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UNIT-CIRCLE-PRESERVING MAPPINGS

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We prove that if a one-to-one mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ ($n\ge 2$) preserves the unit circles, then f is a linear isometry up to translation.

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1. Introduction. Let X and Y be normed spaces. A mapping $f:X\to Y$ is called an isometry if f satisfies the equality

$$\|f(x)-f(y)\| = \|x-y\| \tag{1.1}$$

for all $x,y\in X$. A distance $r>0$ is said to be preserved (conserved) by a mapping $f:X\to Y$ if

$$\|f(x)-f(y)\| = r \quad \forall x,y\in X \text{ with } \|x-y\| = r. \tag{1.2}$$

If f is an isometry, then every distance $r>0$ is conserved by f, and vice versa. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, Aleksandrov [1] had raised a question whether a mapping $f:X\to X$ preserving a distance $r>0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $r=1$ when X is a normed space (see [16]).

Beckman and Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X=\mathbb{R}^n$ (see also [3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 17, 18, 19, 20]).

Theorem 1.1 (Beckman and Quarles). *If a mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ ($2\le n<\infty$) preserves a distance $r>0$, then f is a linear isometry up to translation.*

Recently, Zaks [25] proved the rational analogues of the Beckman-Quarles theorem. Indeed, he assumes that $n=4k(k+1)$ for some $k\ge 1$ or $n=2m^2-1$ for some $m\ge 3$, and he proves that if a mapping $f:\mathbb{Q}^n\to\mathbb{Q}^n$ preserves the unit distance, then f is an isometry (see also [21, 22, 23, 24]).

It seems interesting to investigate whether the “distance $r>0$” in the Beckman-Quarles theorem can be replaced by some properties characterized by “geometrical figures” without loss of its validity.

In [9], the first author proved that if a one-to-one mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ ($n\ge 2$) maps every regular triangle (quadrilateral or hexagon) of side length $a>0$ onto a figure of

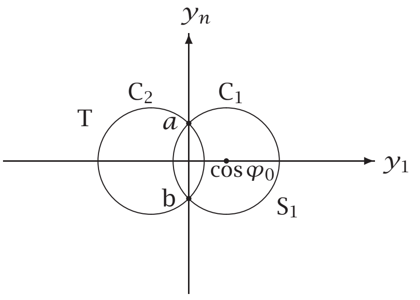

FIGURE 2.1

the same type with side length $b > 0$, then there exists a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ up to translation such that

$$f(x) = \frac{b}{a} I(x). \tag{1.3}$$

Furthermore, the first author proved that if a one-to-one mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps every unit circle onto a unit circle, then f is a linear isometry up to translation (see [10]).

In this connection, we will extend the result of [10] to the n-dimensional cases; more precisely, we prove in this paper that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ maps every unit circle onto a unit circle, then f is a linear isometry up to translation.

2. Preliminaries. We start with any two distinct points a and b in $\mathbb{R}^n$ with the distance between the two less than 2. Let their distance be

$$2c = 2\sin\varphi_0 \quad \text{with } 0 < \varphi_0 < \frac{\pi}{2},\ 0 < c < 1. \tag{2.1}$$

Given such two distinct points whose distance is less than 2, we can choose a coordinate $(y_1, \ldots, y_n)$ for $\mathbb{R}^n$ such that

$$a = (0, \ldots, 0, \sin\varphi_0), \qquad \mathrm{b} = (0, \ldots, 0, -\sin\varphi_0). \tag{2.2}$$

Let the $(n-2)$-dimensional unit sphere contained in the space orthogonal to the y_n-direction be

$$\mathrm{Y} = \{(y_1, \ldots, y_{n-1}, 0) \mid y_1^2 + \cdots + y_{n-1}^2 = 1\}. \tag{2.3}$$

