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# UNIFORMLY SUMMING SETS OF OPERATORS ON SPACES OF CONTINUOUS FUNCTIONS

J. M. DELGADO and CÁNDIDO PIÑEIRO

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Let $X$ and $Y$ be Banach spaces. A set $\mathcal{M}$ of 1-summing operators from $X$ into $Y$ is said to be *uniformly summing* if the following holds: given a weakly 1-summing sequence $(x_n)$ in $X$, the series $\sum_n \|Tx_n\|$ is uniformly convergent in $T \in \mathcal{M}$. We study some general properties and obtain a characterization of these sets when $\mathcal{M}$ is a set of operators defined on spaces of continuous functions.

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**1. Introduction.** Throughout this paper, $X$ and $Y$ will be Banach spaces. If $X$ is a Banach space, $B_X = \{x \in X : \|x\| \le 1\}$ will denote its closed unit ball and $X^*$ will be the topological dual of $X$. Given a real number $p \in [1,\infty)$, a (linear) operator $T : X \to Y$ is said to be *p-summing* if there exists a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} ||Tx_i||^p\right)^{1/p} \le C \cdot \sup\left\{\left(\sum_{i=1}^{n} |\langle x^*, x_i\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\} \tag{1.1}$$

for every finite set $\{x_1,\dots,x_n\} \subset X$. The least $C$ for which the above inequality always holds is denoted by $\pi_p(T)$ (the *p-summing norm* of $T$). The linear space of all $p$-summing operators from $X$ into $Y$ is denoted by $\Pi_p(X,Y)$ which is a Banach space endowed with the $p$-summing norm.

As usual, $\ell_w^p(X)$ will be the Banach space of weakly $p$-summable sequences in $X$, that is, the sequences $(x_n) \subset X$ satisfying $\sum_n |\langle x^*, x_n\rangle|^p < \infty$ for all $x^* \in X^*$; the norm in $\ell_w^p(X)$ is $\epsilon_p(x_n) = \sup\{(\sum_n |\langle x^*, x_n\rangle|^p)^{1/p} : x^* \in B_{X^*}\}$. The set of all strongly $p$-summable sequences in $X$ is denoted by $\ell_a^p(X)$; the norm in this space is $\pi_p(x_n) = (\sum_n \|x_n\|^p)^{1/p}$. If $T \in \Pi_p(X,Y)$, the correspondence $\hat{T} : (x_n) \mapsto (Tx_n)$ always induces a bounded operator from $\ell_w^p(X)$ into $\ell_a^p(Y)$ with $\|\hat{T}\| = \pi_p(T)$ [5, Proposition 2.1].

Families of operators arise in different applications: equations containing a parameter, homotopies of operators, and so forth. In these applications, it may be very interesting to know that, given a set $\mathcal{M} \subset \Pi_p(X,Y)$ and $(x_n) \in \ell_w^p(X)$, the series $\sum_n \|Tx_n\|^p$ is uniformly convergent in $T \in \mathcal{M}$. The main purpose of this paper is to study *uniformly p-summing* sets, that is, those sets $\mathcal{M} \subset \Pi_p(X,Y)$ for which, given $(x_n) \in \ell_w^p(X)$, the series $\sum_n \|Tx_n\|^p$ is uniformly convergent in $T \in \mathcal{M}$. These sets also enjoy some properties that justify their study; the next proposition lists some of them.

**Proposition 1.1.** (a) *Let $(T_k)$ be a sequence in $\Pi_p(X,Y)$. Then, $\widehat{T}_k \xrightarrow{k} 0$ pointwise if and only if $T_k \xrightarrow{k} 0$ pointwise and $(T_k)$ is uniformly $p$-summing.*

(b) *Let $\mathcal{M} \subset \Pi_p(X,Y)$ be a uniformly $p$-summing set. If $\mathcal{M}$ is endowed with the strong operator topology, then the map $T \in \mathcal{M} \mapsto \sum_n \|Tx_n\|^p \in \mathbb{R}$ is continuous for every $(x_n) \in \ell_w^p(X)$.*

A basic argument shows that uniformly $p$-summing sets are bounded for the $p$-summing norm. In fact, if $X$ does not contain any copy of $c_0$, bounded sets and uniformly 1-summing sets are the same. That is the reason for which we only consider operators defined on a $\mathscr{C}(\Omega)$-space, $\Omega$ being a compact Hausdorff space. We recall that every weakly compact operator $T : \mathscr{C}(\Omega) \to Y$ has a representing measure $m_T : \Sigma \to Y$ defined by $m_T(B) = T^{**}(\chi_B)$ for all $B \in \Sigma$, where $\Sigma$ denotes the Borel $\sigma$-field of subsets of $\Omega$ and $\chi_B$ denotes the characteristic function of $B$. The vector measure $m_T$ is regular and countably additive [6, Theorem VI.2.5 and Corollary VI.2.14]. If we denote by $\widetilde{T}$ the operator $T^{**}$ restricted to $B(\Sigma)$ (the space of all bounded Borel-measurable scalar-valued functions defined on $\Omega$), then

$$\widetilde{T}\varphi = \int_\Omega \varphi \, dm_T, \tag{1.2}$$

for all $\varphi \in B(\Sigma)$ (the integral is the elementary Bartle integral [6, Definition I.1.12]).

