta(u^2v+vu^2) &= 2\theta(u^2)\delta(v)+2\theta(v)u\delta(u^2)\\ &= 2\theta(u^2)\delta(v)+4\theta(v)\theta(u)\delta(u), \quad \forall u,v \in J.\end{aligned} \tag{2.5}$$

Hence adding (2.4), (2.5) and using the fact that char $R \neq 2$, we obtain

$$\begin{aligned}\delta(u^2v) &= \theta(u^2)\delta(v)+(\theta(u)\theta(v)+\theta(v)\theta(u))\delta(u)\\ &\quad +\theta(u)\delta([u,v]), \quad \forall u,v \in J.\end{aligned} \tag{2.6}$$

(ii) As in the proof of the case (i), subtracting (2.4) from (2.5), we find that

$$\begin{aligned}\delta(vu^2) &= \theta(u^2)\delta(v) + (3\theta(v)\theta(u) - \theta(u)\theta(v))\delta(u)\\ &\quad - \theta(u)\delta([u,v]), \quad \forall u,v \in J.\end{aligned} \tag{2.7}$$

□

3. Left derivation on Jordan ideal of a prime ring. In [3], there is a more general result which implies that in a 2-torsion-free prime ring R, the existence of a nonzero Jordan left derivation on a Lie ideal U of R forces that either $U \subseteq Z(R)$ or $\delta(U) = (0)$. In the present section, we attempt to generalize the above-mentioned result for Jordan left (θ,θ)-derivation which acts on a Jordan ideal of the ring.

Theorem 3.1. *Let R be a 2-torsion-free prime ring and let J be a Jordan ideal and a subring of R. If θ is an automorphism of R and $\delta : R \to R$ is an additive mapping satisfying $\delta(u^2) = 2\theta(u)\delta(u)$, for all $u \in J$, then either $J \subseteq Z(R)$ or $\delta(J) = (0)$.*

Proof. Suppose that $J \nsubseteq Z(R)$. By Lemma 2.2(iv), we have

$$[\theta(u),\theta(v)]\theta(u)\delta(u) = \theta(u)[\theta(u),\theta(v)]\delta(u), \quad \forall u,v \in J. \tag{3.1}$$

This implies that

$$(\theta(u^2)\theta(v) - 2\theta(u)\theta(v)\theta(u) + \theta(v)\theta(u^2))\delta(u) = 0, \quad \forall u,v \in J. \tag{3.2}$$

Replacing u by $[u,w]$ in (3.2), we get

$$\begin{aligned}&\theta([u,w]^2)\theta(v)\delta([u,w]) - 2\theta([u,w])\theta(v)\theta([u,w])\delta([u,w])\\ &\quad + \theta(v)\theta([u,w]^2)\delta([u,w]) = 0,\end{aligned} \tag{3.3}$$

for all $u,v,w \in J$. Now, application of Lemma 2.3(i) yields that $\theta([u,w]^2)\theta(v)\delta([u,w]) = (0)$, for all $u,v,w \in J$. Since θ is an automorphism of R, the latter expression gives $[u,w]^2 J\theta^{-1}(\delta([u,w])) = (0)$. Hence, by Lemma 2.6, we find that for each pair $u,w \in J$, either $[u,w]^2 = 0$ or $\theta^{-1}(\delta([u,w])) = 0$. Note that the mappings $(u,w) \mapsto [u,w]$ and $(u,w) \mapsto \theta^{-1}(\delta([u,w]))$ satisfy the requirements of Lemma 2.1. Hence, either $[u,w]^2 = 0$, for all $u,w \in J$, or $\theta^{-1}(\delta([u,w])) = 0$, for all $u,w \in J$. If $[u,w]^2 = 0$, for all $u,w \in J$, then by application of Lemma 2.8, J is commutative and hence central, that is, $J \subseteq Z(R)$, a contradiction. Now, we consider the case $\theta^{-1}(\delta([u,w])) = 0$, then $\delta([u,w]) = 0$, that is, $\delta(uw) = \delta(wu)$, for all $u,w \in J$. In view of Lemma 2.2(i), we have

$$\begin{aligned}2\delta((wu)u) &= \delta((wu)u + u(wu))\\ &= 2\theta(w)\theta(u)\delta(u) + 2\theta(u)\delta(wu + uw)\\ &= 2\{\theta(u^2)\delta(w) + \theta(u)\theta(w)\delta(u) + \theta(w)\theta(u)\delta(u)\}, \quad \forall u,w \in J.\end{aligned} \tag{3.4}$$