If we call the center of any unit circle passing through the two points (a and b) o$'$ and the origin of the coordinate o, then the vector $\overrightarrow{\mathrm{oo}'}$ is perpendicular to the y_n-axis and its length must be $\cos\varphi_0$ and therefore $\overrightarrow{\mathrm{oo}'} \in \tilde{\mathrm{Y}} = \cos\varphi_0 \mathrm{Y}$, see Figure 2.1. It means that any unit circle passing through the points a and b has its center in $\tilde{\mathrm{Y}} = \cos\varphi_0 \mathrm{Y}$. Let T be the set of union of all the unit circles passing through the points a and b. More precisely, if we define the following set:

$$\mathrm{T} = \{(\cos\varphi + \cos\varphi_0)y + (0, \ldots, 0, \sin\varphi) \mid y \in \mathrm{Y},\ 0 \leq \varphi < 2\pi\}, \tag{2.4}$$

then it is clear that this is the set of union of all the unit circles which are centered at $\cos\varphi_0 y$ for each fixed $y \in \mathrm{Y}$ and which pass through a and b when $\varphi = \pi \mp \varphi_0$ (see Figure 2.1).

The intersection of T and the y_1-y_n plane consists of two circles, say C_1 (when $y_1 = 1$, i.e., $y = (1,0,\dots,0)$) and C_2 (when $y_1 = -1$, i.e., $y = (-1,0,\dots,0)$, see Figure 2.1). In the following contexts, we will consider the cases $y_1 = 1$ and -1 in connection with T as the circles C_1 and C_2, respectively. Call S_1 the $(n-1)$-dimensional unit sphere containing the circle C_1. If we let the center of C_1 be O and the center of S_1 be $\tilde{\mathrm{O}}$, then it is obvious that $\mathrm{O} = \tilde{\mathrm{O}}$.

(To see this, choose any point $\mathrm{A} \in \mathrm{C}_1$ and its antipodal point B in C_1. Then, by the definition of the antipodal points that they lie exactly the opposite with respect to the center of the circle C_1 whose center is at O, and because they are of the same length 1, we have the following condition that

$$\overrightarrow{\mathrm{OA}} = -\overrightarrow{\mathrm{OB}}, \qquad \overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OB}} = 2\,\overrightarrow{\mathrm{OB}}. \tag{2.5}$$

On the other hand, we have, since the two points A and B lie also on the unit sphere S_1 with its center at $\tilde{\mathrm{O}}$,

$$2 = \left|\overrightarrow{\mathrm{AB}}\right| = \left|\overrightarrow{\mathrm{A\tilde{O}}} + \overrightarrow{\mathrm{\tilde{O}B}}\right| \le \left|\overrightarrow{\mathrm{A\tilde{O}}}\right| + \left|\overrightarrow{\mathrm{\tilde{O}B}}\right| = 1+1 = 2. \tag{2.6}$$

Therefore, by the Cauchy-Schwarz inequality, $\overrightarrow{\mathrm{A\tilde{O}}}$ is a positive multiple of $\overrightarrow{\mathrm{\tilde{O}B}}$, which means $\overrightarrow{\mathrm{A\tilde{O}}} = \overrightarrow{\mathrm{\tilde{O}B}}$ because their lengths are both 1. So,

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{A\tilde{O}}} + \overrightarrow{\mathrm{\tilde{O}B}} = 2\,\overrightarrow{\mathrm{\tilde{O}B}}, \tag{2.7}$$

and therefore $\tilde{\mathrm{O}} = \mathrm{O}$.)

Now, we first show that S_1 and T intersect only at C_1. To make computation simpler we use a new coordinate x for $\mathbb{R}^n$, where

$$x = y - (\cos\varphi_0, 0, \dots, 0). \tag{2.8}$$

In this coordinate (see Figure 2.2), S_1 becomes the unit sphere S centered at the origin,

$$\mathrm{S}_1 = S = \{(x_1,\dots,x_n) \mid x_1^2 + \cdots + x_n^2 = 1\}, \tag{2.9}$$

$$\begin{aligned}\mathrm{T} = \{x = &(\cos\varphi + \cos\varphi_0)y + (0,\dots,0,\sin\varphi)\\ &-(\cos\varphi_0,0,\dots,0) \mid y \in \mathrm{Y},\ 0 \le \varphi < 2\pi\}.\end{aligned} \tag{2.10}$$

With the help of this coordinate we show the following lemma.