It is well known that every $p$-summing operator defined on a Banach space $X$ is weakly compact. In Section 2, we consider 1-summing operators $T$ defined on $\mathscr{C}(\Omega)$; these operators are characterized as those with representing measure $m_T$ having finite variation and $\pi_1(T) = |m_T|(\Omega)$ [6, Theorem VI.3.3]. We show that a set $\mathcal{M} \subset \Pi_1(\mathscr{C}(\Omega), Y)$ is uniformly 1-summing if and only if the family of all variation measures $\{|m_T| : T \in \mathcal{M}\}$ is uniformly bounded and there is a countably additive measure $\mu : \Sigma \to [0,\infty)$ such that $\{|m_T| : T \in \mathcal{M}\}$ is uniformly $\mu$-continuous.

In Section 3, we mention a special class of uniformly $p$-summing operators: *uniformly dominated sets*. The relationship between uniformly summing sets and relatively weak compactness is also studied. Finally, we give some examples and open problems.

## 2. Uniformly 1-summing sets in $\Pi_1(\mathscr{C}(\Omega), Y)$.

Before facing our main theorem, we include three results which correspond to the vector measure theory. These results will be usually invoked along the following lines.

**Proposition 2.1** [6, Proposition I.1.17]**.** *The following statements about a collection $\{m_i : i \in I\}$ of $Y$-valued measures defined on a $\sigma$-field $\Sigma$ are equivalent:*

(a) *the set $\{m_i : i \in I\}$ is uniformly countably additive, that is, if $(E_n)$ is a sequence of pairwise disjoint members of $\Sigma$, then $\lim_n \|\sum_{k \ge n} m_i(E_k)\| = 0$ uniformly in $i \in I$,*

(b) *the set $\{y^* \circ m_i : i \in I,\ y^* \in B_{Y^*}\}$ is uniformly countably additive,*

(c) *if $(E_n)$ is a sequence of pairwise disjoint members of $\Sigma$, then $\lim_n \|m_i(E_n)\| = 0$ uniformly in $i \in I$,*

(d) *if $(E_n)$ is a sequence of pairwise disjoint members of $\Sigma$, then $\lim_n \|m_i\|(E_n) = 0$ uniformly in $i \in I$, where $\|m_i\|$ denotes the semivariation of $m_i$,*

(e) *the set $\{|y^* \circ m_i| : i \in I,\ y^* \in B_{Y^*}\}$ is uniformly countably additive.*

**Theorem 2.2** [6, Theorem I.2.4]. *Let $\{m_i : \Sigma \to Y : i \in I\}$ be a uniformly bounded (with respect to the semivariation) family of countably additive vector measures on a $\sigma$-field $\Sigma$. The family $\{m_i : i \in I\}$ is uniformly countably additive if and only if there exists a positive real-valued countably additive measure $\mu$ on $\Sigma$ such that $\{m_i : i \in I\}$ is uniformly $\mu$-continuous, that is,*

$$\lim_{\mu(E)\to 0} \|m_i(E)\| = 0 \tag{2.1}$$

*uniformly in $i \in I$.*

If $\Omega$ is a compact Hausdorff space and $\Sigma$ denotes the $\sigma$-field of the Borel subsets of $\Omega$, a vector measure $m$ on $\Sigma$ is regular if for each Borel set $E$ and $\varepsilon > 0$ there exists a compact set $K$ and an open set $O$ such that $K \subset E \subset O$ and $\|m\|(O\backslash K) < \varepsilon$.

**Proposition 2.3** [6, Lemma VI.2.13]. *Let $\mathcal{K}$ be a family of regular (countably additive) scalar measures defined on $\Sigma$. Each of the following statements implies all the others:*

(a) *for each pairwise disjoint sequence $(O_n)$ of open subsets of $\Omega$, $\lim_n \mu(O_n) = 0$ uniformly in $\mu \in \mathcal{K}$,*

(b) *for each pairwise disjoint sequence $(O_n)$ of open subsets of $\Omega$, $\lim_n |\mu|(O_n) = 0$ uniformly in $\mu \in \mathcal{K}$,*

(c) *$\mathcal{K}$ is uniformly countably additive,*

(d) *$\mathcal{K}$ is uniformly regular, that is, if $E \in \Sigma$ and $\varepsilon > 0$, then there exists a compact set $K$ and an open set $O$ such that $K \subset E \subset O$ and $\sup_{\mu\in\mathcal{K}} |\mu|(O\backslash K) < \varepsilon$.*

Now, we are able to show our main result. In the proof, we will use the fact that $|m_T|$ is regular when $T : \mathcal{C}(\Omega) \to Y$ is 1-summing [7, Proposition 15.21].

