IJMMS 2004:59, 3151–3170
PII. S016117120431210X
http://ijmms.hindawi.com

HYPERSYMMETRIC FUNCTIONS AND POCHHAMMERS OF 2×2 NONAUTONOMOUS MATRICES

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Received 9 December 2003

We introduce the *hypersymmetric functions* of 2×2 nonautonomous matrices and show that they are related, by simple expressions, to the Pochhammers (factorial polynomials) of these matrices. The *hypersymmetric functions* are generalizations of the *associated elementary symmetric functions*, and for a specific class of 2×2 matrices, having a high degree of symmetry, they reduce to these latter functions. This class of matrices includes rotations, Lorentz boosts, and discrete time generators for the harmonic oscillators. The *hypersymmetric functions* are defined over four sets of independent indeterminates using a triplet of interrelated binary partitions. We work out the algebra of this triplet of partitions and then make use of the results in order to simplify the expressions for the *hypersymmetric functions* for a special class of matrices. In addition to their obvious applications in matrix theory, in coupled difference equations, and in the theory of symmetric functions, the results obtained here also have useful applications in problems involving successive rotations, successive Lorentz transformations, discrete harmonic oscillators, and linear two-state systems.

2000 Mathematics Subject Classification: 15A24, 05E05, 39A10.

1. Introduction. In a previous article [4], we introduced the *associated elementary symmetric functions* $U_m^n(x_j,y_j;\ j\in\mathbb{N}_n)$, and showed that, combined with the ordinary *elementary symmetric functions* $T_m^n(x_j;\ j\in\mathbb{N}_n)$, they provide the native mathematical language of trigonometry, and more generally polygonometry. In this paper, we introduce the *hypersymmetric functions* $W_m^n(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j\in\mathbb{N}_n)\equiv W_m^n(\omega(j);\ j\in\mathbb{N}_n)$, where $\omega(j)$ is a 2×2 nonautonomous matrix whose elements are $\{\alpha_j,\beta_j,\gamma_j,\eta_j\}$, and show that they are related by simple expressions to the Pochhammers (factorial polynomials) of the matrix $\omega(j)$. For a specific class of matrices having a high degree of symmetry, the *hypersymmetric functions* reduce to the *associated elementary symmetric function*. This class of matrices includes rotations, Lorentz boosts, and discrete time generators for the harmonic oscillator.

The *hypersymmetric functions* are defined over four sets of independent indeterminates using the binary partition ℓ, of m into n parts, where $\ell=\{\ell_1,\ell_2,\dots,\ell_n\}$ and $\ell_i\in\{0,1\}$, as well as its associated dual partitions h and $\bar{h}$. The algebra of the resulting triplet of interrelated binary partitions ℓ, h, and $\bar{h}$ is worked out in Appendix B, and the results are used to simplify the expressions for the *hypersymmetric functions* for a special class of matrices.

The results obtained here have bearing on matrix theory, coupled difference equations [3, 5, 12], and symmetric functions [16, 18]. They also have useful applications in problems involving successive rotations, successive Lorentz transformations [2], discrete harmonic oscillators [6], and linear two-state systems. They are based on the analytic solution for two coupled first-order difference equations with variable coefficients [3, 5], which, in turn, is based on the discrete path approach to linear recursion relations [1] (see also [7, 9, 10]).

In Section 2, we introduce the triplet of binary *associated partitions* underlying the definition of the *hypersymmetric functions*. In Section 3, we introduce the *hypersymmetric functions*. In Section 4, we derive an expression for the Pochhammers of 2×2 nonautonomous matrices in terms of the *hypersymmetric functions*. In Section 5, we apply the above expression to a class of matrices with a high degree of symmetry. Appendix A deals with integer decomposition. Appendix B gives proofs of the main results on the interrelated triplet of partitions. Appendix C sets out the notation, and lists the needed identities, for the symmetric functions. Appendix D sets out the notation for Pochhammers and for the Pauli matrices.

2. Associated ordered partitions

2.1. The general case. Let $\mathbb{N}_q = \{1,2,\dots,q\}$ denote, as usual, the set of the first q positive integers, with $\mathbb{N}_0 = \varnothing$, and let $\lambda(n,m,q)$ denote an ordered partition of m into n parts belonging to the set $\{0,\mathbb{N}_q\}$. That is

$$\lambda(n,m,q) = (\lambda_1,\lambda_2,\dots,\lambda_n), \quad m = \sum_{i=1}^{n}\lambda_i, \quad \lambda_i \in \{0,1,2,\dots,q\}. \tag{2.1}$$

Corresponding to every ordered partition $\lambda(n,m,q)$, we introduce $q+1$ associated partitions $h(\lambda,n,q,s)$ defined by

$$h(\lambda,n,q,s) = (h_1,h_2,\dots,h_n), \quad h_j = \left(s+\sum_{i=1}^{j}\lambda_i\right)\operatorname{mod}(q+1), \quad s = 0,1,2,\dots,q. \tag{2.2}$$

The ordered partition $\lambda(n,m,q)$, combined with its $q+1$ associated ordered partitions $h(\lambda,n,q,s)$, constitute a $q+2$ multiplet of interrelated ordered partitions.

2.2. The case $q = 1$: the triplet of partitions. In the special case $q = 1$, we use the simplified notation $\ell \equiv \lambda(n,m,1)$, $h \equiv h(\ell,n,1,0)$, and $\bar{h} \equiv h(\ell,n,1,1)$, so that

$$\ell = (\ell_1,\ell_2,\dots,\ell_n), \quad \sum_{i=1}^{n}\ell_i = m, \quad \ell_i \in \{0,1\}, \tag{2.3}$$

$$h = (h_1,h_2,\dots,h_n), \quad h_j = \left(\sum_{i=1}^{j}\ell_i\right)\operatorname{mod}2, \quad h_i \in \{0,1\}, \tag{2.4}$$

$$\bar{h} = (\bar{h}_1,\bar{h}_2,\dots,\bar{h}_n), \quad \bar{h}_j = \left(1+\sum_{i=1}^{j}\ell_i\right)\operatorname{mod}2, \quad \bar{h}_i \in \{0,1\}. \tag{2.5}$$

It is easy to see that the two associated partitions h and $\bar{h}$ are dual to each other, and that their components obey the relations

$$h_i\bar{h}_i = 0, \qquad h_i + \bar{h}_i = 1. \tag{2.6}$$

Hence, they are also orthogonal,

$$h \bullet \bar{h} = \sum_{i=1}^{n} h_i\bar{h}_i = 0 \tag{2.7}$$

and, making use of identity (B.2) of Appendix B, the sum of the squares of their norms is given by

$$h \bullet h + \bar{h} \bullet \bar{h} = \sum_{i=1}^{n} (h_i + \bar{h}_i) = n. \tag{2.8}$$

Furthermore, we have the following two important identities:

$$\begin{aligned} \ell \bullet h &= \sum_{i=1}^{n} \ell_i h_i = \left\lfloor \frac{m+1}{2} \right\rfloor, \\ \ell \bullet \bar{h} &= \sum_{i=1}^{n} \ell_i \bar{h}_i = \left\lfloor \frac{m}{2} \right\rfloor. \end{aligned} \tag{2.9}$$

The proofs of identities (2.9) are given in Appendix B, and they make use of the theorem on integer decomposition derived in Appendix A.

