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AN APPLICATION OF THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS FOR LOW REYNOLDS NUMBER FLOW PAST A CYLINDER OF ARBITRARY CROSS SECTION

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We study the low Reynolds number flow of an incompressible Newtonian fluid of infinite expanse past a cylinder of arbitrary cross section by using the method of matched asymptotic expansions. The analysis that will be made in this paper is equivalent to that developed by Power (1990) in order to solve the resulting inner (or Stokes) problems with the completed double-layer boundary integral equation method (CDLBIEM) due to Power and Miranda (1987). We will solve these problems by the boundary integral method developed by Hsiao and Kress (1985).

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1. Introduction. The method of matched asymptotic expansions has been developed by Kaplun [2] and Proudman and Pearson [11] for the problem of the low Reynolds number flow past a circular cylinder or a sphere. This method was used by many authors to treat several low Reynolds number flow problems. For example, Umemura [13] obtained a matched asymptotic analysis of low Reynolds number flow past two equal cylinders. The same method was also applied by Shintani et al. [12] to study the low Reynolds number flow due to a uniform stream at infinity past an elliptic cylinder. Lee and Leal [6] treated the low Reynolds number flow past cylindrical bodies of arbitrary cross section. In addition to the method of matched asymptotic expansions, they used the boundary integral formulation of Youngren and Acrivos [14], in order to obtain the corresponding solutions of the Stokes and Oseen approximations. Power [7] developed a matched asymptotic analysis for low Reynolds number flow past a cylinder of arbitrary cross section by using the completed double layer boundary integral equation method (CDLBIEM) to solve the resulting inner problems, and the singularity method to treat the resulting outer problems (see also [9, Section 6.3]).

Note that Youngren and Acrivos [14] proposed a boundary integral method in order to treat the unbounded Stokes flow due to the motion of a solid particle of arbitrary shape in an incompressible Newtonian fluid of infinite expanse. This method uses the direct boundary integral representation of an exterior Stokes flow, in which the variables are the boundary velocity and traction. Also, the method leads to a set of Fredholm integral equations of the first kind with unknown boundary traction. The method of Youngren and Acrivos [14] has been applied by many authors to obtain the numerical solutions of several problems dealing with solid particles and drops in Stokes flows, the motion of a particle near a solid wall or a fluid interface, particle-particle interactions, Stokes flows in containers, and so forth (see, e.g., [4, 9, 10]). However, the direct boundary integral representations, in particular, those encountered in the method of Youngren and Acrivos [14], lead to a set of Fredholm integral equations of the first kind, which, after the discretization of the involved surface integrals, is ill-conditioned at a large number of boundary elements. On the other hand, it is known that the boundary integral methods which lead to Fredholm integral equations of the second kind are more preferable than those which lead to Fredholm integral equations of the first kind, since the Fredholm integral equations of the second kind give rise to numerical solutions that are more stable than those due to Fredholm integral equations of the first kind. The indirect boundary integral methods are designed so that they provide a set of Fredholm integral equations of the second kind, and therefore they are always well-behaved numerically. In particular, effective iterative solution procedures can be applied to solve large scale problems with indirect formulations. An alternative indirect boundary integral formulation was proposed by Power and Miranda [8] for the three-dimensional exterior Stokes flow around a solid particle (see also [9]). Power and Miranda's method is a completion plus a deflation procedure that leads to a bounded and invertible integral operator (with a spectral radius strictly less than one), and therefore iterative solution strategies are guaranteed to converge to a unique solution. Karrila and Kim [3] called Power and Miranda's method the *completed double-layer boundary integral equation method* because of the involved completion procedure. This method applies to both two- and three-dimensional Stokes flow problems. For the two-dimensional Stokes flow problem due to the motion of a cylinder of arbitrary shape in an unbounded domain, there are two equivalent integral formulations available: one was provided by Hsiao and Kress [1] and uses a combination of double- and single-layer potentials. This formulation leads to a system of Fredholm integral equations of the second kind that has a unique continuous solution. The second formulation was developed by Power [7] and is given in terms of a double-layer potential and two singularities located inside the cylinder.

In this paper, we study the low Reynolds number flow of an incompressible Newtonian fluid of infinite expanse past a cylinder of arbitrary cross section by using the method of matched asymptotic expansions and the method of Hsiao and Kress [1] in order to solve the resulting inner (or Stokes) problems.

2. Inner and outer expansions. We consider the problem of determining the low Reynolds number flow of an incompressible Newtonian fluid of infinite expanse past a stationary cylinder of arbitrary cross section. At infinity the flow is a uniform stream with velocity U_∞ in the direction of the x_1-axis.