**Theorem 2.4.** *Let $\mathcal{M} \subset \Pi_1(\mathcal{C}(\Omega), Y)$ be a bounded set. The following statements are equivalent:*

(a) *$\mathcal{M}$ is uniformly 1-summing,*

(b) *the family of nonnegative measures $\{|m_T| : T \in \mathcal{M}\}$ is uniformly countably additive,*

(c) *given $\varepsilon > 0$ and a disjoint sequence $(E_n)$ of Borel subsets of $\Omega$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sum_{n\geq n_0} \|m_T(E_n)\| < \varepsilon, \tag{2.2}$$

*for all $T \in \mathcal{M}$.*

**Proof.** (a)⇒(b). According to [6, Lemma VI.2.13], it suffices to show that $\lim_{n\to\infty} |m_T|(O_n) = 0$ uniformly in $T \in \mathcal{M}$, for all disjoint sequences $(O_n)$ of open subsets of $\Omega$. By contradiction, suppose that there exists $\varepsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers such that

$$|m_{T_n}|(O_{k_n}) > 2\varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

Now we consider the operators $S_n : \mathscr{C}(\Omega, O_{k_n}) \to Y$ defined by

$$S_n\varphi = \int_{O_{k_n}} \varphi \, dm_{T_n}, \tag{2.4}$$

for all $\varphi \in \mathscr{C}(\Omega, O_{k_n})$, where $\mathscr{C}(\Omega, O_{k_n})$ is the closed subspace of $\mathscr{C}(\Omega)$ formed by all continuous functions $\varphi$ on $\Omega$ such that $\varphi$ vanishes in $\Omega \backslash O_{k_n}$. It is known that $\pi_1(S_n) = |m_{T_n}|(O_{k_n})$, for all $n \in \mathbb{N}$ [7, Theorem 19.3]. For each $n \in \mathbb{N}$, we can choose a finite set $\{\varphi_1^n, \dots, \varphi_{p_n}^n\} \subset \mathscr{C}(\Omega, O_{k_n})$ satisfying $\epsilon_1(\varphi_i^n)_{i=1}^{p_n} \le 1$ and

$$\sum_{i=1}^{p_n} \|S_n\varphi_i^n\| > \pi_1(S_n) - \varepsilon. \tag{2.5}$$

Since the open sets $O_{k_n}$ are disjoint, it follows that the sequence $(\varphi_1^1, \dots, \varphi_{p_1}^1, \varphi_1^2, \dots, \varphi_{p_2}^2, \dots)$ belongs to $\ell_w^1(\mathscr{C}(\Omega))$. Nevertheless, for all $n \in \mathbb{N}$, we have

$$\sum_{m \ge n} \sum_{i=1}^{p_m} \|T_n\varphi_i^m\| \ge \sum_{i=1}^{p_n} \|T_n\varphi_i^n\| = \sum_{i=1}^{p_n} \|S_n\varphi_i^n\| > \pi_1(S_n) - \varepsilon = |m_{T_n}|(O_{k_n}) - \varepsilon > \varepsilon. \tag{2.6}$$

This denies (a) and proves that (a) implies (b).

(b)⇒(c). Again we proceed by contradiction. Suppose $(E_n)$ is a disjoint sequence of Borel subsets of $\Omega$ for which there exists $\varepsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers so that

$$\sum_{i=k_n+1}^{k_{n+1}} \|m_{T_n}(E_i)\| > \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

If we put $B_n = \bigsqcup_{i=k_n+1}^{k_{n+1}} E_i$, the above inequality yields $|m_{T_n}|(B_n) > \varepsilon$. So, in view of [6, Proposition I.1.17], the family $\{|m_T| : T \in \mathcal{M}\}$ is not uniformly countably additive.

(c)⇒(b). We need to prove

$$\lim_{n\to\infty} |m_T|(E_n) = 0 \quad \text{uniformly in } T \in \mathcal{M}, \tag{2.8}$$

for all disjoint sequences $(E_n)$ of Borel subsets of $\Omega$. Suppose (b) fails. Then, there exists $\varepsilon > 0$, a sequence $(T_n)$ in $\mathcal{M}$, and a strictly increasing sequence $(k_n)$ of natural numbers satisfying

$$|m_{T_n}|(E_{k_n}) > \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

For each $n \in \mathbb{N}$, we choose a finite partition $\{E_1^n, \dots, E_{p_n}^n\}$ of $E_{k_n}$ for which

$$\sum_{i=1}^{p_n} \|m_{T_n}(E_i^n)\| > \varepsilon. \tag{2.10}$$

Then, the disjoint sequence $(E_1^1, \dots, E_{p_1}^1, E_1^2, \dots, E_{p_2}^2, \dots)$ does not satisfy (c).

