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FOURIER TRANSFORM AND DISTRIBUTIONAL REPRESENTATION OF THE GAMMA FUNCTION LEADING TO SOME NEW IDENTITIES

M. ASLAM CHAUDHRY and ASGHAR QADIR

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We present a Fourier transform representation of the gamma functions, which leads naturally to a distributional representation for them. Both of these representations lead to new identities for the integrals of gamma functions multiplied by other functions, which are also presented here.

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1. Introduction. Considering the time since Euler extended the domain of the factorial function from the natural numbers, $\mathbb{N}$, to $\mathbb{C}$ by defining the gamma function [1, page 1, (1.1)] and its extensive application to a wide variety of problems [1, pages 357–433], one would not expect to find any significant new statements about the gamma function. However, to the best of our knowledge, there is no distributional representation of this function. We present a Fourier transform representation that leads naturally to a distributional representation. Both these representations lead to new identities for the integrals of products of gamma functions with other functions, which are not included in the standard lists of properties of the gamma function [2, 3, 4].

2. The Fourier transform and distributional representations. Denoting the inner product of two functions relative to the weight factor 1, over the domain $(-\infty,\infty)$, and using t_+^{α} to denote the function t^{α} for $t>0$ and 0 for $t\leq 0$, the gamma function can be represented by [1, page 1, (1.1)]

$$\Gamma(\sigma+i\tau)=\langle t_+^{\alpha+i\tau-1},e^{-t}\rangle. \tag{2.1}$$

Replacing t by e^x, we can rewrite (2.1) as

$$\Gamma(\sigma+i\tau)=\langle e^{i\tau x},e^{\sigma x}\exp(-e^x)\rangle. \tag{2.2}$$

Writing

$$e^{\sigma x}\exp(-e^x):=f_\sigma(x)\quad(\sigma>0), \tag{2.3}$$

we see that the gamma function can be regarded as the Fourier transform of $f_\sigma(x)$:

$$F[f_\sigma(x);\tau]=\Gamma(\sigma+i\tau), \tag{2.4}$$

where we define $F[\varphi;\tau]:=\langle e^{ix\tau},\varphi(x)\rangle$.

This is the Fourier transform representation of the gamma function.

Using the series expansion

$$\exp(-e^{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{nx} \tag{2.5}$$

and the relationship [5, page 253]

$$F[e^{\sigma x};\tau] = 2\pi\delta(\tau - i\sigma), \tag{2.6}$$

we can rewrite (2.1) as a series of delta functions:

$$\Gamma(\sigma + i\tau) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(\tau - i(\sigma + n)), \tag{2.7}$$

which is our distributional representation of the gamma function. This representation is only meaningful when defined as the inner product of the Γ with a function that is continuous and has compact support (an element of the space of test functions).

3. Some identities based on the Fourier transform representation. Using the Parseval identity [5, page 232] for the Fourier transform representation (2.4), we find

$$\left\langle F[f_\sigma, \tau], \overline{F[f_\rho, \tau]} \right\rangle = 2\pi \langle f_\sigma, \overline{f_\rho} \rangle. \tag{3.1}$$

The right-hand side of this equation can be obtained by straightforward integration to yield

$$\int_{-\infty}^{+\infty} \Gamma(\sigma + i\tau)\Gamma(\rho - i\tau)\,d\tau = \pi 2^{1-\sigma-\rho}\Gamma(\sigma + \rho). \tag{3.2}$$

As a special case, taking $\rho = \sigma$, we get the "norm squared" of Γ:

$$\int_{-\infty}^{+\infty} |\Gamma(\sigma + i\tau)|^2 d\tau = \left\langle \Gamma(\sigma + i\tau), \overline{\Gamma(\sigma + i\tau)} \right\rangle = \pi 2^{1-2\sigma}\Gamma(2\sigma). \tag{3.3}$$

To verify that these formulae are consistent with known results, take $\rho = \sigma = 1/2$ to give [4, page 387]

$$\int_0^{\infty} \operatorname{sec}h\, bx\, dx = \frac{\pi}{2b}, \tag{3.4}$$

$\rho = \sigma = 1$ to give [4, page 391]

$$\int_0^{\infty} x \operatorname{cosec}h\, bx\, dx = \frac{\pi^2}{4b^2}, \tag{3.5}$$

and $\rho = 3/4$, $\sigma = 1/4$ to give

$$\int_{-\infty}^{+\infty} [\cosh bx + \imath \sinh bx]^{-1} = \frac{\pi b}{\sqrt{2}}. \tag{3.6}$$