Since R is 2-torsion-free, we get $\delta((wu)u) = \theta(u^2)\delta(w) + \theta(u)\theta(w)\delta(u) + \theta(w)\theta(u)\delta(u)$, for all $u,w \in J$. By Lemma 2.9(ii), we obtain $[\theta(u),\theta(w)]\delta(u) = 0$, for all $u,w \in J$. Replacing w by wv in the latter expression, we get $[\theta(u),\theta(w)]\theta(v)\delta(u) = 0$, that is, $[u,w]J\theta^{-1}(\delta(u)) = (0)$. Thus, by Lemma 2.6, we find that for each $u \in J$ either

$[u,w]=0$ or $\theta^{-1}(\delta(u))=0$. Since θ is an automorphism, we have either $[u,w]=0$ or $\delta(u)=0$, for all $w\in J$. Now let $J_1=\{u\in J\mid [u,w]=0,\text{ for all } w\in J\}$ and $J_2=\{u\in J\mid \delta(u)=0\}$. Clearly, J_1 and J_2 are additive subgroups of J whose union is J. Hence, by Brauer's trick, either $J=J_1$ or $J=J_2$. If $J=J_1$, then $[u,w]=0$, for all $u,w\in J$, that is, J is commutative, and hence by Lemma 2.7, $J\subseteq Z(R)$, again a contradiction. Hence, we have the remaining possibility that $\delta(u)=0$, for all $u\in J$, that is, $\delta(J)=(0)$. This completes the proof of the theorem. □

Remark 3.2. In the hypotheses of the above theorem, if we assume only that J is a subring of R, then neither J is central nor $\delta(J)=(0)$. This is shown by the following example.

Example 3.3. Let S be a ring such that the square of each element in S is zero, but the product of some elements in S is nonzero. Further, suppose that $R=\{\left(\begin{smallmatrix}x & y\\ 0 & 0\end{smallmatrix}\right)\mid x,y\in S\}$. Consider $J=\{\left(\begin{smallmatrix}0 & y\\ 0 & 0\end{smallmatrix}\right)\mid y\in S\}$, then J is a subring of R. Define mappings $\delta:R\to R$ and $\theta:R\to R$ as follows:

$$\delta\begin{pmatrix}x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & y\\ 0 & 0\end{pmatrix},\qquad \theta\begin{pmatrix}x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix}x & -y\\ 0 & 0\end{pmatrix}. \tag{3.5}$$

It is easy to verify that δ is a Jordan left (θ,θ)-derivation, but neither $J\subseteq Z(R)$ nor $\delta(J)=(0)$.

Corollary 3.4. *Let R be a 2-torsion-free prime ring. If $\delta:R\to R$ is a nonzero additive mapping satisfying $\delta(x^2)=2x\delta(x)$, for all $x\in R$, then R is commutative.*

The following example demonstrates that to have R prime is essential in the hypothesis of the above result.

Example 3.5. Consider a ring R, as in Example 3.3, and define mappings $\delta:R\to R$ and $\theta:R\to R$ as follows:

$$\delta\begin{pmatrix}x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & -x\\ 0 & 0\end{pmatrix},\qquad \theta\begin{pmatrix}x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix}x & -y\\ 0 & 0\end{pmatrix}. \tag{3.6}$$

Then, with $J=R$, it can be easily seen that $\delta(x^2)=2\theta(x)\delta(x)$, for all $x\in R$, but R is not commutative.

4. Left derivation as a homomorphism or as an antihomomorphism. Let S be a nonempty subset of a ring R and d a derivation of R. If $d(xy)=d(x)d(y)$ (resp., $d(xy)=d(y)d(x)$) holds for all $x,y\in S$, then we say that d acts as a homomorphism (resp., antihomomorphism) on S.

In 1989, Bell and Kappe [4] proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an antihomomorphism on a nonzero right ideal I of R, then $d=0$ on R. Further, this result was extended for (θ,ϕ)-derivation in [2] as follows.

Theorem 4.1. *Let R be a prime ring, I a nonzero right ideal of R, and let θ, ϕ be automorphisms of R. Suppose that $\delta : R \to R$ is a (θ,ϕ)-derivation of R.*

(i) *If δ acts as a homomorphism on I, then $\delta = 0$ on R.*

(ii) *If δ acts as an antihomomorphism on I, then $\delta = 0$ on R.*

In the present section, our objective is to extend the above result for left (θ,θ)-derivation of a prime ring R which acts as a homomorphism or as an antihomomorphism on a Jordan ideal J of R. In fact, we prove the following theorem.