(b)⇒(a). According to [6, Theorem I.2.4] there exists a countably additive measure $\mu:\Sigma\to[0,\infty)$ so that

$$\lim_{\mu(E)\to 0}|m_T|(E)=0 \quad \text{uniformly in } T\in\mathcal{M}. \tag{2.11}$$

Hence, given $\varepsilon>0$, there exists $\delta>0$ such that, if $E\in\Sigma$ verifies $\mu(E)<\delta$, then $|m_T|(E)<\varepsilon/2$, for all $T\in\mathcal{M}$.

Next, given $(\varphi_n)\in\ell_w^1(\mathscr{C}(\Omega))$ with $\epsilon_1(\varphi_n)\le 1$, notice that the series $\sum_{n=1}^{\infty}|\varphi_n(t)|$ is convergent for all $t\in\Omega$. Put $f_n(t)=\sum_{k=1}^{n}|\varphi_k(t)|$ and $f(t)=\lim_{n\to\infty}f_n(t)$. By Egorov's theorem, the sequence $(f_n)$ is quasi-uniformly convergent to $f$. Then, there exists $E\in\Sigma$ such that $\mu(E)<\delta$ and

$$f_{n|\Omega\backslash E}\longrightarrow f_{|\Omega\backslash E} \tag{2.12}$$

uniformly. If $C=\sup\{|m_T|(\Omega):T\in\mathcal{M}\}$, there exists $n_0\in\mathbb{N}$ so that

$$\sum_{n\ge n_0}|\varphi_n(t)|<\frac{\varepsilon}{2C},\quad \forall t\in\Omega\backslash E. \tag{2.13}$$

Now,

$$\begin{aligned}\sum_{n\ge n_0}\|T\varphi_n\| &= \sum_{n\ge n_0}\left\|\int_\Omega \varphi_n(t)\,dm_T\right\|\\ &\le \sum_{n\ge n_0}\left\|\int_E \varphi_n(t)\,dm_T\right\|+\sum_{n\ge n_0}\left\|\int_{\Omega\backslash E}\varphi_n(t)\,dm_T\right\|\\ &\le \sum_{n\ge n_0}\int_E|\varphi_n(t)|\,d|m_T|+\sum_{n\ge n_0}\int_{\Omega\backslash E}|\varphi_n(t)|\,d|m_T|\\ &= \int_E\left(\sum_{n\ge n_0}|\varphi_n(t)|\right)d|m_T|+\int_{\Omega\backslash E}\left(\sum_{n\ge n_0}|\varphi_n(t)|\right)d|m_T|\\ &\le |m_T|(E)+\frac{\varepsilon}{2C}|m_T|(\Omega\backslash E)\\ &<\varepsilon.\end{aligned} \tag{2.14}$$

□

We denote by $\mathscr{V}(X,Y)$ the class of completely continuous operators from $X$ into $Y$, that is, the class of operators which map weakly convergent sequences in $X$ into norm-convergent sequences in $Y$. A set $\mathcal{M}\subset\mathscr{V}(X,Y)$ is said to be *uniformly completely continuous* if, given a weakly convergent sequence $(x_n)$ in $X$, $(Tx_n)$ is norm convergent uniformly in $T\in\mathcal{M}$. The following result gives some characterizations of uniformly completely continuous sets in $\mathscr{V}(\mathscr{C}(\Omega),Y)$. Recall that an operator $T$ defined on $\mathscr{C}(\Omega)$ is completely continuous if and only if $T$ is weakly compact [6, Corollary VI.2.17], so $m_T$ is countably additive and regular, too.

**Theorem 2.5.** *Let $\mathcal{M}\subset\mathscr{V}(\mathscr{C}(\Omega),Y)$ be a bounded set for the operator norm. The following statements are equivalent:*

(a) *$\mathcal{M}$ is uniformly completely continuous,*

(b) *the family $\{m_T:T\in\mathcal{M}\}$ is uniformly countably additive,*

(c) $\mathcal{M}^* = \{T^* : T \in \mathcal{M}\}$ *is collectively weakly compact, that is, the set* $\bigcup_{T\in\mathcal{M}} T^*(B_{Y^*})$ *is relatively weakly compact in* $\mathcal{C}(\Omega)^*$.