3. Hypersymmetric functions. The *hypersymmetric functions* $W_m^n(\alpha_j,\beta_j,\gamma_j,\delta_j;\ j \in \mathbb{N}_n)$ and $\bar{W}_m^n(\alpha_j,\beta_j,\gamma_j,\delta_j;\ j \in \mathbb{N}_n)$ are defined over four sets of independent indeterminates $\mathcal{A}_n = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$, $\mathcal{B}_n = \{\beta_1,\beta_2,\ldots,\beta_n\}$, $\mathcal{C}_n = \{\gamma_1,\gamma_2,\ldots,\gamma_n\}$, and $\mathcal{D}_n = \{\eta_1,\eta_2,\ldots,\eta_n\}$, by using the triplet of ordered partitions ℓ, h, and $\bar{h}$, according to

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n) &= \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m \\ \ell_\nu\in\{0,1\}}} \prod_{i=1}^{n} \alpha_i^{(1-\ell_i)(1-h_i)} \beta_i^{\ell_i(1-h_i)} \gamma_i^{\ell_i h_i} \eta_i^{(1-\ell_i)h_i}, \\ \bar{W}_m^n(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n) &= \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m \\ \ell_\nu\in\{0,1\}}} \prod_{i=1}^{n} \alpha_i^{(1-\ell_i)(1-\bar{h}_i)} \beta_i^{\ell_i(1-\bar{h}_i)} \gamma_i^{\ell_i \bar{h}_i} \eta_i^{(1-\ell_i)\bar{h}_i}, \end{aligned} \tag{3.1}$$

where $h = \{h_1,h_2,\ldots,h_n\}$ and $\bar{h} = \{\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_n\}$ are defined by (2.4) and (2.5), respectively. We will also make use of the more compact notation

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n) &\equiv W_m^n(\omega(j);\ j \in \mathbb{N}_n), \\ \bar{W}_m^n(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n) &\equiv \bar{W}_m^n(\omega(j);\ j \in \mathbb{N}_n), \end{aligned} \tag{3.2}$$

where the matrix $\omega(j)$ is given by

$$\omega(j) = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \eta_j \end{pmatrix}. \tag{3.3}$$

4. Matrix Pochhammers

4.1. Pochhammers in terms of the evolution matrix. In [5], the Pochhammers (factorial polynomials) of a 2×2 nonautonomous matrix $M(n)$ were obtained in terms of the *evolution matrix* $S(n,n_0)$ according to

$$[M(n)]^{(n-n_0)} \equiv M(n)M(n-1)\cdots M(n_0+1) = S(n,n_0), \tag{4.1}$$

where the elements of the *evolution matrix* $S(n,n_0)$ are given in terms of the *structure functions* $B_j^{n_0}(n,m)$ by [5]

$$S_{ij}(n,n_0) = \delta_{ij}\sum_{\substack{m=0\\ m\text{ even}}}^{n-n_0} B_j^{n_0}(n-n_0,m) + (1-\delta_{ij})\sum_{\substack{m=0\\ m\text{ odd}}}^{n-n_0} B_j^{n_0}(n-n_0,m) \tag{4.2}$$

and the *structure functions* $B_j^{n_0}(n,m)$ are defined by

$$B_j^{n_0}(n,m) = \sum_{\substack{\ell_1+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}}\prod_{p=1}^{n} F_{\ell_p}\left(n_0+p,\left(j-1+\sum_{s=1}^{p}\ell_s\right)\bmod 2\right) \tag{4.3}$$

with the *transmission coefficients* $F_\ell(p,h)$ given by

$$F_\ell(p,h) = [M_{h+1\ h+1}(p)]^{1-\ell}[M_{h+1\ 2-h}(p)]^{\ell}, \quad \ell=0,1,\ h=0,1. \tag{4.4}$$

4.2. Pochhammers in terms of the hypersymmetric functions. It is not difficult to show that the *transmission coefficients* (4.4) can alternatively be rewritten as

$$\begin{gathered} F_\ell(p,h) = [M_{11}(p)]^{(1-\ell)(1-h)}[M_{12}(p)]^{\ell(1-h)}[M_{21}(p)]^{\ell h}[M_{22}(p)]^{(1-\ell)h}, \\ \ell=0,1, \qquad h=0,1. \end{gathered} \tag{4.5}$$

Combining identities (4.3) and (4.5) for the *structure functions* $B_j^{n_0}(n,m)$, comparing with definitions (3.1) for the *hypersymmetric functions* W_m^n and $\bar{W}_m^n$, and making use of the notation introduced in (3.2), we obtain

$$\begin{aligned} B_1^{n_0}(n,m) &= W_m^n(M(n_0+p);\ p\in\mathbb{N}_n), \\ B_2^{n_0}(n,m) &= \bar{W}_m^n(M(n_0+p);\ p\in\mathbb{N}_n). \end{aligned} \tag{4.6}$$

Substituting from identities (4.6) into identity (4.2), we obtain the elements of the *evolution matrix* in terms of the *hypersymmetric functions* as

$$S_{11}(n,n_0) = \sum_{\substack{m=0\\ m \text{ even}}}^{n-n_0} W_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right), \tag{4.7}$$

$$S_{22}(n,n_0) = \sum_{\substack{m=0\\ m \text{ even}}}^{n-n_0} \bar{W}_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right), \tag{4.8}$$

$$S_{12}(n,n_0) = \sum_{\substack{m=0\\ m \text{ odd}}}^{n-n_0} \bar{W}_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right), \tag{4.9}$$

$$S_{21}(n,n_0) = \sum_{\substack{m=0\\ m \text{ odd}}}^{n-n_0} W_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right), \tag{4.10}$$

and due to identity (4.1), we finally obtain

$$\begin{aligned}
&[M(n)]^{(n-n_0)}\\
&\quad\equiv M(n)M(n-1)\cdots M(n_0+1)\\
&\quad= \sum_{\substack{m=0\\ m \text{ even}}}^{n-n_0} \begin{pmatrix} W_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right) & 0\\ 0 & \bar{W}_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right)\end{pmatrix}\\
&\qquad+ \sum_{\substack{m=0\\ m \text{ odd}}}^{n-n_0} \begin{pmatrix} 0 & \bar{W}_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right)\\ W_m^{n-n_0}\left(M(n_0+p);\ p \in \mathbb{N}_{n-n_0}\right) & 0\end{pmatrix}.
\end{aligned} \tag{4.11}$$