Lemma 2.1. $\mathrm{T} \cap \mathrm{S}_1 = \mathrm{C}_1$.

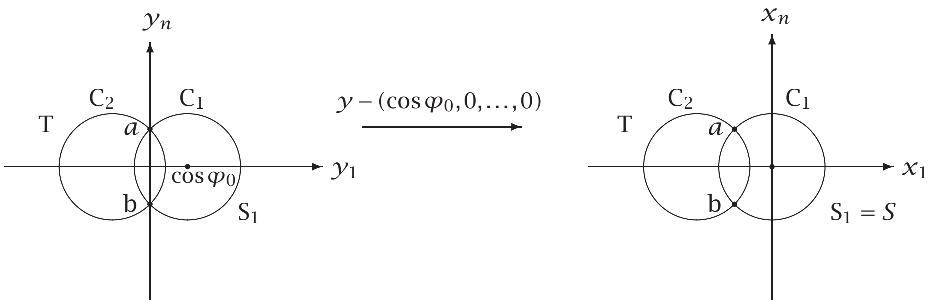

FIGURE 2.2

PROOF. If any element in T has distance 1 from the origin of the x-coordinate, then we have

$$\begin{aligned} 1 &= \left[(\cos\varphi+\cos\varphi_0)y_1-\cos\varphi_0\right]^2+(\cos\varphi+\cos\varphi_0)^2y_2^2 \\ &\quad +\cdots+(\cos\varphi+\cos\varphi_0)^2y_{n-1}^2+\sin^2\varphi \\ &= (\cos\varphi+\cos\varphi_0)^2-2\cos\varphi_0(\cos\varphi+\cos\varphi_0)y_1+\cos^2\varphi_0+\sin^2\varphi \\ &= 1+2\cos^2\varphi_0(1-y_1)+2\cos\varphi_0\cos\varphi(1-y_1). \end{aligned} \tag{2.11}$$

Therefore, we have

$$0 = 2\cos\varphi_0(1-y_1)(\cos\varphi+\cos\varphi_0). \tag{2.12}$$

With $y_1 = 1$, T in (2.10) represents the unit circle C_1 in the x_1-x_n plane. If

$$\cos\varphi = -\cos\varphi_0, \qquad \text{i.e., } \varphi = \pi\mp\varphi_0, \tag{2.13}$$

then it follows from (2.10) that

$$\mathrm{T} = \{x = (-\cos\varphi_0,0,\dots,0,\pm\sin\varphi_0)\} = \{a,\mathrm{b}\} \tag{2.14}$$

which also belong to C_1. □

Now, consider, as in Figure 2.3, the origin e and $\tilde{\mathrm{e}} = (-2,0,\dots,0)$ in the x-coordinate and the unit circle C_1 passing through e and ẽ in the x_1-x_n plane. Choose a point $\mathrm{d}\in C_1$, $\mathrm{d}\notin\{\mathrm{e},\tilde{\mathrm{e}}\}$. We parameterize all the unit circles passing through the points e and d. We assume the x_n-coordinate of d is negative.

By triangle inequality, the distance between e and d is less than 2, say $2\sin\varphi_0$, with $0<\varphi_0<\pi/2$. Choose a new coordinate y for $\mathbb{R}^n$ and consider two points

$$\mathrm{e}' = (0,\dots,0,\sin\varphi_0), \qquad \mathrm{d}' = (0,\dots,0,-\sin\varphi_0), \tag{2.15}$$

(see Figure 2.4).