The flow is governed by the continuity and steady Navier-Stokes equations, which in nondimensional form are given by

$$\begin{gathered} \nabla\cdot\mathbf{v}=0 \quad \text{in } D, \\ -\nabla p+\nabla^2\mathbf{v}-\mathrm{Re}(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{0} \quad \text{in } D, \end{gathered} \tag{2.1}$$

where D is the two-dimensional unbounded domain exterior to the cross section of the cylinder in the x_1x_2-plane. Let Γ denote the boundary of this domain, assumed to be a simple closed Lyapunov curve (i.e., Γ has a continuously varying normal vector; more exactly, there exists $\alpha \in (0,1]$ such that Γ is of class $C^{1,\alpha}$). Also, let D_0 denote the bounded domain inside Γ. Equations (2.1) are nondimensionalized with respect to the characteristic variables $U_c = U_\infty$, $l_c = a$ (a characteristic cylinder radius), and $p_c = \mu U_\infty / a$ (the characteristic pressure). Also, the Reynolds number is defined by $\mathrm{Re} = \rho a U_\infty / \mu$, where ρ and μ are the density and dynamic viscosity of the fluid.

We have to require the following boundary and asymptotic conditions:

$$\mathbf{v}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Gamma, \tag{2.2}$$

$$\mathbf{v}(\mathbf{x}) \longrightarrow \mathbf{i}_1, \quad p(\mathbf{x}) \longrightarrow 0 \quad \text{as } |\mathbf{x}| \longrightarrow \infty, \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{i}_1$ denotes the unit vector along the x_1-axis of a frame of Cartesian coordinates whose origin is inside Γ.

According to the method of Kaplun [2] and Proudman and Pearson [11], the region around the cylinder is divided into two separate but overlapping regions, called the *inner and outer regions*. In the inner region, where $\mathrm{Re} \ll 1$ (and hence the inertial term is small), we consider the following expansions (see also [7] and [9, Section 6.3.4]):

$$\begin{aligned} \mathbf{v} &= f_0(\mathrm{Re})\mathbf{v}^0 + f_1(\mathrm{Re})\mathbf{v}^1 + \cdots, \\ p &= f_0(\mathrm{Re})p^0 + f_1(\mathrm{Re})p^1 + \cdots, \end{aligned} \tag{2.4}$$

such that

$$\frac{f_{n+1}(\mathrm{Re})}{f_n(\mathrm{Re})} \longrightarrow 0 \quad \text{as } \mathrm{Re} \longrightarrow 0. \tag{2.5}$$

The leading-order terms $\mathbf{v}^0$ and p^0 of these expansions satisfy the Stokes system of equations

$$\nabla \cdot \mathbf{v}^0 = 0, \qquad -\nabla p^0 + \nabla^2 \mathbf{v}^0 = \mathbf{0}, \tag{2.6}$$

whereas the first-order terms $\mathbf{v}^1$ and p^1 satisfy the following equations:

$$\nabla \cdot \mathbf{v}^1 = 0, \qquad -\nabla p^1 + \nabla^2 \mathbf{v}^1 - \mathrm{Re}\,\frac{(f_0(\mathrm{Re}))^2}{f_1(\mathrm{Re})}(\mathbf{v}^0 \cdot \nabla)\mathbf{v}^0 = \mathbf{0}. \tag{2.7}$$

On the other hand, in the outer region, where $|\mathbf{x}| \geq \mathbb{O}(\mathrm{Re}^{-1})$, the inertia term is not negligible, and hence it must be taken into consideration. Therefore, in this region the expansions (2.4) are not valid. For this reason, we introduce the new characteristic variables $\hat{l}_c = l_c / \mathrm{Re}$, $\hat{p}_c = \mu U_c / \hat{l}_c = \mathrm{Re}\, p_c$ (the characteristic pressure), and $v_c = U_\infty$ (the characteristic velocity). Also, we denote by $\hat{\mathbf{x}}$, $\hat{\mathbf{v}}$, and $\hat{p}$ the position vector of an arbitrary point and, respectively, the velocity and pressure fields corresponding to the outer region. Then the governing equations take the form

$$\frac{\partial \hat{v}_i}{\partial \hat{x}_i} = 0, \qquad -\frac{\partial \hat{p}}{\partial \hat{x}_j} + \frac{\partial^2 \hat{v}_j}{\partial \hat{x}_i \partial \hat{x}_i} - \hat{v}_i \frac{\partial \hat{v}_j}{\partial \hat{x}_i} = 0, \tag{2.8}$$