Theorem 4.2. *Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal and a subring of R. Suppose that θ is an automorphism of R and $\delta : R \to R$ is a left (θ,θ)-derivation of R.*

(i) *If δ acts as a homomorphism on J, then $\delta = 0$ on R.*

(ii) *If δ acts as an antihomomorphism on J, then $\delta = 0$ on R.*

Proof. (i) By our hypotheses, we have

$$\delta(u)\delta(v) = \delta(uv) = \theta(u)\delta(v) + \theta(v)\delta(u), \quad \forall u,v \in J. \tag{4.1}$$

Replacing u by uv in (4.1), we find that

$$\delta(uv)\delta(v) = \theta(uv)\delta(v) + \theta(v)\delta(uv), \quad \forall u,v \in J. \tag{4.2}$$

Now, application of (4.1) yields that $\theta(u)\delta(v)\delta(v) = \theta(uv)\delta(v)$, for all $u,v \in J$. This implies that

$$\theta(u)(\delta(v) - \theta(v))\delta(v) = 0, \quad \forall u,v \in J. \tag{4.3}$$

Thus, $\theta(J)(\delta(v) - \theta(v))\delta(v) = (0)$, for all $v \in J$. Since θ is an automorphism and J is a nonzero Jordan ideal of R, $\theta(J)$ is also a nonzero Jordan ideal of R. Application of Lemma 2.6 yields that $(\delta(v) - \theta(v))\delta(v) = 0$, for all $v \in J$ and hence $\delta(v^2) = \theta(v)\delta(v)$, for all $v \in J$. Since δ is a left (θ,θ)-derivation, we have $\theta(v)\delta(v) = 0$, for all $v \in J$. On linearizing the latter relation, we find that

$$\theta(v)\delta(u) + \theta(u)\delta(v) = 0, \quad \forall u,v \in J. \tag{4.4}$$

Again, replacing u by vu in (4.4), we get $\theta(v)\theta(u)\delta(v) = 0$, for all $u,v \in J$, that is, $vJ\theta^{-1}(\delta(v)) = (0)$, for all $v \in J$. Application of Lemma 2.6 yields that either $v = 0$ or $\theta^{-1}(\delta(v)) = 0$. But $v = 0$ also gives that $\theta^{-1}(\delta(v)) = 0$, that is, $\delta(v) = 0$, for all $v \in J$. Further, replace v by $v \circ r$ to get $2\theta(v)\delta(r) = 0$, for all $v \in J$ and $r \in R$. Since R is 2-torsion-free and $\theta(J)$ is a nonzero Jordan ideal of R, application of Lemma 2.6 yields the required result.

(ii) If d acts as an antihomomorphism on J, then

$$\begin{aligned}\delta(u)\delta(v) &= \delta(vu) = \theta(v)\delta(u) + \theta(u)\delta(v)\\ &= \theta(u)\delta(v) + \theta(v)\delta(u) = \delta(uv) = \delta(v)\delta(u),\end{aligned} \tag{4.5}$$

and hence δ also acts as a homomorphism on J. Therefore, in view of (i) we get the required result. □

Remark 4.3. We feel that Theorem 3.1 (resp., Theorem 4.2) could be proved for Jordan left (θ,ϕ)-derivation (resp., left (θ,ϕ)-derivation) of a prime ring. However, we did not succeed to settle it.

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References

[1] M. Ashraf and N. Rehman, *On Lie ideals and Jordan left derivations of prime rings*, Arch. Math. (Brno) **36** (2000), no. 3, 201–206.
[2] M. Ashraf, N. Rehman, and M. A. Quadri, *On (σ,τ)-derivations in certain classes of rings*, Rad. Mat. **9** (1999), no. 2, 187–192.
[3] M. Ashraf, N. Rehman, and A. Shakir, *On Jordan left derivations of Lie ideals in prime rings*, Southeast Asian Bull. Math. **25** (2001), no. 3, 379–382.
[4] H. E. Bell and L.-C. Kappe, *Rings in which derivations satisfy certain algebraic conditions*, Acta Math. Hungar. **53** (1989), no. 3-4, 339–346.
[5] M. Brešar and J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 7–16.
[6] ______, *Jordan (θ,ϕ)-derivations*, Glas. Mat. Ser. III **26(46)** (1991), no. 1-2, 13–17.
[7] Q. Deng, *On Jordan left derivations*, Math. J. Okayama Univ. **34** (1992), 145–147.
[8] M. F. Smiley, *Jordan homomorphisms onto prime rings*, Trans. Amer. Math. Soc. **84** (1957), 426–429.
[9] J. Vukman, *Jordan left derivations on semiprime rings*, Math. J. Okayama Univ. **39** (1997), 1–6.

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