4.3. Closure relations. The closure relations for the *hypersymmetric functions* can be obtained via identities (4.7) through (4.10), by setting $n_0 = 0$ leading to

$$\begin{aligned}
\sum_{i=1}^{2}\left[\begin{pmatrix}\alpha_n & \beta_n\\ \gamma_n & \eta_n\end{pmatrix}\right]_{i1}^{(n)} &= \sum_{m=0}^{n} W_m^n\left(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n\right),\\
\sum_{i=1}^{2}\left[\begin{pmatrix}\alpha_n & \beta_n\\ \gamma_n & \eta_n\end{pmatrix}\right]_{i2}^{(n)} &= \sum_{m=0}^{n} \bar{W}_m^n\left(\alpha_j,\beta_j,\gamma_j,\eta_j;\ j \in \mathbb{N}_n\right),
\end{aligned} \tag{4.12}$$

or alternatively

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\left[\begin{pmatrix}\alpha_n & \beta_n\\ \gamma_n & \eta_n\end{pmatrix}\right]_{ij}^{(n)} = \sum_{m=0}^{n}\left\{\begin{array}{l} W_m^n\left(\alpha_k,\beta_k,\gamma_k,\eta_k;\ k \in \mathbb{N}_n\right)\\ +\bar{W}_m^n\left(\alpha_k,\beta_k,\gamma_k,\eta_k;\ k \in \mathbb{N}_n\right)\end{array}\right\}. \tag{4.13}$$

5. Matrices with a high degree of symmetry. In the special case $\eta_j = a\alpha_j$ and $\gamma_j = b\beta_j$, the matrix $\omega(j)$ reduces to

$$\tilde{\omega}(j;a,b) = \begin{pmatrix} \alpha_j & \beta_j \\ b\beta_j & a\alpha_j \end{pmatrix} \tag{5.1}$$

and the corresponding *hypersymmetric functions* reduce to

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,b\beta_j,a\alpha_j;\ j\in\mathbb{N}_n) &= \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} a^{\sum_{i=1}^n(1-\ell_i)h_i} b^{\sum_{i=1}^n \ell_i h_i} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}, \\ \bar{W}_m^n(\alpha_j,\beta_j,b\beta_j,a\alpha_j;\ j\in\mathbb{N}_n) &= \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} a^{\sum_{i=1}^n(1-\ell_i)\bar{h}_i} b^{\sum_{i=1}^n \ell_i \bar{h}_i} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}. \end{aligned} \tag{5.2}$$

Making use of identities (2.8) and (2.9), the above identities (5.2) can be rewritten as

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,b\beta_j,a\alpha_j;\ j\in\mathbb{N}_n) &= \left(\frac{b}{a}\right)^{\lfloor (m+1)/2\rfloor} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} a^{h\bullet h} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}, \\ \bar{W}_m^n(\alpha_j,\beta_j,b\beta_j,a\alpha_j;\ j\in\mathbb{N}_n) &= \left(\frac{b}{a}\right)^{\lfloor m/2\rfloor} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} a^{\bar{h}\bullet\bar{h}} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}. \end{aligned} \tag{5.3}$$

5.1. Pochhammers in terms of the associated elementary symmetric functions

5.1.1. The case $a = 1$. In the case $a = 1$, the matrix $\tilde{\omega}(j;a,b)$ further reduces to

$$\tilde{\omega}(j;1,b) = \begin{pmatrix} \alpha_j & \beta_j \\ b\beta_j & \alpha_j \end{pmatrix} \tag{5.4}$$

and the *hypersymmetric functions* W_m^n and $\bar{W}_m^n$, as given by identities (5.3), reduce to

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,b\beta_j,\alpha_j;\ j\in\mathbb{N}_n) &= b^{\lfloor (m+1)/2\rfloor} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}, \\ \bar{W}_m^n(\alpha_j,\beta_j,b\beta_j,\alpha_j;\ j\in\mathbb{N}_n) &= b^{\lfloor m/2\rfloor} \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m\\ \ell_\nu\in\{0,1\}}} \prod_{i=1}^n \alpha_i^{(1-\ell_i)}\beta_i^{\ell_i}. \end{aligned} \tag{5.5}$$

Due to definition (C.4) for the *associated elementary symmetric functions* U_m^n (see Appendix C), the above identities (5.5) can be rewritten as

$$\begin{aligned} W_m^n(\alpha_j,\beta_j,b\beta_j,\alpha_j;\ j\in\mathbb{N}_n) &= b^{\lfloor (m+1)/2\rfloor} U_m^n(\beta_j,\alpha_j;\ j\in\mathbb{N}_n), \\ \bar{W}_m^n(\alpha_j,\beta_j,b\beta_j,\alpha_j;\ j\in\mathbb{N}_n) &= b^{\lfloor m/2\rfloor} U_m^n(\beta_j,\alpha_j;\ j\in\mathbb{N}_n). \end{aligned} \tag{5.6}$$

Consequently, identity (4.11) reduces to

$$\left[\begin{pmatrix}\alpha_n & \beta_n\\ b\beta_n & \alpha_n\end{pmatrix}\right]^{(n-n_0)} = \sum_{\substack{m=0\\ m \text{ even}}}^{n-n_0}\begin{pmatrix}b^{\lfloor (m+1)/2\rfloor} & 0\\ 0 & b^{\lfloor m/2\rfloor}\end{pmatrix} U_m^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right) + \sum_{\substack{m=0\\ m \text{ odd}}}^{n-n_0}\begin{pmatrix}0 & b^{\lfloor m/2\rfloor}\\ b^{\lfloor (m+1)/2\rfloor} & 0\end{pmatrix} U_m^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right) \tag{5.7}$$

and can be rewritten as

$$\left[\begin{pmatrix}\alpha_n & \beta_n\\ b\beta_n & \alpha_n\end{pmatrix}\right]^{(n-n_0)} = I\sum_{k=0}^{\lfloor (n-n_0)/2\rfloor} b^k U_{2k}^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right) + \left(\sigma_+ + b\sigma_-\right)\sum_{k=0}^{\lfloor (n-n_0-1)/2\rfloor} b^k U_{2k+1}^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right), \tag{5.8}$$

where I is the 2×2 identity matrix, σ_+ and σ_- are the standard linear combinations of Pauli matrices as given by identities (D.2) and (D.3) of Appendix D.