where the summation convention rule after the repeated indices is used. The solution $(\hat{\mathbf{v}},\hat{p})$ is expressed in the form (see also [7])

$$\begin{aligned}\hat{\mathbf{v}} &= \hat{f}_0(\mathrm{Re})\hat{\mathbf{v}}^0+\hat{f}_1(\mathrm{Re})\hat{\mathbf{v}}^1+\cdots,\\ \hat{p} &= \hat{f}_0(\mathrm{Re})\hat{p}^0+\hat{f}_1(\mathrm{Re})\hat{p}^1+\cdots,\end{aligned} \tag{2.9}$$

such that

$$\frac{\hat{f}_{n+1}(\mathrm{Re})}{\hat{f}_n(\mathrm{Re})}\longrightarrow 0 \quad \text{as } \mathrm{Re}\longrightarrow 0. \tag{2.10}$$

Clearly, the first term in each of the above asymptotic expansions corresponds to the uniform flow. Hence, we consider

$$\hat{f}_0(\mathrm{Re})=1, \qquad (\hat{\mathbf{v}}^0,\hat{p}^0)=(\mathbf{i}_1,0). \tag{2.11}$$

Therefore, the governing equations for $(\hat{\mathbf{v}}^1,p^1)$ are

$$\frac{\partial\hat{v}_j^1}{\partial\hat{x}_j}=0, \qquad -\frac{\partial\hat{p}^1}{\partial\hat{x}_j}+\frac{\partial^2\hat{v}_j^1}{\partial\hat{x}_k\partial\hat{x}_k}-\frac{\partial\hat{v}_j^1}{\partial\hat{x}_1}=0, \tag{2.12}$$

that is, the continuity and Oseen's equations.

We require that the boundary condition (2.2) be satisfied by the first of the expansions (2.4), and that the uniform stream conditions at infinity (2.3) be satisfied by the asymptotic expansions (2.9). Additionally, we have to apply the matching principle in the overlapping domain between the inner and outer regions, from which we obtain other asymptotic conditions for each expansion and the possibility to compute successive terms of these expansions.

3. The solution of the leading-order problem in the inner region. We next determine the solution $(\mathbf{v}^0,p^0)$ of the leading-order problem in the inner region by using the boundary integral method of Hsiao and Kress [1]. Therefore, we consider the following boundary integral representation of the flow field $\mathbf{v}^0$:

$$\begin{aligned}\mathbf{v}^0(\mathbf{x}) &= \mathbf{V}\left(\mathbf{x},-\frac{\mathbf{F}}{4\pi|\Gamma|}\right)+\eta_0\mathbf{V}\left(\mathbf{x},\frac{1}{4\pi}\left(\Phi-\frac{1}{|\Gamma|}\int_\Gamma\Phi\,dl\right)\right)\\ &\quad+\mathbf{W}\left(\mathbf{x},\frac{1}{4\pi}\Phi\right)-\eta_1\int_\Gamma\Phi(\mathbf{y})\,dl(\mathbf{y}),\end{aligned} \tag{3.1}$$

where $\mathbf{V}(\cdot,\Psi)$ is the single-layer potential with continuous density Ψ, given by

$$V_j(\mathbf{x},\Psi)=\int_\Gamma\mathcal{G}_{ji}(\mathbf{x}-\mathbf{y})\psi_i(\mathbf{y})\,dl(\mathbf{y}), \tag{3.2}$$

$\mathbf{W}(\cdot,\mathbf{h})$ is the double-layer potential with continuous density $\mathbf{h}$, given by

$$W_j(\mathbf{x},\mathbf{h})=\int_\Gamma K_{ij}^S(\mathbf{y},\mathbf{x})h_i(\mathbf{y})\,dl(\mathbf{y})=\int_\Gamma S_{ijk}^S(\mathbf{y}-\mathbf{x})n_k(\mathbf{y})h_i(\mathbf{y})\,dl(\mathbf{y}), \tag{3.3}$$

$\mathscr{G}_{ji}$ are the components of the two-dimensional Stokeslet $\mathscr{G}$, and S^S_{ijk} are the components of the stress tensor $\mathbf{S}^S$ associated with the two-dimensional Stokeslet. These components are given by (see, e.g., [5, 10])

$$\mathscr{G}_{ji}(\mathbf{x}) = -\delta_{ji}\ln|\mathbf{x}| + \frac{x_j x_i}{|\mathbf{x}|^2}, \qquad S^S_{ijk}(\mathbf{x}) = -4\frac{x_i x_j x_k}{|\mathbf{x}|^4}. \tag{3.4}$$

In addition, $|\mathbf{F}|$ is the total force exerted by the flow $(\mathbf{v}^0, p^0)$ on Γ, $\mathbf{n}$ is the outward unit normal vector to Γ, η_0 and η_1 are two real constants such that $\eta_0 > 0$ and $\eta_1 \neq 0$, $|\Gamma| = \int_\Gamma dl$ is the length of the curve Γ, and Φ is an unknown continuous vector density on Γ. The total force $\mathbf{F}$ will be determined from the matching principle.