**Proof.** (a)⇒(b). By [6, Proposition I.1.17], the family $\{m_T : T \in \mathcal{M}\}$ is uniformly countably additive if and only if $\mathcal{N} = \{y^* \circ m_T : T \in \mathcal{M},\ y^* \in B_{Y^*}\}$ is. According to [6, Lemma VI.1.13], we have to prove that

$$\lim_{n\to\infty} y^* \circ m_T(O_n) = 0 \quad \text{uniformly in } \mathcal{N}, \tag{2.15}$$

for all disjoint sequences $(O_n)$ of open subsets of $\Omega$. By contradiction, suppose there exists such a sequence $(O_n)$ for which $\lim_{n\to\infty} y^* \circ m_T(O_n) = 0$ but not uniformly in $\mathcal{N}$. Then, there exists $\varepsilon > 0$ and sequences $(y_n^*) \subset B_{Y^*}$, $(T_n) \in \mathcal{M}$, and $(O_{k_n}) \subset (O_n)$ such that

$$|y_n^* \circ m_{T_n}(O_{k_n})| > \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.16}$$

Now, using the regularity of each $m_{T_n}$, we can find a sequence of compact sets $(H_n)$ with $H_n \subset O_{k_n}$ and

$$\|m_{T_n}\|(O_{k_n} \backslash H_n) < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}, \tag{2.17}$$

($\|m\|$ denotes the semivariation of $m$, that is, $\|m\|(E) = \sup\{|y^* \circ m|(E) : y^* \in B_{Y^*}\}$). By Urysohn's lemma, for every $n \in \mathbb{N}$ there exists a continuous function $\varphi_n : \Omega \to [0,1]$ such that $\varphi_n(H_n) = 1$ and $\varphi_n(\Omega \backslash O_{k_n}) = 0$. Obviously, the series $\sum_{n=1}^{\infty} \varphi_n$ is unconditionally convergent in $\mathcal{C}(\Omega)$. Since $\mathcal{M}$ is uniformly completely continuous, there exists $n_0 \in \mathbb{N}$ such that

$$\|T\varphi_n\| < \frac{\varepsilon}{2}, \quad \forall n \ge n_0,\ \forall T \in \mathcal{M}. \tag{2.18}$$

Then, we have

$$\begin{aligned}
\|m_{T_n}(O_{k_n})\| &\le \|m_{T_n}(O_{k_n}) - T_n\varphi_n\| + \|T_n\varphi_n\| \\
&= \left\| \int_\Omega \chi_{O_{k_n}}\, dm_{T_n} - \int_\Omega \varphi_n\, dm_{T_n} \right\| + \|T_n\varphi_n\| \\
&= \left\| \int_{O_{k_n}} (1-\varphi_n)\, dm_{T_n} \right\| + \|T_n\varphi_n\| \\
&= \left\| \int_{O_{k_n}\backslash H_n} (1-\varphi_n)\, dm_{T_n} \right\| + \|T_n\varphi_n\| \\
&\le \|m_{T_n}\|(O_{k_n}\backslash H_n) + \|T_n\varphi_n\| \\
&< \varepsilon,
\end{aligned} \tag{2.19}$$

for all $n \ge n_0$. This is in contradiction with (2.16).

(b)⇒(a). By [6, Theorem I.2.4], there exists a scalar countably additive measure $\mu : \Sigma \to [0,\infty)$ such that $\{m_T : T \in \mathcal{M}\}$ is uniformly $\mu$-continuous. Then, if $(\varphi_n)$ is a sequence that tends to zero weakly in $\mathscr{C}(\Omega)$, it is obvious that zero is the pointwise limit of the sequence $(\varphi_n(t))$. Now, using Egorov's theorem and proceeding along similar lines as the proof of (b)⇒(a) in Theorem 2.4, the proof concludes.

(b)⇔(c). The set $\bigcup_{T\in\mathcal{M}} T^*(B_{Y^*}) = \{y^* \circ m_T : T \in \mathcal{M},\ y^* \in B_{Y^*}\} \subset \mathscr{C}(\Omega)^*$ is relatively weakly compact if and only if it is bounded and uniformly countably additive [4, Theorem VII.13]. A call to [6, Proposition I.1.17] makes clear that $\bigcup_{T\in\mathcal{M}} T^*(B_{Y^*})$ is uniformly countably additive if and only if condition (b) is satisfied. □

**Corollary 2.6.** *If $\mathcal{M} \subset \Pi_1(\mathscr{C}(\Omega), Y)$ is uniformly 1-summing, then $\mathcal{M}$ is uniformly completely continuous.*

The converse of the last result is not true in general.

**Proposition 2.7.** *Suppose that the cardinal of $\Omega$ is infinite. The following statements are equivalent:*

(a) *each subset of $\Pi_1(\mathscr{C}(\Omega), Y)$ uniformly completely continuous is uniformly 1-summing,*

(b) *$Y$ is finite-dimensional.*

**Proof.** (a)⇒(b). By contradiction, suppose there is an unconditionally summable serie $\sum_k y_k$ in $Y$ such that $\sum_k \|y_k\| = \infty$. Let $(\omega_k)$ be a sequence in $\Omega$ with $\omega_k \neq \omega_l$ when $k \neq l$. For each $m \in \mathbb{N}$ consider the operator $T_m : \mathscr{C}(\Omega) \to Y$ defined by

$$T_m\varphi = \sum_{k=1}^{m} \varphi(\omega_k) y_k. \tag{2.20}$$

It is not difficult to show that $\mathcal{M} = (T_m)$ is uniformly completely continuous. Nevertheless,

$$\pi_1(T_m) = \sum_{k=1}^{m} \|y_k\| \xrightarrow{m} \infty, \tag{2.21}$$

so $\mathcal{M}$ cannot be uniformly 1-summing because it is not $\pi_1$-bounded.