5.1.2. The case $a = b = 1$. In the case $a = b = 1$, $\tilde{\omega}(j;a,b)$ takes the highly symmetric form:

$$\tilde{\omega}(j;1,1) = \begin{pmatrix}\alpha_j & \beta_j\\ \beta_j & \alpha_j\end{pmatrix}, \tag{5.9}$$

and the *hypersymmetric functions* W_m^n and $\bar{W}_m^n$ become degenerate and both reduce to the corresponding *associated elementary symmetric functions* U_m^n,

$$W_m^n\left(\alpha_j,\beta_j,\beta_j,\alpha_j;\ j\in\mathbb{N}_n\right) = \bar{W}_m^n\left(\alpha_j,\beta_j,\beta_j,\alpha_j;\ j\in\mathbb{N}_n\right) = U_m^n\left(\beta_j,\alpha_j;\ j\in\mathbb{N}_n\right), \tag{5.10}$$

while identity (5.8) reduces to

$$\left[\begin{pmatrix}\alpha_n & \beta_n\\ \beta_n & \alpha_n\end{pmatrix}\right]^{(n-n_0)} = I\sum_{k=0}^{\lfloor (n-n_0)/2\rfloor} U_{2k}^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right) + \sigma_1\sum_{k=0}^{\lfloor (n-n_0-1)/2\rfloor} U_{2k+1}^{n-n_0}\left(\beta_{n_0+j},\alpha_{n_0+j};\ j\in\mathbb{N}_{n-n_0}\right), \tag{5.11}$$

where I is the 2×2 identity matrix, and σ_1 is the first Pauli matrix as defined by identity (D.2) of Appendix D, and where we have also made use of identity (D.4) of this appendix.

The class of matrices $\tilde{\omega}(j;1,1)$ includes the Lorentz boosts. By imposing the group property on successive collinear Lorentz boosts, and making use of identity (5.11), we recover the relativistic law of addition of multiple collinear velocities derived in [2].

5.1.3. The case $a = -b = 1$. In the case $a = -b = 1$, the matrix $\tilde{\omega}(j;a,b)$ takes the asymmetric form

$$\tilde{\omega}(j;1,-1) = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}, \tag{5.12}$$

and the *hypersymmetric functions* W_m^n and $\bar{W}_m^n$ reduce, respectively, to

$$W_m^n(\alpha_j,\beta_j,-\beta_j,\alpha_j;\ j \in \mathbb{N}_n) = (-1)^{\lfloor (m+1)/2 \rfloor} U_m^n(\beta_j,\alpha_j;\ j \in \mathbb{N}_n), \tag{5.13}$$

$$\bar{W}_m^n(\alpha_j,\beta_j,-\beta_i,\alpha_j;\ j \in \mathbb{N}_n) = (-1)^{\lfloor m/2 \rfloor} U_m^n(\beta_j,\alpha_j;\ j \in \mathbb{N}_n), \tag{5.14}$$

while identity (5.8) reduces to

$$\begin{aligned} \left[\begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n & \alpha_n \end{pmatrix}\right]^{(n-n_0)} &= I \sum_{k=0}^{\lfloor (n-n_0)/2 \rfloor} (-1)^k U_{2k}^{n-n_0}(\beta_{n_0+j},\alpha_{n_0+j};\ j \in \mathbb{N}_{n-n_0}) \\ &\quad + i\sigma_2 \sum_{k=0}^{\lfloor (n-n_0-1)/2 \rfloor} (-1)^k U_{2k+1}^{n-n_0}(\beta_{n_0+j},\alpha_{n_0+j};\ j \in \mathbb{N}_{n-n_0}), \end{aligned} \tag{5.15}$$

where I is the 2×2 identity matrix, and σ_2 is the second Pauli matrix as defined by identity (D.2) of Appendix D, and where we have made use of identity (D.4) of this appendix.

5.1.4. Successive rotations. The class of matrices $\tilde{\omega}(j;1,-1)$ includes the rotation matrix $R_{\hat{z}}(\theta_k)$ by an angle θ_k about the $\hat{z}$ axis. In the case of successive rotations about the same axis, we have

$$[R_{\hat{z}}(\theta_n)]^{(n)} \equiv R_{\hat{z}}(\theta_n) R_{\hat{z}}(\theta_{n-1}) \cdots R_{\hat{z}}(\theta_1) = R_{\hat{z}}\left(\sum_{k=1}^{n} \theta_k\right). \tag{5.16}$$

Setting $n_0 = 0$ in identity (5.15), and combining it with identity (5.16), we obtain

$$\begin{aligned} &\begin{pmatrix} \cos(\theta_1+\theta_2+\cdots+\theta_n) & \sin(\theta_1+\theta_2+\cdots+\theta_n) \\ -\sin(\theta_1+\theta_2+\cdots+\theta_n) & \cos(\theta_1+\theta_2+\cdots+\theta_n) \end{pmatrix} \\ &\quad = \left[\begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix}\right]^{(n)} \\ &\quad = I \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k U_{2k}^n(\sin\theta_j,\cos\theta_j;\ j \in \mathbb{N}_n) \\ &\qquad + i\sigma_2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k U_{2k+1}^n(\sin\theta_j,\cos\theta_j;\ j \in \mathbb{N}_n), \end{aligned} \tag{5.17}$$

leading to the following identities for the trigonometric functions of multiple distinct angles:

$$\begin{aligned}\cos(\theta_1+\theta_2+\cdots+\theta_n) &= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k U_{2k}^n(\sin\theta_j,\cos\theta_j;\ j\in\mathbb{N}_n),\\ \sin(\theta_1+\theta_2+\cdots+\theta_n) &= \sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k U_{2k+1}^n(\sin\theta_j,\cos\theta_j;\ j\in\mathbb{N}_n),\end{aligned}\tag{5.18}$$

which are identities (D.10) and (C.10), respectively, of [4].

5.2. Pochhammers in terms of the elementary symmetric functions

5.2.1. The case $a=1$ and $\alpha_i=1$. In the case $a=1$ and $\alpha_i=1$, the matrix $\tilde{\omega}(j;a,b)$ further reduces to

$$\tilde{\omega}(j;1,b)\big|_{\alpha_j=1}=\begin{pmatrix}1 & \beta_j\\ b\beta_j & 1\end{pmatrix},\tag{5.19}$$

while W_m^n and $\bar{W}_m^n$, as given, respectively, by identities (5.6), reduce to

$$\begin{aligned}W_m^n(1,\beta_j,b\beta_j,1;\ j\in\mathbb{N}_n) &= b^{\lfloor (m+1)/2\rfloor}T_m^n(\beta_j;\ j\in\mathbb{N}_n),\\ \bar{W}_m^n(1,\beta_j,b\beta_j,1;\ j\in\mathbb{N}_n) &= b^{\lfloor m/2\rfloor}T_m^n(\beta_j;\ j\in\mathbb{N}_n),\end{aligned}\tag{5.20}$$

where we have made use of definition (C.3), and identity (C.5), of Appendix C. Furthermore, and again due to identity (C.5) of Appendix C, identity (5.8) reduces to

$$\begin{aligned}\left[\begin{pmatrix}1 & \beta_n\\ b\beta_n & 1\end{pmatrix}\right]^{(n-n_0)} &= I\sum_{k=0}^{\lfloor (n-n_0)/2\rfloor} b^k T_{2k}^{n-n_0}(\beta_{n_0+j};\ j\in\mathbb{N}_{n-n_0})\\ &\quad+(\sigma_+ + b\sigma_-)\sum_{k=0}^{\lfloor (n-n_0-1)/2\rfloor} b^k T_{2k+1}^{n-n_0}(\beta_{n_0+j};\ j\in\mathbb{N}_{n-n_0}).\end{aligned}\tag{5.21}$$