The corresponding boundary integral representation of the pressure field p^0 is given by

$$p^0(\mathbf{x}) = P^s\left(\mathbf{x}, -\frac{\mathbf{F}}{4\pi|\Gamma|}\right) + \eta_0 P^s\left(\mathbf{x}, \frac{1}{4\pi}\left(\Phi - \frac{1}{|\Gamma|}\int_\Gamma \Phi\, dl\right)\right) + P^d\left(\mathbf{x}, \frac{1}{4\pi}\Phi\right), \tag{3.5}$$

where $P^s(\cdot,\Psi)$ is the pressure field associated with the single-layer potential $\mathbf{V}(\cdot,\Psi)$, that is,

$$P^s(\mathbf{x},\Psi) = \int_\Gamma \Pi^S_i(\mathbf{x}-\mathbf{y})\psi_i(\mathbf{y})\,dl(\mathbf{y}), \tag{3.6}$$

and $P^d(\cdot,\mathbf{h})$ is the pressure field associated with the double-layer potential $\mathbf{W}(\cdot,\mathbf{h})$, that is,

$$P^d(\mathbf{x},\mathbf{h}) = \int_\Gamma \Lambda^S_{ik}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})h_i(\mathbf{y})\,dl(\mathbf{y}), \tag{3.7}$$

Π^S_i and Λ^S_{ik} being the components of the pressure vector Π^S and of the pressure tensor $\mathbf{\Lambda}^S$, respectively, associated with the two-dimensional Stokeslet. These components are given by the formulas (see, e.g., [10])

$$\Pi^S_i(\mathbf{x}) = 2\frac{x_i}{|\mathbf{x}|^2}, \qquad \Lambda^S_{ik}(\mathbf{x}) = -4\frac{\delta_{ik}}{|\mathbf{x}|^2} + 8\frac{x_i x_k}{|\mathbf{x}|^4}. \tag{3.8}$$

Note that the single-layer potential $\mathbf{V}(\cdot,\Psi)$ is continuous across the Lyapunov contour Γ, but the double-layer potential $\mathbf{W}(\cdot,\mathbf{h})$ has a jump provided by the following limiting values on both sides of Γ:

$$W^{\pm}_j(\mathbf{x}_0,\mathbf{h}) = \pm 2\pi h_j(\mathbf{x}_0) + \int_\Gamma^{\mathrm{PV}} S^S_{ijk}(\mathbf{y}-\mathbf{x}_0)n_k(\mathbf{y})h_i(\mathbf{y})\,dl(\mathbf{y}), \quad \mathbf{x}_0 \in \Gamma, \tag{3.9}$$

where the plus sign applies for the external side of Γ (in the direction of the unit normal vector) and the minus sign applies for the internal side of Γ. Also, the symbol PV means the principal value of the double-layer potential at an arbitrary point $\mathbf{x}_0 \in \Gamma$ (note that the kernel K^S_{ij} of W_j is weakly singular, but the corresponding double-layer integral has a well-defined value at any point of Γ. For more details, see, for example, [9, Chapter 5]).

Now, applying the boundary condition (2.2) to the flow field defined by the boundary integral representation (3.1), and using the above-mentioned properties of single- and double-layer potentials, we obtain the following Fredholm integral equation of the second kind with unknown continuous vector density $\Phi = (\phi_1, \phi_2)$:

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}^d + \eta_0 \mathbf{K}^s \mathbf{M} - \eta_1 |\Gamma| (\mathbf{I} - \mathbf{M})\right)\Phi = \mathbf{V}\left(\cdot, \frac{\mathbf{F}}{4\pi|\Gamma|}\right) \quad \text{on } \Gamma, \tag{3.10}$$

where $\mathbf{I} : C^0(\Gamma) \to C^0(\Gamma)$ is the identity operator, $\mathbf{M} : C^0(\Gamma) \to C^0(\Gamma)$ is the operator given by

$$\mathbf{M}\mathbf{h} = \mathbf{h} - \frac{1}{|\Gamma|}\int_\Gamma \mathbf{h}\, dl, \quad \mathbf{h} \in C^0(\Gamma), \tag{3.11}$$

and $\mathbf{K}^s : C^0(\Gamma) \to C^0(\Gamma)$ and $\mathbf{K}^d : C^0(\Gamma) \to C^0(\Gamma)$ are the single- and double-layer integral operators given by the relations

$$(\mathbf{K}^s\mathbf{h})(\mathbf{x}) = \mathbf{V}\left(\mathbf{x}, \frac{1}{4\pi}\mathbf{h}\right), \qquad (\mathbf{K}^d\mathbf{h})_j(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma^{\mathrm{PV}} S_{ijk}(\mathbf{y}-\mathbf{x}) n_k(\mathbf{y}) h_i(\mathbf{y})\, dl(\mathbf{y}) \tag{3.12}$$

for $\mathbf{h} \in C^0(\Gamma)$ and $\mathbf{x} \in \Gamma$. Note that both single- and double-layer integral operators are compact on $C^0(\Gamma)$.

By using the notation $\Phi = |\mathbf{F}|\Psi$, the above equation becomes

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}^d + \eta_0 \mathbf{K}^s \mathbf{M} - \eta_1 |\Gamma| (\mathbf{I} - \mathbf{M})\right)\Psi = \mathbf{V}\left(\cdot, \frac{\mathbf{i}_1}{4\pi|\Gamma|}\right) \quad \text{on } \Gamma. \tag{3.13}$$

We mention that Hsiao and Kress [1] proved that (3.13) possesses a unique continuous solution Ψ. Therefore, this density provides the unique continuous solution Φ of (3.10).

4. The solution of the first-order problem in the outer region. As in [7], we take into account the fact that far from the cylindrical body the role of the cylinder is similar to that of a point force. Consequently, if we allow some point force located at the origin to act on the fluid, then the first-order problem (i.e., the Oseen problem) could be satisfied in the outer region. Therefore, it is sufficient to consider the two-dimensional Oseenlet, that is, the fundamental solution of Oseen's equation, which in outer variables is given by (see [6, 7] and [9, page 238])

$$\begin{aligned}\mathcal{G}_{ij}^O(\hat{\mathbf{x}}) = &-\frac{1}{4\pi}e^{\hat{x}_1/2}K_0\left(\frac{|\hat{\mathbf{x}}|}{2}\right)\delta_{ij} \\ &-\frac{1}{4\pi}\left[e^{\hat{x}_1/2}K_1\left(\frac{|\hat{\mathbf{x}}|}{2}\right) - \frac{2}{|\hat{\mathbf{x}}|}\right]\left[\frac{\hat{x}_i}{|\hat{\mathbf{x}}|}\delta_{ij} + \frac{\hat{x}_2\delta_{i1} - \hat{x}_1\delta_{i2}}{|\hat{\mathbf{x}}|}\delta_{2j}\right],\end{aligned} \tag{4.1}$$

where K_0 and K_1 are the modified Bessel functions of the second kind and of orders 0 and 1, respectively. Note that the above fundamental solution can be derived if we include the term $-\mathbf{i}_j\delta(\hat{\mathbf{x}})$ on the left-hand side of the second equation of (2.12), and then use the method of Fourier transform. Here, δ denotes the Dirac distribution or the delta function in $\mathbb{R}^2$.

5. The matching principle for the inner and outer expansions. We have seen that the zeroth-order solution for the outer region is a uniform flow described by $\hat{f}_0(\mathrm{Re}) = 1, (\hat{\mathbf{v}}^0, \hat{p}^0) = (\mathbf{i}_1, 0)$. Therefore, the matching principle requires that the zeroth-order solution for the inner region should become

$$\lim_{\substack{\mathrm{Re}\to 0\\ |\mathbf{x}|\gg 1}} f_0(\mathrm{Re})(\mathbf{v}^0, p^0) = (\mathbf{i}_1, 0). \tag{5.1}$$

On the other hand, the singular behavior of the flow field $\mathbf{v}^0$ at large distances is provided by the two-dimensional Stokeslet, and hence it is of logarithmic type (see (3.1)). Therefore, for large $|\mathbf{x}| = \mathcal{O}(\mathrm{Re}^{-1})$, where $\mathrm{Re} \to 0$, we have the relation

$$f_0(\mathrm{Re}) v_i^0(\mathbf{x}) \sim \frac{f_0(\mathrm{Re})}{4\pi} (\ln \mathrm{Re}) F_i, \tag{5.2}$$

which shows that the matching condition (5.1) is satisfied to leading order if

$$\frac{4\pi}{\ln \mathrm{Re}} \mathbf{i}_1 = f_0(\mathrm{Re}) \mathbf{F}. \tag{5.3}$$