(b)⇒(a). This follows easily in view of conditions (b) in Theorems 2.4 and 2.5. □

We have showed that the converse of Corollary 2.6 is not true in general. However, a direct argument using Theorems 2.4 and 2.5 leads up to conclude that every uniformly completely continuous set $\mathcal{M} \subset \Pi_1(\mathscr{C}(\Omega), Y)$ verifying the following condition is uniformly 1-summing:

(i) given $T \in \mathcal{M}$ and a finite subset $\{(\varphi_1, y_1^*), \dots, (\varphi_m, y_m^*)\}$ of $\mathscr{C}(\Omega) \times B_{Y^*}$, there exist $S \in \mathcal{M}$ and $z^* \in B_{Y^*}$ such that $|\langle y_n^*, T\varphi_n\rangle| \le |\langle z^*, S\varphi_n\rangle|$, $n = 1, \dots, m$.

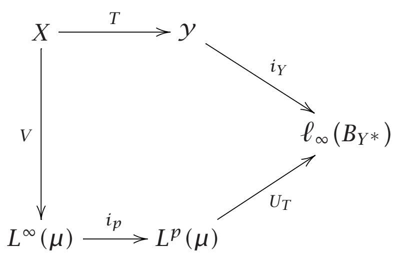
**3. Final notes and examples.** The Grothendieck-Pietsch domination theorem states that an operator $T: X \to Y$ is $p$-summing if and only if there exists a positive Radon measure $\mu$ defined on the (weak*) compact space $B_{X^*}$ such that

$$\|Tx\|^p \le \int_{B_{X^*}} |\langle x^*, x\rangle|^p \, d\mu(x^*), \tag{3.1}$$

for all $x \in X$ [5, Theorem 2.12]. Since the appearance of this theorem, there is a great interest in finding out the structure of uniformly $p$-dominated sets. A subset $\mathcal{M}$ of $\Pi_p(X,Y)$ is *uniformly $p$-dominated* if there exists a positive Radon measure $\mu$ such that the inequality (3.1) holds for all $x \in X$ and all $T \in \mathcal{M}$. In [3, 8, 9], the reader can find some of the most recent steps given on this subject. Now we are going to show that these sets are uniformly $p$-summing.

**Proposition 3.1.** *If $\mathcal{M} \subset \Pi_p(X,Y)$ is a uniformly $p$-dominated set, then $\mathcal{M}^{**} = \{T^{**}\colon\ T \in \mathcal{M}\}$ is uniformly $p$-summing.*

**Proof.** Let $\mu$ be a measure for which $\mathcal{M}$ is uniformly $p$-dominated. In a similar way as in the Pietsch factorization theorem [5, Theorem 2.13], we can obtain, for all $T \in \mathcal{M}$, operators $U_T\colon L_p(\mu) \to \ell_\infty(B_{Y^*})$, $\|U_T\| \le \mu(B_{X^*})^{1/p}$, and an operator $V\colon X \to L_\infty(\mu)$ such that the following diagram is commutative:



(3.2)

Here, $i_p$ is the canonical injection from $L_\infty(\mu)$ into $L_p(\mu)$ and $i_Y$ is the isometry from $Y$ into $\ell_\infty(B_{Y^*})$ defined by $i_Y(y) = (\langle y^*, y\rangle)_{y^* \in B_{Y^*}}$. Notice that $i_p^{**}$ can be viewed as $i_p$ composed with the canonical projection $P\colon L_\infty(\mu)^{**} \to L_\infty(\mu)$ which is simply the adjoint of the usual embedding $L_1(\mu) \to L_1(\mu)^{**}$. By weak compactness, we may and do consider $T^{**}$ as a map from $X^{**}$ into $Y$ for which

$$i_Y \circ T^{**} = U_T \circ i_p \circ P \circ V^{**}. \tag{3.3}$$

Given $\varepsilon > 0$ and $(x_n^{**}) \in \ell_w^p(X^{**})$, we can choose $n_0 \in \mathbb{N}$ so that

$$\sum_{n \ge n_0} \|i_p \circ P \circ V^{**}(x_n^{**})\|^p < \frac{\varepsilon}{\mu(B_{X^*})}, \tag{3.4}$$

because $i_p \circ P \circ V^{**}$ is $p$-summing. Then, we have

$$\begin{aligned} \sum_{n \ge n_0} \|T^{**}x_n^{**}\|^p &= \sum_{n \ge n_0} \|i_Y \circ T^{**}(x_n^{**})\|^p = \sum_{n \ge n_0} \|U_T \circ i_p \circ P \circ V^{**}(x_n^{**})\|^p \\ &\le \mu(B_{X^*}) \sum_{n \ge n_0} \|i_p \circ P \circ V^{**}(x_n^{**})\|^p < \varepsilon, \end{aligned} \tag{3.5}$$

for all $T \in \mathcal{M}$. So, $\mathcal{M}^{**}$ is uniformly $p$-summing. □

It is easy to show that the study of uniformly $p$-summing sets can be reduced to the behavior of its sequences. Indeed, a bounded set $\mathcal{M}$ in $\Pi_p(X,Y)$ is uniformly $p$-summing if and only if every sequence $(T_n)$ in $\mathcal{M}$ admits a uniformly $p$-summing subsequence. Thus, it seems to be interesting to make clear the relationship between uniformly $p$-summing sets and relatively weakly compact sets. For $p=1$, we have the following result.