5.2.2. The simple harmonic oscillator. The formulation of Newton's second law of motion for the discretized one-dimensional harmonic oscillator, using the Euler method, can be cast in the form of two coupled finite difference equations for the position x_n and the velocity v_n (see [6, identity 27])

$$\begin{pmatrix}x_n\\ v_n\end{pmatrix}\begin{pmatrix}1 & \Delta t_n\\ -\omega^2\Delta t_n & 1\end{pmatrix}\begin{pmatrix}x_{n-1}\\ v_{n-1},\end{pmatrix},\tag{5.22}$$

where $\omega=\sqrt{k/m}$ is the classical angular frequency, k being the spring constant, and m the oscillating mass, and where we have allowed for the possibility of using variable time intervals. The matrix

$$H(\Delta t_n)=\begin{pmatrix}1 & \Delta t_n\\ -\omega^2\Delta t_n & 1\end{pmatrix}\tag{5.23}$$

in the above identity (5.22) advances the system forward one step in time (by an amount Δt_n). $H(\Delta t_n)$ is of the form of matrix $\tilde{\omega}(j;1,b)|_{\alpha_j=1}$, as given by identity (5.19), with

$\beta_j = \Delta t_j$ and $b = -\omega^2$. Hence, making use of identity (5.21), we obtain the solution for the classical simple harmonic oscillator in the form

$$\begin{aligned}\begin{pmatrix} x_n \\ v_n \end{pmatrix} &= \left[\begin{pmatrix} 1 & \Delta t_n \\ -\omega^2 \Delta t_n & 1 \end{pmatrix}\right]^{(n)} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \sum_{k=0}^{\lfloor n/2 \rfloor} (i\omega)^{2k} T_{2k}^{n}(\Delta t_j;\ j \in \mathbb{N}_n) \\ &\quad + \begin{pmatrix} v_0 \\ -\omega^2 x_0 \end{pmatrix} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (i\omega)^{2k} T_{2k+1}^{n}(\Delta t_j;\ j \in \mathbb{N}_n),\end{aligned} \tag{5.24}$$

where we have made use of the identity

$$(\sigma_+ - \omega^2 \sigma_-) \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} v_0 \\ -\omega^2 x_0 \end{pmatrix}. \tag{5.25}$$

In the case of constant time intervals, identity (C.11) of Appendix C leads to

$$T_m^n(\Delta t;\ j \in \mathbb{N}_n) = \binom{n}{m} (\Delta t)^m, \tag{5.26}$$

consequently identity (5.24) reduces to

$$\begin{aligned}\begin{pmatrix} x_n \\ v_n \end{pmatrix} &= \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} (\Delta t \omega)^{2k} \\ &\quad + \frac{1}{\omega} \begin{pmatrix} v_0 \\ -\omega^2 x_0 \end{pmatrix} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} (\Delta t \omega)^{2k+1}\end{aligned} \tag{5.27}$$

which is equivalent to [6, identities (59) and (60)].

6. Conclusion. The main result of the paper is identity (4.11) expressing Pochhammers in terms of hypersymmetric functions. There are in addition a number of secondary results. Identities (2.9) concern the algebra of the triplet of partitions; identities (5.10) and (5.13) give the reduction of the hypersymmetric functions to the associated elementary symmetric functions; identities (5.20) give the reduction of the hypersymmetric functions to the ordinary elementary symmetric functions. The applications are to successive rotations (5.17), and to the time evolution of the discrete harmonic oscillator (5.24). The main definitions are those of the hypersymmetric functions (3.1), and the associated partitions (2.2) on which these functions are based.

The approach used here in the case of 2×2 matrices, whereby hypersymmetric functions are first defined using associated partitions, and then related to Pochhammers, may help indicate the way to obtaining similar analytic evaluations of Pochhammers of $n \times n$ nonautonomous matrices.

Appendices

A. Integer decomposition

Theorem A.1. *Given any two positive integers m and q,*

$$m = \sum_{j=0}^{q-1} \left\lfloor \frac{m+j}{q} \right\rfloor. \tag{A.1}$$

Proof. m can be written as

$$m = kq + i, \quad i \in \{0,1,2,\dots,q-1\},\ k \in \mathcal{N} \equiv \{0,1,2,\dots\}. \tag{A.2}$$

Substituting the above expression for m into the sum on the right-hand side of identity (A.1), we obtain

$$\sum_{j=0}^{q-1} \left\lfloor \frac{m+j}{q} \right\rfloor = \sum_{j=0}^{q-1} \left\lfloor \frac{kq+i+j}{q} \right\rfloor = \sum_{j=0}^{q-1} \left(k + \left\lfloor \frac{i+j}{q} \right\rfloor \right) = kq + \sum_{j=0}^{q-1} \left\lfloor \frac{i+j}{q} \right\rfloor. \tag{A.3}$$

We now make the change of variable $j \to j' = j - (q-i)$ so that

$$\sum_{j=0}^{q-1} \left\lfloor \frac{i+j}{q} \right\rfloor = \sum_{j'=-(q-i)}^{i-1} \left\lfloor \frac{q+j'}{q} \right\rfloor = \sum_{j'=-(q-i)}^{-1} \left\lfloor \frac{q+j'}{q} \right\rfloor + \sum_{j'=0}^{i-1} \left\lfloor \frac{q+j'}{q} \right\rfloor. \tag{A.4}$$

But

$$\sum_{j'=-(q-i)}^{-1} \left\lfloor \frac{q+j'}{q} \right\rfloor = 0, \qquad \sum_{j'=0}^{i-1} \left\lfloor \frac{q+j'}{q} \right\rfloor = i. \tag{A.5}$$

Hence,

$$\sum_{j=0}^{q-1} \left\lfloor \frac{m+j}{q} \right\rfloor = kq + i = m, \tag{A.6}$$

which completes the proof of Theorem A.1. For the special case $q = 2$, identity (A.1) gives

$$m = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m+1}{2} \right\rfloor. \tag{A.7}$$

□

B. The algebra of partitions. In this appendix, we study the algebra of a partition $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ and its associated dual partitions $h = (h_1, h_2, \dots, h_n)$ and $\bar{h} = (\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n)$, where $\ell_i, h_i, \bar{h}_i \in \{0,1\}$ are binary variables defined by

$$\sum_{i=1}^{n} \ell_i = m, \qquad h_j = \left(\sum_{i=1}^{j} \ell_i \right) \bmod 2, \qquad \bar{h}_j = \left(1 + \sum_{i=1}^{j} \ell_i \right) \bmod 2. \tag{B.1}$$

The dual associated partitions obey relations (2.6).