Consequently, the complete velocity field up to the leading-order solution of the Stokes problem is given by

$$\begin{aligned} &f_0(\mathrm{Re}) \mathbf{v}^0(\mathbf{x}) \\ &= \frac{4\pi}{\ln \mathrm{Re}} \left\{ \mathbf{V}\left(\mathbf{x}, -\frac{\mathbf{i}_1}{4\pi|\Gamma|}\right) + \eta_0 \mathbf{V}\left(\mathbf{x}, \frac{1}{4\pi}\left(\Psi - \frac{1}{|\Gamma|}\int_\Gamma \Psi\, dl\right)\right) + \mathbf{W}\left(\mathbf{x}, \frac{1}{4\pi}\Psi\right) - \eta_1 \int_\Gamma \Psi\, dl \right\}, \end{aligned} \tag{5.4}$$

where Ψ is the unique continuous solution of (3.13). Moreover, we have

$$\begin{aligned} &f_0(\mathrm{Re}) p^0(\mathbf{x}) \\ &= \frac{4\pi}{\ln \mathrm{Re}} \left\{ P^s\left(\mathbf{x}, -\frac{\mathbf{i}_1}{4\pi|\Gamma|}\right) + \eta_0 P^s\left(\mathbf{x}, \frac{1}{4\pi}\left(\Psi - \frac{1}{|\Gamma|}\int_\Gamma \Psi\, dl\right)\right) + P^d\left(\mathbf{x}, \frac{1}{4\pi}\Psi\right) \right\}. \end{aligned} \tag{5.5}$$

Furthermore, taking into account the asymptotic expansions (2.9) and the expressions (5.4) and (5.5), we find that

$$\hat{f}_1(\mathrm{Re}) = \frac{4\pi}{\ln \mathrm{Re}} \tag{5.6}$$

and that $(\hat{\mathbf{v}}^1, \hat{p}^1)$ is the flow due to an Oseenlet located at the origin and oriented in the x_1-direction. Therefore, we have

$$\begin{aligned} &\hat{f}_1(\mathrm{Re}) \hat{v}_i^1(\hat{\mathbf{x}}) \\ &= \frac{4\pi}{\ln \mathrm{Re}} \left\{ -\frac{1}{4\pi} e^{\hat{x}_1/2} K_0\left(\frac{|\hat{\mathbf{x}}|}{2}\right) \delta_{i1} - \frac{1}{4\pi}\left[e^{\hat{x}_1/2} K_1\left(\frac{|\hat{\mathbf{x}}|}{2}\right) - \frac{2}{|\hat{\mathbf{x}}|}\right] \frac{\hat{x}_i}{|\hat{\mathbf{x}}|} \delta_{i1} \right\}. \end{aligned} \tag{5.7}$$

In order to study the asymptotic behavior of the inner flow velocity field far from the origin, we expand the two-dimensional Stokeslet $\mathcal{G}(\mathbf{x}-\mathbf{y})$ in a Taylor series with respect to $\mathbf{y}$ about the origin. Then we obtain the following asymptotic expansion of the inner flow velocity field:

$$\begin{aligned} f_0(\mathrm{Re})v_j^0(\mathbf{x}) &= \frac{4\pi}{\ln\mathrm{Re}}\left\{-\mathcal{G}_{ji}(\mathbf{x})\frac{\delta_{i1}}{4\pi}+\cdots+W_j\left(\mathbf{x},\frac{1}{4\pi}\Psi\right)-\eta_1\int_\Gamma \psi_j\,dl\right\} \\ &= \frac{4\pi}{\ln\mathrm{Re}}\left\{-\frac{1}{4\pi}\mathcal{G}_{j1}(\mathbf{x})+\cdots+W_j\left(\mathbf{x},\frac{1}{4\pi}\Psi\right)-\eta_1\int_\Gamma \psi_j\,dl\right\}. \end{aligned} \tag{5.8}$$