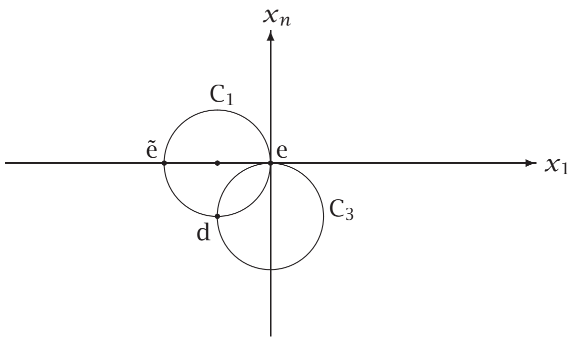

FIGURE 2.3

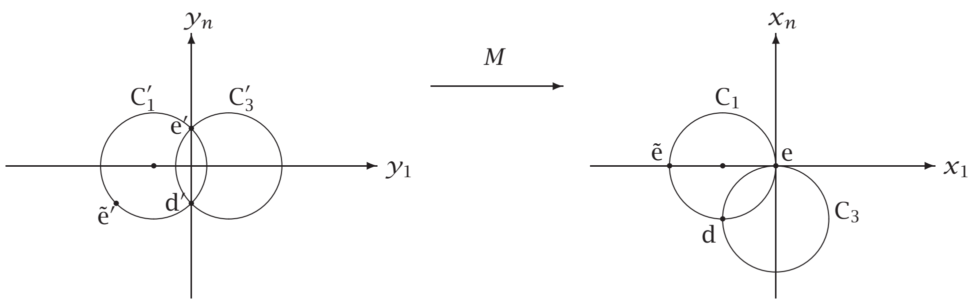

FIGURE 2.4

To get a parameterization of the unit circles passing through e and d, we consider the mapping M defined by

$$x = My = \begin{bmatrix} \cos\varphi_0 & 0 & \cdots & 0 & \sin\varphi_0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\sin\varphi_0 & 0 & \cdots & 0 & \cos\varphi_0 \end{bmatrix} \left[y + (\cos\varphi_0, 0, \ldots, 0)\right]^t - (1, 0, \ldots, 0)^t. \tag{2.16}$$

This transformation M is an isometry (since it is a composition of a rotation and translations) and sends

$$\{y = (0, \ldots, 0, \pm\sin\varphi_0)\} = \{e', d'\} \tag{2.17}$$

to

$$\{x = (0, \ldots, 0),\ x = (\cos(-2\varphi_0) - 1, 0, \ldots, 0, \sin(-2\varphi_0))\} = \{e, d\} \tag{2.18}$$

and therefore it sends any unit circle passing through e′ and d′ to a unit circle passing through e and d.

Therefore, by comparing Figure 2.4 with Figure 2.1 and considering (2.4), all the unit circles passing through e and d can be parameterized as

$$\{x = My \mid y = (\cos\varphi + \cos\varphi_0)y' + (0,\dots,0,\sin\varphi),\ y' \in \mathrm{Y},\ 0 \le \varphi < 2\pi\}. \tag{2.19}$$

With the help of this parameterization, we are ready to show the following lemma.

Lemma 2.2. *For* $\mathrm{d} \in \mathrm{C}_1$, $\mathrm{d} \notin \{\mathrm{e}, \tilde{\mathrm{e}}\}$, *any unit circle in* $\mathbb{R}^n$, *which passes through* d *and* e, *has some point whose* x_1*-coordinate is positive, except the circle* C_1.