TABLE B.1. Proof by cases of Lemma B.1.

$(\sum_{i=1}^{n}\ell_i, \ell_{i+1})$	(even, 0)	(even, 1)	(odd, 0)	(odd, 1)
h_i according to definition (B.1)	0	0	1	1
h_{i+1} according to definition (B.1)	0	1	1	0
h_{i+1} according to Lemma B.1	0	1	1	0

B.1. Binary identities. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be an ordered set of independent indeterminates μ_i belonging to $\{0,1\}$. Then, we have the following useful identities:

$$\mu_i^n = \mu_i, \tag{B.2}$$

$$\mu^2 = \mu \bullet \mu = \sum_{i=1}^{n} \mu_i, \tag{B.3}$$

$$(1-\mu_i)\mu_i = 0, \tag{B.4}$$

$$(2\mu_i - 1)\mu_i = \mu_i, \tag{B.5}$$

$$1 - (-1)^{\mu_i} - 2\mu_i = 0. \tag{B.6}$$

Identities (B.3) through (B.5) follow directly from identity (B.2), while identity (B.6) can be proved by inspection.

B.2. Recursive representations of the associated partitions

LEMMA B.1. *The components of the associated partition h are given recursively by*

$$h_{i+1} = h_i + (-1)^{h_i}\ell_{i+1}. \tag{B.7}$$

PROOF. It is possible to give an analytic proof of this lemma by making use of the results of Section B.1 above. On the other hand, the shortest proof is by cases, and it is given in Table B.1. □

LEMMA B.2. *The components of the associated partition h are given recursively by*

$$h_k = \sum_{i=1}^{k} \ell_i (h_i - \bar{h}_i). \tag{B.8}$$

PROOF. We give a proof by induction. For $k = 1$, identity (B.8) reduces to

$$(1-\ell_1)h_1 + \ell_1\bar{h}_1 = 0. \tag{B.9}$$

But from definition (B.1), we have $h_1 = \ell_1$, and $\bar{h}_1 = 1 - \ell_1$. Hence, the above equation reduces to $2(1-\ell_1)\ell_1 = 0$, and is satisfied due to identity (B.4). Hence, identity (B.8) is valid for $k = 1$.

Next, we assume that identity (B.8) is valid for k. Then, combining identities (B.7) and (B.8) we obtain

$$h_{k+1} = \sum_{i=1}^{k} \ell_i (h_i - \bar{h}_i) + (-1)^{h_k} \ell_{k+1}, \tag{B.10}$$

which can be rewritten as

$$h_{k+1} = \sum_{i=1}^{k+1} \ell_i (h_i - \bar{h}_i) + [(-1)^{h_k} - (h_{k+1} - \bar{h}_{k+1})] \ell_{k+1}, \tag{B.11}$$

or, more conveniently, as

$$h_{k+1} = \sum_{i=1}^{k+1} \ell_i (h_i - \bar{h}_i) + R, \tag{B.12}$$

where, making use of identity (2.6), R is given by

$$R = [1 + (-1)^{h_k} - 2h_{k+1}] \ell_{k+1}, \tag{B.13}$$

and due to identities (B.2) and (B.7) reduces to

$$R = [1 - (-1)^{h_k} - 2h_k] \ell_{k+1}. \tag{B.14}$$

Due to identity (B.6), $R = 0$, leading to

$$h_{k+1} = \sum_{i=1}^{k+1} \ell_i (h_i - \bar{h}_i). \tag{B.15}$$

Hence, identity (B.8) is valid for all integer $k \geq 1$, and the proof of Lemma B.2 is complete. □

Corollary B.3. *The components of the associated partition h are given recursively by*

$$h_k = \sum_{i=1}^{k} \ell_i (2h_i - 1). \tag{B.16}$$

Identity (B.16) follows from Lemma B.2 and the fact that $h_i + \bar{h}_i = 1$ (identity (2.6)).

Corollary B.4. *The components of the associated partition h are given recursively by*

$$h_k = \frac{1}{(2\ell_k - 1)} \left[\ell_k - \sum_{i=1}^{k-1} \ell_i (2h_i - 1) \right]. \tag{B.17}$$

Identity (B.17) follows from Corollary B.3 by grouping together the terms containing h_k.

B.3. Scalar products of partitions. The scalar products of h and $\bar{h}$ obey the identities

$$h \bullet \bar{h} = \sum_{i=1}^{n} h_i \bar{h}_i = 0, \qquad h \bullet h + \bar{h} \bullet \bar{h} = \sum_{i=1}^{n} (h_i + \bar{h}_i) = n, \tag{B.18}$$

where we have made use of identities (2.6) and (B.3).

Lemma B.5. *Let $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ be a partition of m into n parts, with $\sum_{i=1}^{n} \ell_i = m$ and $\ell_i \in \{0,1\}$. Let $h = (h_1, h_2, \ldots, h_n)$ and $\bar{h} = (\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)$ be its associated dual partitions. Let $\ell' = (\ell'_1, \ell'_2, \ldots, \ell'_n)$ with $\sum_{i=1}^{n} \ell'_i = m+1$ and $\ell'_i \in \{0,1\}$ be a partition of $m+1$ into n parts, obtained from the partition $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ by changing the value of one ℓ_i from 0 to 1. Let this be $\ell_k = 0$. Let $h' = (h'_1, h'_2, \ldots, h'_n)$ and $\bar{h}' = (\bar{h}'_1, \bar{h}'_2, \ldots, \bar{h}'_n)$ be the dual partitions associated with ℓ'. Then,*

$$\ell' \bullet \bar{h}' = \ell \bullet h, \tag{B.19}$$

or, explicitly,

$$\sum_{i=1}^{n} \ell'_i \bar{h}'_i = \sum_{i=1}^{n} \ell_i h_i. \tag{B.20}$$

Proof. From the conditions of the lemma, we have

$$\sum_{i=1}^{n} \ell_i = m, \quad \ell_i \in \{0,1\}, \tag{B.21}$$

$$\sum_{i=1}^{n} \ell'_i = m+1, \quad \ell'_i \in \{0,1\}, \tag{B.22}$$

$$\ell'_i = \begin{cases} \ell_i & \text{for } i \neq k,\ \ell_k = 0, \\ 1 & \text{for } i = k,\ \ell_k = 0, \end{cases} \tag{B.23}$$

$$h'_i = \begin{cases} h_i & \text{for } i < k, \\ 1 - h_i & \text{for } i \geq k. \end{cases} \tag{B.24}$$