The above expansion shows that the asymptotic form of the inner flow velocity field far from the origin is the velocity field due to a Stokeslet located at the origin plus a constant vector. Moreover, we use the fact that the outer flow velocity field up to the first-order approximation at large distances from the origin is the velocity field of a uniform flow in the x_1-direction plus the velocity field due to an Oseenlet located at the origin and oriented in the x_1-direction. Therefore, we find that the mismatch between $f_0(\mathrm{Re})\mathbf{v}^0$, from the inner region, and the sum $\hat{f}_0(\mathrm{Re})\hat{\mathbf{v}}^0+\hat{f}_1(\mathrm{Re})\hat{\mathbf{v}}^1$, from the outer region, has the same terms as the mismatch provided by Proudman and Pearson [11], up to the same order of approximation for the singular perturbation solution of a uniform flow past a circular cylinder at small Reynolds number (see also [7] and [9, page 240]). This mismatch denoted by

$$\Delta \equiv \hat{f}_0(\mathrm{Re})\hat{\mathbf{v}}^0+\hat{f}_1(\mathrm{Re})\hat{\mathbf{v}}^1-f_0(\mathrm{Re})\mathbf{v}^0 \tag{5.9}$$

is hence given by

$$\Delta \sim \frac{1}{\ln\mathrm{Re}}\left\{(\gamma_0-\ln 4)\mathbf{i}_1+4\pi\eta_1\int_\Gamma \Psi\,dl\right\}, \tag{5.10}$$

where $\gamma_0 = 0.5772\ldots$ is the Euler constant and Ψ is the unique continuous solution of (3.13). Further, according to the fact that this mismatched uniform flow is $\mathcal{O}((\ln\mathrm{Re})^{-1})$, we deduce that $(\mathbf{v}^1,p^1)$ will be a solution of the Stokes equation, since the term $(\mathbf{v}^0\cdot\nabla)\mathbf{v}^0$ in the second equation of (2.7) is asymptotically negligible with respect to any inverse power of $\ln\mathrm{Re}$. Consequently, the first-order inner velocity and pressure fields are also given by (3.1) and (3.5) with

$$f_1(\mathrm{Re}) = \frac{4\pi}{(\ln\mathrm{Re})^2}, \qquad \mathbf{F} = (\gamma_0-\ln 4)\mathbf{i}_1+4\pi\eta_1\int_\Gamma \Psi\,dl. \tag{5.11}$$

Hence the vector density of the corresponding double-layer potential for the first-order approximation is given by the unique continuous solution of (3.10) with the constant vector $\mathbf{F}$ given by the second equation of (5.11). Accordingly, the hydrodynamic force $\mathbf{F}_T$ acting on the cylinder is provided by the single-layer potentials $\mathbf{V}(\cdot,-(4\pi)^{-1}|\Gamma|^{-1}\mathbf{i}_1)$ and $\mathbf{V}(\cdot,-(4\pi)^{-1}|\Gamma|^{-1}\mathbf{F})$, where $\mathbf{F}$ has been mentioned previously. The components $F_{T,j}$ of this force, up to $\mathcal{O}((\ln\mathrm{Re})^{-2})$, are given by

$$F_{T,j} = \frac{4\pi}{\ln\mathrm{Re}}\delta_{1j}+\frac{4\pi}{(\ln\mathrm{Re})^2}\left\{(\gamma_0-\ln 4)\delta_{1j}+4\pi\eta_1\int_\Gamma \psi_j\,dl\right\}+\mathcal{O}((\ln\mathrm{Re})^{-3}). \tag{5.12}$$

From (5.12) it follows that the contribution of $\mathbb{O}((\ln \mathrm{Re})^{-1})$ to the hydrodynamic force does not depend on the cylinder geometry. However, the contribution of $\mathbb{O}((\ln \mathrm{Re})^{-2})$ to the drag force depends on the cylinder geometry, since it contains the term $\int_\Gamma \psi_1\, dl$. Also, the lift force of $\mathbb{O}((\ln \mathrm{Re})^{-2})$ depends on the cylinder geometry by the fact that it is expressed in terms of $\int_\Gamma \psi_2\, dl$, where $\Psi = (\psi_1, \psi_2)$ is the unique continuous solution of (3.13). Consequently, the dependency of the hydrodynamic force on the cylinder geometry appears in the second-order approximation and is provided by the matching asymptotic procedure.

In the case $\eta_1 = 1/(4\pi)$, the asymptotic formula (5.12) reduces to that obtained by Power [7]; see also [9, page 240].