Proof. Without loss of generality, we can assume the x_n-coordinate of d is negative. Note that with $\varphi = \pi \mp \varphi_0$ in (2.19), $y = (0,\dots,0,\pm\sin\varphi_0)$ are the points e′ or d′ in the y-coordinate and further $\varphi = \pi \mp \varphi_0$ means that

$$x = (0,\dots,0) = \mathrm{e}, \qquad x = (\cos(-2\varphi_0) - 1, 0,\dots,0, \sin(-2\varphi_0)) = \mathrm{d} \tag{2.20}$$

in the x-coordinate, regardless of $y' \in \mathrm{Y}$. Any unit circle passing through e and d is given as $x = My$ with y given as in (2.19), that is,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 & 0 & \cdots & 0 & \sin\varphi_0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\sin\varphi_0 & 0 & \cdots & 0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} (\cos\varphi + \cos\varphi_0)y_1' + \cos\varphi_0 \\ (\cos\varphi + \cos\varphi_0)y_2' \\ \vdots \\ (\cos\varphi + \cos\varphi_0)y_{n-1}' \\ \sin\varphi \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{2.21}$$

The first coordinate is

$$x_1 = \cos\varphi_0(\cos\varphi + \cos\varphi_0)y_1' + \cos^2\varphi_0 + \sin\varphi_0 \sin\varphi - 1. \tag{2.22}$$

We show that for $y_1' \ne -1$ ($y_1' = -1$ means the circle C_1' in the y-coordinate and the circle C_1 in the x-coordinate, see Figure 2.4), there is always some φ near $\pi - \varphi_0$ (i.e., near the point e) such that the above x_1 becomes positive.

Let

$$\theta = (\pi - \varphi_0) - \varphi = \pi - (\varphi + \varphi_0), \tag{2.23}$$

and so

$$\varphi = \pi - (\theta + \varphi_0). \tag{2.24}$$

Then, the above is

$$\begin{aligned}x_1 &= -\cos\varphi_0\cos(\theta+\varphi_0)y_1' + \cos^2\varphi_0(1+y_1') + \sin\varphi_0\sin(\theta+\varphi_0) - 1\\ &= -\cos\varphi_0[\cos\theta\cos\varphi_0 - \sin\theta\sin\varphi_0]y_1' + \sin\varphi_0[\sin\theta\cos\varphi_0 + \cos\theta\sin\varphi_0]\\ &\qquad -1 + \cos^2\varphi_0(1+y_1')\\ &= \sin\theta\sin\varphi_0\cos\varphi_0(1+y_1') + \cos\theta\sin^2\varphi_0 - \cos\theta\cos^2\varphi_0 y_1'\\ &\qquad -1 + \cos^2\varphi_0(1+y_1')\\ &= \sin\theta\sin\varphi_0\cos\varphi_0(1+y_1') + \cos\theta - \cos\theta\cos^2\varphi_0(1+y_1')\\ &\qquad -[1-\cos^2\varphi_0(1+y_1')]\\ &= \sin\theta\sin\varphi_0\cos\varphi_0(1+y_1') - [1-\cos^2\varphi_0(1+y_1')](1-\cos\theta).\end{aligned} \tag{2.25}$$

$\theta = 0$ ($\varphi = \pi - \varphi_0$) means the intersection point e and the above x_1 becomes 0 as it should. Assume

$$\theta \neq 0 \quad (-\pi - \varphi_0 < \theta < 0,\ 0 < \theta \leq \pi - \varphi_0). \tag{2.26}$$

Then, x_1 is positive if and only if

$$\sin\theta\sin\varphi_0\cos\varphi_0(1+y_1') > [1-\cos^2\varphi_0(1+y_1')](1-\cos\theta), \tag{2.27}$$

that is,

$$\frac{\sin\theta}{1-\cos\theta} > \frac{1-\cos^2\varphi_0(1+y_1')}{\sin\varphi_0\cos\varphi_0(1+y_1')} \tag{2.28}$$

(recall $y_1' \neq -1$ and $0 < \varphi_0 < \pi/2$). In other words, the x_1-coordinate is positive if and only if

$$\cot\frac{\theta}{2} > \frac{1-\cos^2\varphi_0(1+y_1')}{\sin\varphi_0\cos\varphi_0(1+y_1')}. \tag{2.29}$$

Therefore, for $y_1' \neq -1$ (i.e., except the circle C_1), the x_1-coordinate is positive for small enough $\theta > 0$. □

3. Main theorem.

In the previous section, we introduced all preliminary lemmas for the main result of this paper. Now, we prove our main theorem.