Furthermore, due to identity (2.6), we have

$$h'_i + \bar{h}'_i = 1, \tag{B.25}$$

and making use of identities (B.24) and (B.25), we obtain

$$\bar{h}'_j = h_j \quad \text{for } j \geq k. \tag{B.26}$$

Due to identities (B.23), (B.25), and (B.24), and the fact that $\ell_k = 0$, we have

$$\ell' \bullet \bar{h}' = \sum_{i=1}^{n} \ell'_i \bar{h}'_i = \bar{h}'_k + \sum_{i=1}^{n} \ell_i \bar{h}'_i = \bar{h}'_k + \sum_{i=1}^{k-1} \ell_i \bar{h}_i + \sum_{i=k}^{n} \ell_i (1 - \bar{h}_i) \tag{B.27}$$

and due to identities (2.6) and (B.26), the above identity can be rewritten as

$$\ell' \bullet \bar{h}' = h_k + \sum_{i=1}^{k-1} \ell_i (1-h_i) + \sum_{i=k}^{n} \ell_i h_i. \tag{B.28}$$

But

$$\sum_{i=k}^{n} \ell_i h_i = \ell \bullet h - \sum_{i=1}^{k-1} \ell_i h_i, \tag{B.29}$$

hence,

$$\ell' \bullet \bar{h}' = \ell \bullet h + h_k + \sum_{i=1}^{k-1} \ell_i (1-2h_i), \tag{B.30}$$

and due to identity (B.16), and the fact that $\ell_k = 0$,

$$h_k + \sum_{i=1}^{k-1} \ell_i (1-2h_i) = 0. \tag{B.31}$$

Hence we recover identity (B.19), and the proof of Lemma B.5 is completed. □

Theorem B.6. *Let $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ be an ordered partition of m into n parts belonging to $\{0,1\}$, and let $h = (h_1, h_2, \ldots, h_n)$ be an associated partition given by $h_j = (\sum_{i=1}^{j} \ell_i) \bmod 2$. Then, the scalar product of ℓ and h is given by*

$$\ell \bullet h = \sum_{i=1}^{n} \ell_i h_i = \left\lfloor \frac{m+1}{2} \right\rfloor. \tag{B.32}$$

Proof. We prove Theorem B.6 by induction on m.

The case $m = 1$. For $m = 1$, $\ell_i = \delta_{ik}$ for some value of k in the range $k \in \{1, 2, \ldots, n\}$, and

$$h_j = \left(\sum_{i=1}^{j} \ell_i \right) \bmod 2 = \begin{cases} 0 & \text{for } j < k, \\ 1 & \text{for } j \geq k. \end{cases} \tag{B.33}$$

Hence,

$$\sum_{i=1}^{n} \ell_i h_i = \sum_{i=k}^{n} \ell_i = \sum_{i=k}^{n} \delta_{ik} = 1, \tag{B.34}$$

which is the result predicted by $\lfloor (m+1)/2 \rfloor$ evaluated at $m = 1$. Hence, Theorem B.6 is valid for $m = 1$, for all positive integer values of n.

The case $m = n$. For $m = n$, $\ell_i = 1$ and

$$h_j = \left(\sum_{i=1}^{j} \ell_i \right) \bmod 2 = j \bmod 2 = \begin{cases} 0 & \text{for } j \text{ even}, \\ 1 & \text{for } j \text{ odd}. \end{cases} \tag{B.35}$$

Hence,

$$\sum_{i=1}^{n} \ell_i h_i = \sum_{i=1}^{n} h_i = \sum_{s=1}^{\lfloor n/2 \rfloor} h_{2s} + \sum_{s=1}^{\lfloor (n+1)/2 \rfloor} h_{2s-1} = \sum_{s=1}^{\lfloor (n+1)/2 \rfloor} 1 = \left\lfloor \frac{n+1}{2} \right\rfloor \tag{B.36}$$

and since $m = n$, then Theorem B.6 is also valid for the case $m = n$, for all positive integer values of n.

The case $1 \le m \le n-1$. Next, we assume identity (B.32) to be valid for $m \le n-1$, and evaluate $\ell' \bullet h'$, where ℓ' is a partition of $m+1$ into n parts, as defined by (B.22) and (B.23). Then, using identity (B.25), we have

$$\ell' \bullet h' = \sum_{i=1}^{n} \ell_i' h_i' = \sum_{i=1}^{n} \ell_i' (1 - \bar{h}_i') = \sum_{i=1}^{n} \ell_i' - \sum_{i=1}^{n} \ell_i' \bar{h}_i'. \tag{B.37}$$

But according to Lemma B.5,

$$\sum_{i=1}^{n} \ell_i' \bar{h}_i' = \sum_{i=1}^{n} \ell_i h_i. \tag{B.38}$$

Hence, making use of identity (B.22), we have

$$\ell' \bullet h' = (m+1) - \sum_{i=1}^{n} \ell_i h_i. \tag{B.39}$$

According to the induction hypothesis,

$$\sum_{i=1}^{n} \ell_i h_i = \left\lfloor \frac{m+1}{2} \right\rfloor. \tag{B.40}$$

Hence,

$$\ell' \bullet h' = (m+1) - \left\lfloor \frac{m+1}{2} \right\rfloor = \left\lfloor \frac{m+2}{2} \right\rfloor, \tag{B.41}$$

where the last step follows from Theorem A.1 of Appendix A. Hence identity (B.32) is also true for $m+1$, and the above three results combined (for $m = 1$, $m = n$, and $1 \le m \le n-1$), guarantee the validity of Theorem B.6 for all positive integer $m \le n$. □

Theorem B.7. *Let $\ell = (\ell_1, \ell_2, \dots, \ell_n)$ be an ordered partition of m into n parts belonging to $\{0,1\}$, and let $\bar{h} = (\bar{h}_1, \bar{h}_2, \dots, \bar{h}_n)$ be an associated partition given by $\bar{h}_j = (1 + \sum_{i=1}^{j} \ell_i) \bmod 2$. Then, the scalar product of ℓ and $\bar{h}$ is given by*

$$\ell \bullet \bar{h} = \sum_{i=1}^{n} \ell_i \bar{h}_i = \left\lfloor \frac{m}{2} \right\rfloor. \tag{B.42}$$

Proof.

$$\ell \bullet \bar{h} = \sum_{i=1}^{n} \ell_i \bar{h}_i = \sum_{i=1}^{n} \ell_i (\bar{h}_i + h_i) - \sum_{i=1}^{n} \ell_i h_i. \tag{B.43}$$

But according to identities (B.1), (2.6), as well as Theorem B.6, we have

$$\sum_{i=1}^{n} \ell_i = m, \qquad (\bar{h}_i + h_i) = 1, \qquad \sum_{i=1}^{n} \ell_i h_i = \left\lfloor \frac{m+1}{2} \right\rfloor. \tag{B.44}$$