**Proposition 3.2.** *Every relatively weakly compact set in* $\Pi_1(X,Y)$ *is uniformly* 1-*summing.*

**Proof.** Let $\mathcal{M}$ be a relatively weakly compact set in $\Pi_1(X,Y)$. Given $\hat{x}=(x_n)\in \ell_w^1(X)$, consider the (weak-weak) continuous operator $U_{\hat{x}}:\Pi_1(X,Y)\to \ell_a^1(Y)$ defined by $U_{\hat{x}}(T)=(Tx_n)$. Then, $U_{\hat{x}}(\mathcal{M})$ is relatively weakly compact in $\ell_a^1(Y)$; according to [2, Theorem 2], we can conclude that $\mathcal{M}$ is uniformly 1-summing. □

Proposition 3.2 does not remain true if $p=2$. For example, for each $\beta=(\beta_n)\in\ell_2$ consider the operator $T_\beta\colon c_0\to\ell_2$ defined by $T(\alpha_n)=(\alpha_n\cdot\beta_n)$ and put $\mathcal{M}=\{T_\beta:\beta\in B_{\ell_2}\}\subset\Pi_2(c_0,\ell_2)$ [5, Theorem 3.5]. If we consider $\ell_2$ as a subspace of $\Pi_2(c_0,\ell_2)$, the set $\mathcal{M}=B_{\ell_2}$ is relatively weakly compact. Nevertheless, no matter how we choose $k\in\mathbb{N}$,

$$\sum_{n\ge k}\left\|T_{e_k}e_n\right\|^2=1, \tag{3.6}$$

so $\mathcal{M}$ cannot be uniformly 2-summing.

Now we show that there are uniformly $p$-summing sets failing to be relatively weakly compact.

**Proposition 3.3.** *If every uniformly $p$-summing set is relatively weakly compact in* $\Pi_p(X,Y)$*, then $Y$ is reflexive.*

**Proof.** Fixing $x_0^*\in X^*$ with $\|x_0^*\|=1$, the isometry $y\in Y\mapsto x_0^*\otimes y\in x_0^*\otimes Y$ allows us to see $Y$ as a subspace of $\Pi_p(X,Y)$. If $\varepsilon>0$ and $(x_n)\in\ell_w^p(X)$, there exists $n_0\in\mathbb{N}$ so that

$$\sum_{n\ge n_0}\left|\langle x_0^*,x_n\rangle\right|^p<\varepsilon; \tag{3.7}$$

hence, for every $y\in B_Y$,

$$\sum_{n\ge n_0}\left\|(x_0^*\otimes y)(x_n)\right\|^p=\sum_{n\ge n_0}\left|\langle x_0^*,x_n\rangle\right|^p\|y\|^p<\varepsilon. \tag{3.8}$$

This yields that $B_Y$ is uniformly $p$-summing and, by hypothesis, weakly compact. □

The converse of Proposition 3.3 is not always true. By contradiction, suppose every uniformly 1-summing set in $\Pi_1(\ell_1,\ell_2)$ is relatively weakly compact. Because $\ell_1$ does not contain any copy of $c_0$, every bounded set in $\Pi_1(\ell_1,\ell_2)$ is relatively weakly compact. Then, we conclude that $\Pi_1(\ell_1,\ell_2)$ is reflexive, which is not possible since $\ell_1^*$, viewed as a subspace of $\Pi_1(\ell_1,\ell_2)$, is not.

However, if $p=1$ and $X=\mathscr{C}(\Omega)$, the reflexivity of $Y$ is a sufficient condition for a uniformly 1-summing set to be relatively weakly compact. Indeed, if $rbvca(\Sigma,Y)$ denotes the set of all regular, countably additive, $Y$-valued measures $m$ on $\Sigma$ with bounded variation, recall that relatively weakly compact sets $\mathcal{M}$ in $rbvca(\Sigma, Y)$ are those verifying the following conditions: (i) $\mathcal{M}$ is bounded; (ii) the family of nonnegative measures $\{|m| : m \in \mathcal{M}\}$ is uniformly countably additive; and (iii) for each $E \in \Sigma$, the set $\{m(E) : m \in \mathcal{M}\}$ is relatively weakly compact in $Y$ [6, Theorem IV.2.5]. Having in mind the identification between $\Pi_1(\mathscr{C}(\Omega), Y)$ and $rbvca(\Sigma, Y)$, and making use of the characterization of uniformly 1-summing sets obtained in Theorem 2.4, we conclude the next characterization.