Use the Fourier convolution of $f_\sigma(x)$ with $f_\rho(x)$ to write

$$(f_\sigma * f_\rho)(x) = e^{\sigma x}\int_{-\infty}^{+\infty} e^{(\rho-\sigma)t}\exp\left[-e^t - e^x e^{-t}\right]dt. \tag{3.7}$$

Putting $e^t = \tau$, we can write (3.7) in terms of the generalized incomplete gamma function [1, page 43, (2.65)]:

$$(f_\sigma * f_\rho)(x) = e^{\sigma x}\Gamma(\rho-\sigma;0;e^x) = e^{\rho x}\Gamma(\sigma-\rho;0;e^x), \tag{3.8}$$

as the convolution is symmetric. Since the Fourier transform of the convolution equals the product of Fourier transforms,

$$F\left[e^{\sigma x}\Gamma(\rho-\sigma;0;e^x);\tau\right] = \Gamma(\sigma+i\tau)\Gamma(\rho+i\tau), \tag{3.9}$$

which relates the Fourier transform of the generalized incomplete gamma function with a product of gamma function. Further, since

$$F^2[f;x] = 2\pi f(-x), \tag{3.10}$$

we find

$$F\left[\Gamma(\sigma+i\tau)\Gamma(\rho+i\tau);x\right] = 2\pi e^{-\sigma x}\Gamma(\rho-\sigma;0;e^{-x}). \tag{3.11}$$

From (3.1) and (3.11), we find that

$$\int_{-\infty}^{+\infty}\left|\Gamma(\sigma+i\tau)\Gamma(\rho+i\tau)\right|^2 d\tau = \int_{-\infty}^{+\infty} e^{-2\sigma x}\Gamma^2(\rho-\sigma;0;e^{-x})dx. \tag{3.12}$$

Since the generalized incomplete gamma function is related to the Macdonald function by

$$\Gamma(\alpha;0;b) = 2b^{\alpha/2}K_\alpha(2\sqrt{b}), \tag{3.13}$$

then

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\Gamma(\sigma+i\tau)\Gamma(\rho+i\tau)\right|^2 d\tau &= 2\pi\int_{-\infty}^{+\infty} 4e^{-(\sigma+\rho)x}K^2_{\rho-\sigma}(2e^{-x/2})dx \\ &= 8\pi\int_0^{\infty} t^{\sigma+\rho-1}K^2_{\rho-\sigma}(2\sqrt{t})dt,\end{aligned} \tag{3.14}$$

putting t in place of e^{-x}.

The right-hand side in (3.14) is the standard integral [2, page 334, (45)] that yields the closed-form evaluation

$$\int_{-\infty}^{+\infty}\left|\Gamma(\sigma+i\tau)\Gamma(\rho+i\tau)\right|^2 d\tau = 2\pi\Gamma^2(\rho+\sigma)B(2\sigma,2\rho) \quad (\rho,\sigma>0). \tag{3.15}$$

As a special case, taking $\rho=\sigma$, we find

$$\int_{-\infty}^{+\infty}\left|\Gamma(\sigma+i\tau)\right|^4 d\tau = 2\pi\Gamma^4\frac{(2\sigma)}{\Gamma(4\sigma)}. \tag{3.16}$$

To verify that this result yields known results as special cases, take $\sigma = 1/2$ to obtain [4, page 395, (3.527)(3)]

$$\int_0^\infty \operatorname{sec} h^2 x\, dx = 1, \tag{3.17}$$

and $\sigma = 1$ to obtain [4, page 396, (12)]

$$\int_0^\infty x^2 \operatorname{csc} h^2 x\, dx = \frac{\pi^2}{6}. \tag{3.18}$$

4. New identities based on the distributional representation. Let Σ be the space of all entire functions φ for which the series $\sum_{n=0}^{\infty}(-1)^n/n!\phi(z+n)$ converges for all z.