Hence, making use of identity (A.7) of Appendix A, we have

$$\ell \bullet \bar{h} = m - \left\lfloor \frac{m+1}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor. \tag{B.45}$$

This completes the proof of Theorem B.7. □

C. The symmetric functions. Let $\mathscr{X}_n = \{x_1, x_2, \dots, x_n\}$ and $\mathscr{Y}_n = \{y_1, y_2, \dots, y_n\}$ be two sets of independent indeterminates, with $\mathscr{X}_0 = \mathscr{Y}_0 = \varnothing$. The *constrained symmetric functions* $T^n_{mq}(x_j;\ j \in \mathbb{N}_n)$ are defined over $\mathscr{X}_n$ according to

$$T^n_{mq}(x_j;\ j \in \mathbb{N}_n) \equiv T^n_{mq}(\mathscr{X}_n) = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=m \\ \lambda_\nu \in \{0,1,2,\dots,q\}}} \prod_{i=1}^{n} x_i^{\lambda_i}, \tag{C.1}$$

where $\mathbb{N}_q = \{1, 2, \dots, q\}$ denotes the set of the first q positive integers, with $\mathbb{N}_0 = \varnothing$. The *elementary symmetric functions* $T^n_m(x_j;\ j \in \mathbb{N}_n)$ are defined over $\mathscr{X}_n$ by

$$T^n_m(x_j;\ j \in \mathbb{N}_n) \equiv T^n_{m1}(x_j;\ j \in \mathbb{N}_n), \tag{C.2}$$

or explicitly by

$$T^n_m(x_j;\ j \in \mathbb{N}_n) \equiv T^n_m(\mathscr{X}_n) = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m \\ \ell_\nu \in \{0,1\}}} \prod_{i=1}^{n} x_i^{\ell_i}, \tag{C.3}$$

and the *associated elementary symmetric functions* $U^n_m(x_j, y_j;\ j \in \mathbb{N}_n)$ are defined over $\mathscr{X}_n$ and $\mathscr{Y}_n$ according to [4]

$$U^n_m(x_j, y_j;\ j \in \mathbb{N}_n) \equiv U^n_m(\mathscr{X}_n, \mathscr{Y}_n) = \sum_{\substack{\ell_1+\ell_2+\cdots+\ell_n=m \\ \ell_\nu \in \{0,1\}}} \prod_{i=1}^{n} x_i^{\ell_i} y_i^{1-\ell_i}. \tag{C.4}$$

The sums in identities (C.3) and (C.4) are taken over all partitions of m into n parts $\{\ell_1,\ell_2,\dots,\ell_n\}$ subject to the constraint $\ell_j \in \{0,1\}$. The variables n and m are nonnegative integers, and for $m > n$, $T_m^n = U_m^n = 0$. It is easy to see that these symmetric functions are related by [4]

$$T_m^n(x_1,x_2,\dots,x_n) = U_m^n(x_1,x_2,\dots,x_n;1,1,\dots,1), \tag{C.5}$$

$$U_m^n(x_j,y_j;\ j\in\mathbb{N}_n) = \left\{\prod_{j=1}^{n} y_j\right\} T_m^n\left(\frac{x_j}{y_j};\ j\in\mathbb{N}_n\right). \tag{C.6}$$

They obey the scaling laws

$$T_m^n(ax_j;\ j\in\mathbb{N}_n) \equiv a^m T_m^n(x_j;\ j\in\mathbb{N}_n), \tag{C.7}$$

$$U_m^n(ax_j,by_j;\ j\in\mathbb{N}_n) = a^m b^{n-m} U_m^n(x_j,y_j;\ j\in\mathbb{N}_n), \tag{C.8}$$

and the closure relations

$$[(x_n+1)]^{(n)} = \prod_{j=1}^{n}(x_j+1) = \sum_{m=0}^{n} T_m^n(x_j;\ j\in\mathbb{N}_n), \tag{C.9}$$

$$[(x_n+y_n)]^{(n)} = \prod_{j=1}^{n}(x_j+y_j) = \sum_{m=0}^{n} U_m^n(x_j,y_j;\ j\in\mathbb{N}_n) = \sum_{m=0}^{n} U_m^n(y_j,x_j;\ j\in\mathbb{N}_n). \tag{C.10}$$

When all the n variables (independent indeterminates) are identical, they collapse to

$$T_m^n\underbrace{(x,x,\dots,x)}_{n} = \binom{n}{m} x^m, \tag{C.11}$$

$$U_m^n(\underbrace{x,x,\dots,x}_{n};\underbrace{y,y,\dots,y}_{n}) = \binom{n}{m} x^m y^{n-m}. \tag{C.12}$$

Note that the closure relation (C.9), for the *elementary symmetric functions* $T_m^n(x_j;\ j\in\mathbb{N}_n)$, is a special case of the generating equation (see, e.g., [16]),

$$\prod_{i=1}^{n}(1+tx_i) = \sum_{m=0}^{n} t^m T_m^n(x_j;\ j\in\mathbb{N}_n). \tag{C.13}$$

Furthermore, by combining identities (C.8) and (C.10), it is seen that the closure relation (C.10), for the *associated elementary symmetric functions* $U_m^n(x_j,y_j;\ j\in\mathbb{N}_n)$, is a special case of the generating equation

$$\prod_{j=1}^{n}(tx_j+sy_j) = \sum_{m=0}^{n} t^m s^{n-m} U_m^n(x_j,y_j;\ j\in\mathbb{N}_n). \tag{C.14}$$

D. Notation

D.1. Notation for Pochhammers. We refer to expressions

$$[M(n)]^{(k)} = M(n)M(n-1)\cdots M(n-k+1),$$
$$[M(n)]^{[k]} = M(n)M(n+1)\cdots M(n+k-1), \tag{D.1}$$

as descending and (respectively) ascending *matrix Pochhammers* or *matrix factorial polynomials*. When $M(k)$ is a scalar discrete function M_k, they reduce to ordinary *Pochhammers*, also called *factorial polynomials* and *shifted factorials*. If, in addition, $M_k = k$, they further reduce to ordinary *factorials*. For the different notations and definitions used in the literature, see Elaydi [13, pages 49–51], Hildebrand [14, pages 262–264], Milne-Thomson [17, pages 25–27], Jordan [15, pages 45–56], Berge [11, pages 19–24], and [8, Appendix A].

D.2. Notation for the Pauli matrices. The set $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$,

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{D.2}$$

formed by the identity and the three Pauli matrices, provides a basis for the set of 2×2 matrices. It is also convenient to introduce the Pauli matrices $\sigma_\pm$ defined by

$$\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{D.3}$$

The identities inverse to (D.3) are

$$\sigma_1 = (\sigma_+ + \sigma_-), \quad i\sigma_2 = (\sigma_+ - \sigma_-). \tag{D.4}$$

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