**Corollary 3.4.** *The following statements are equivalent:*
(a) *$Y$ is reflexive,*
(b) *every set $\mathcal{M}$ in $\Pi_1(\mathscr{C}(\Omega), Y)$ is uniformly 1-summing if and only if $\mathcal{M}$ is relatively weakly compact.*

It is well known that a linear operator $T$ is 1-summing if and only if $T^{**}$ is. So, it is natural to ask if a set $\mathcal{M}$ is uniformly 1-summing whenever $\mathcal{M}^{**} = \{T^{**} : T \in \mathcal{M}\}$ is. Unfortunately, we are going to show that this is not true in general. It suffices to take $X$ as the separable $\mathscr{L}_\infty$-space of Bourgain and Delbaen [1]. This space has the Radon–Nikodym property, so it does not contain any copy of $c_0$. Nevertheless, $X^*$ is isomorphic to $\ell_1$ and, therefore, $X^{**}$ contains a copy of $c_0$. Let $(e_n)$ be the canonical basis of $\ell_1$ and $J : \ell_1 \to X^*$ an isomorphism. Put $T_n = Je_n \in \Pi_1(X, \mathbb{R})$; the set $\mathcal{M} = \{T_n : n \in \mathbb{N}\}$ is uniformly 1-summing since it is bounded and $X$ does not contain any copy of $c_0$. Notice that the elements of $\mathcal{M}^{**}$ are the linear forms $x^{**} \in X^{**} \mapsto \langle x^{**}, Je_n\rangle \in \mathbb{R}$, for all $n \in \mathbb{N}$. If $(e_n^*)$ is the canonical basis of $c_0$, then $((J^*)^{-1}(e_n^*)) \in \ell_w^1(X^{**})$; hence, no matter how we choose $k \in \mathbb{N}$, it turns out that

$$\sum_{n\geq k} \left| T_k^{**}\left((J^*)^{-1}(e_n^*)\right) \right| = \sum_{n\geq k} \left| \left\langle (J^*)^{-1}(e_n^*), Je_k \right\rangle \right| = \sum_{n\geq k} \left| \langle e_n^*, e_k\rangle \right| = 1, \tag{3.9}$$

and $\mathcal{M}^{**}$ cannot be uniformly 1-summing.

Nevertheless, if $\mathcal{M}$ is a set of operators defined on $c_0$, then it is true that $\mathcal{M}$ is uniformly 1-summing if and only if $\mathcal{M}^{**}$ is too. To see this, notice that for a 1-summing operator $T : (\alpha_n) \in c_0 \mapsto \sum_{n=1}^{\infty} \alpha_n x_n \in X$, the second adjoint $T^{**} : \ell_\infty \to X$ is defined by $T^{**}(\beta_n) = \sum_{n=1}^{\infty} \beta_n x_n$, for all $(\beta_n) \in \ell_\infty$.

When $\mathcal{M}$ is a set of operators defined on a $\mathscr{C}(\Omega)$-space, we do not know whether $\mathcal{M}^{**}$ inherits the property or not. Anyway, we are going to prove the following weaker result. We inject isometrically $B(\Sigma)$ into $\mathscr{C}(\Omega)^{**}$ in the natural way.

**Proposition 3.5.** *If $\mathcal{M} \subset \Pi_1(\mathscr{C}(\Omega), X)$ is uniformly 1-summing, then $\widetilde{\mathcal{M}} = \{\widetilde{T} : B(\Sigma) \to X : T \in \mathcal{M}\}$ is uniformly 1-summing too.*

**Proof.** The argument is similar to the one used in the proof of (b)⇒(a) in Theorem 2.4. □

Finally, we give an example to show that Corollary 2.6 is not true if $\mathscr{C}(\Omega)$ is replaced by a general Banach space $X$. It suffices to take $X = \ell_2$ and $\mathcal{M} = \{e_n^* : n \in \mathbb{N}\}$, where $(e_n^*)$ is the unit basis of $\ell_2^* \simeq \ell_2$. The set $\mathcal{M}$ is bounded in $\Pi_1(\ell_2, \mathbb{R})$ and, therefore, uniformly 1-summing but it is not uniformly completely continuous.

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J. M. Delgado: Departamento de Matemáticas, Facultad de Ciencias Experimentales, Campus Universitario del Carmen, Avda. de las Fuerzas Armadas, 21071 Huelva, Spain
*E-mail address*: jmdelga@uhu.es

Cándido Piñeiro: Departamento de Matemáticas, Facultad de Ciencias Experimentales, Campus Universitario del Carmen, Avda. de las Fuerzas Armadas, 21071 Huelva, Spain
*E-mail address*: candido@uhu.es