We call the members of the space Σ the test functions. It is to be noted that Σ is a nontrivial space as e^z is in Σ.

If $\phi(z)$ is a test function, then (2.7) directly yields

$$\langle \Gamma(\sigma+i\tau),\ \phi(\rho+i\tau)\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\phi(\rho-\sigma-n). \tag{4.1}$$

The above action of the gamma function is well defined for all φ in the space Σ. Moreover,

$$\langle \Gamma(\sigma+i\tau),\ \varphi(\tau)\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\varphi(i(\sigma+n)) \quad \forall \varphi \in \Sigma. \tag{4.2}$$

It is to be noted that new identities may be obtained by performing the permissible parametric differentiation in the old identities. For example, differentiate (3.2) relative to σ to obtain

$$\int_{-\infty}^{+\infty} \Gamma^{(1)}(\sigma+i\tau)\Gamma(\rho-i\tau)d\tau = \pi 2^{1-\sigma-\rho}[\Gamma(\sigma+\rho)\ln 2 + \Gamma^{(1)}(\sigma+\rho)]. \tag{4.3}$$

Replacing $\Gamma^{(1)}$ by the product $\psi\Gamma$, (4.3) becomes

$$\int_{-\infty}^{+\infty} \psi(\sigma+i\tau)\Gamma(\sigma+i\tau)\Gamma(\rho-i\tau)d\tau = \pi 2^{1-\sigma-\rho}\Gamma(\sigma+\rho)[\ln 2 + \psi(\sigma+\rho)]. \tag{4.4}$$

In particular, choosing $\rho = 1-\sigma$, as $\psi(1) = \gamma$ is the Euler constant, we find an interesting representation

$$\int_{-\infty}^{+\infty} \psi(\sigma+i\tau)\Gamma(\sigma+i\tau)\Gamma(1-\sigma-i\tau)d\tau = \pi(\ln 2+\gamma) \quad (0<\sigma<1), \tag{4.5}$$

which is independent of σ. That this is consistent may be verified by differentiating (4.4) relative to σ, and then integrating by parts.

The action of the gamma function, with complex argument, over the known functions will provide new identities via the relations given in (4.1) and (4.2). For this purpose, we consider the action of the gamma function on the entire function $e^{\pi\tau}$.

Again, directly from (4.2),

$$\langle \Gamma(\sigma+i\tau),\, e^{\pi\tau}\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{i\pi(\sigma+n)} = 2\pi \exp(1+i\pi\sigma). \tag{4.6}$$

Thus we have

$$\int_{-\infty}^{+\infty} e^{\pi\tau}\Gamma(\sigma+i\tau)\,d\tau = 2\pi \exp(1+i\pi\sigma) \tag{4.7}$$

which further yields

$$\int_{-\infty}^{+\infty} \cosh\pi\tau\,\Gamma(\sigma+i\tau)\,d\tau = 2\pi e\cos(\pi\sigma), \tag{4.8}$$

$$\int_{-\infty}^{+\infty} \sinh\pi\tau\,\Gamma(\sigma+i\tau)\,d\tau = 2\pi i e\sin(\pi\sigma). \tag{4.9}$$

Similarly, we have the identity

$$\langle \Gamma(\sigma+i\tau),\, K_{\varrho+i\tau}(x)\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K_{\varrho-\sigma-n}(x) \tag{4.10}$$

involving the Macdonald function.

Following the distributional representation, we consider the action of the gamma function on the entire function $1/(\Gamma(\sigma+i\tau))$. We note that

$$\left\langle \Gamma(\sigma+i\tau),\, \frac{1}{\Gamma(\varrho+i\tau)}\right\rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\Gamma(\varrho-\sigma-n)}. \tag{4.11}$$

These are just some examples of the various identities that can be obtained by the Fourier transform and the distributional representations of the gamma function. It is anticipated that the present Fourier transform and distributional representation of the gamma function would provide further insight to the properties of the gamma function in addition to leading to new identities.

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M. Aslam Chaudhry: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

E-mail address: maslam@kfupm.edu.sa

Asghar Qadir: Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia; Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, Rawalpindi, Pakistan

E-mail address: aqdirs@comsats.net.pk