6. Inertial effects. The inner and outer asymptotic expansions (2.4) and (2.9), for which it can be proved that $f_n(\mathrm{Re}) = (\ln \mathrm{Re})^{-(n+1)}$ and $\hat{f}_n(\mathrm{Re}) = (\ln \mathrm{Re})^{-n}$, do not include the inertial effects of order $\mathbb{O}(\mathrm{Re})$ that are asymptotically smaller than the terms of order $\mathbb{O}((\ln \mathrm{Re})^{-n})$ for each n as $\mathrm{Re} \to 0$. Proudman and Pearson [11] proposed a procedure for removing this inconvenience, which consists in the addition of the following series to the expansions (2.4) and (2.9) (see also [7] and [9, page 242]):

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathrm{Re}^m (\ln \mathrm{Re})^{-n} \mathbf{v}^{m,n}, \qquad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathrm{Re}^m (\ln \mathrm{Re})^{-n} p^{m,n}, \tag{6.1}$$

and, respectively,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathrm{Re}^m (\ln \mathrm{Re})^{-n} \hat{\mathbf{v}}^{m,n}, \qquad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathrm{Re}^m (\ln \mathrm{Re})^{-n} \hat{p}^{m,n}. \tag{6.2}$$

We have to require that these expansions satisfy the same boundary and matching conditions as in the previous analysis. The terms $\mathbf{v}^{m,n}$ and $p^{m,n}$ can be determined by using similar arguments as before. These terms satisfy either the homogeneous Stokes equation or nonhomogeneous versions of this equation. For example, the terms $\mathbf{v}^{0,0}$ and $p^{0,0}$ lead to a nonhomogeneous equation for $\mathbf{v}^{1,2}$ and $p^{1,2}$, whose particular solution $(\mathbf{v}_p^{1,2}, p_p^{1,2})$ can be expressed in terms of the two-dimensional Stokeslet and its associated pressure vector as follows:

$$\begin{aligned} \mathbf{v}_p^{1,2}(\mathbf{x}) &= \int_D \mathcal{G}(\mathbf{x}-\mathbf{y}) \cdot \left(\mathbf{v}^{0,0}(\mathbf{y}) \cdot \nabla \mathbf{v}^{0,0}(\mathbf{y})\right) d\mathbf{y}, \\ p_p^{1,2}(\mathbf{x}) &= \int_D \Pi^S(\mathbf{x}-\mathbf{y}) \cdot \left(\mathbf{v}^{0,0}(\mathbf{y}) \cdot \nabla \mathbf{v}^{0,0}(\mathbf{y})\right) d\mathbf{y}, \end{aligned} \tag{6.3}$$

where D is the flow domain. Note that the particular solution $(\mathbf{v}_p^{1,2}, p_p^{1,2})$ has to be added to the general solution given by (3.1) and (3.5) in order to complete the inner solution $(\mathbf{v}^{1,2}, p^{1,2})$.

If we use arguments similar to those for the terms $(\mathbf{v}^{m,n}, p^{m,n})$, all the terms $(\hat{\mathbf{v}}^{m,n}, \hat{p}^{m,n})$ can be computed too. They satisfy either the homogeneous Oseen equation or nonhomogeneous versions of this equation. These outer higher-order approximations were obtained by Proudman and Pearson [11]. Both inner and outer solutions can be completed by applying the above matching asymptotic method.

7. Conclusions. In this paper, we have applied the method of matched asymptotic expansions to the low Reynolds number flow of an incompressible Newtonian fluid past a cylinder of arbitrary cross section. The hydrodynamic force on the cylinder is expressed in terms of the unique continuous solution of the Fredholm integral equation of the second kind (3.13). We note that this equation is uniquely solvable when the cylinder cross-sectional boundary is an arbitrary simple closed Lyapunov curve and the parameters η_0 and η_1 satisfy the conditions $\eta_0 > 0$ and $\eta_1 \neq 0$. For $\eta_1 = 1/(4\pi)$, the hydrodynamic force $\mathbf{F}_T$ given by formula (5.12) is identical to the corresponding result due to Power [7]. Note that the matched asymptotic analysis developed by Power is based on the CDLBIEM, in order to solve the resulting inner problems, and on the singularity method applied to solve the resulting outer problems (see also [9, Section 6.3.4]).

The matched asymptotic analysis developed in this paper differs from that of Power [7], since we have used the compound double-layer method due to Hsiao and Kress [1] instead of the CDLBIEM, which is the basis of Power's approach.

Finally, we note that another matched asymptotic analysis of low Reynolds number flow past a cylinder of arbitrary cross section was obtained by Lee and Leal [6]. This analysis uses the boundary integral method due to Youngren and Acrivos [14], which reduces the resulting inner problems to integral equations of the first kind.

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