Theorem 3.1. *If a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ maps every unit circle onto a unit circle, then f is a linear isometry up to translation.*

Proof. We show f preserves the distance 2. Suppose the distance between $a = f(A)$ and $b = f(B)$ is less than 2, while the distance between A and B is 2—see Figure 3.1. Then, we show it leads to a contradiction.

Let the distance between a and b be $2c$ $(0 < c < 1)$. Choose any unit circle C passing through A and B and let $f(C) = C_1$. Choose a coordinate for a and b as in Figure 3.1 such that C_1 lies in the x_1-x_n plane and

$$a = \left(-1-\sqrt{1-c^2}, 0, \ldots, 0, c\right), \qquad b = \left(-1-\sqrt{1-c^2}, 0, \ldots, 0, -c\right). \tag{3.1}$$

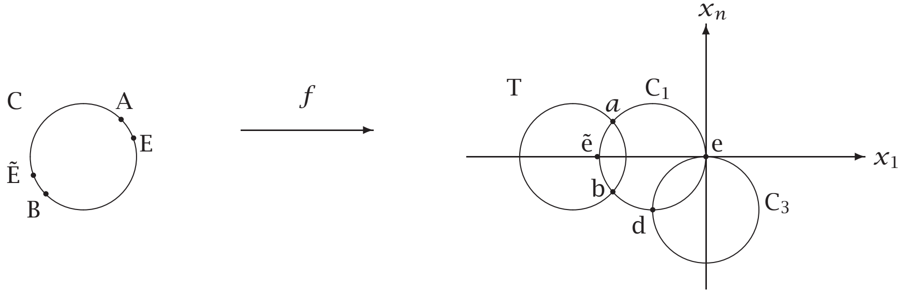

FIGURE 3.1

Let

$$\mathrm{e} = (0,\ldots,0), \qquad \tilde{\mathrm{e}} = (-2,0,\ldots,0). \tag{3.2}$$

Let $f(\mathrm{E}) = \mathrm{e}$ and Ẽ the antipodal point (in C) of E and let $f(\tilde{\mathrm{E}}) = \mathrm{d}$. Let the union of all the unit circles passing through a and b be T and the $(n-1)$-dimensional unit sphere passing through A and B be S and the $(n-1)$-dimensional unit sphere passing through e and ẽ be S_1.

Then, it is clear that any point P on S ($\mathrm{P} \notin \{\mathrm{A},\mathrm{B}\}$) lies in some unit circle determined by the three points A, B, and P. To see this, if we call O the common center of C and S, and let

$$\langle \overrightarrow{\mathrm{OP}}, \overrightarrow{\mathrm{OA}} \rangle = \sin\varphi_0 \quad \left(-\frac{\pi}{2} < \varphi_0 < \frac{\pi}{2}\right), \tag{3.3}$$

then the unit circle determined by these three points is parameterized as

$$\overrightarrow{\mathrm{OV}}(\varphi) = \cos\varphi\left(\frac{\overrightarrow{\mathrm{OP}} - \sin\varphi_0\,\overrightarrow{\mathrm{OA}}}{\cos\varphi_0}\right) + \sin\varphi\,\overrightarrow{\mathrm{OA}} \quad (-\pi < \varphi \le \pi). \tag{3.4}$$

Note that

$$\left\{\left(\frac{\overrightarrow{\mathrm{OP}} - \sin\varphi_0\,\overrightarrow{\mathrm{OA}}}{\cos\varphi_0}\right), \overrightarrow{\mathrm{OA}}\right\} \tag{3.5}$$

are orthonormal to each other and

$$\begin{gathered} \overrightarrow{\mathrm{OV}}(\varphi_0) = \overrightarrow{\mathrm{OP}}, \qquad \overrightarrow{\mathrm{OV}}\left(\frac{\pi}{2}\right) = \overrightarrow{\mathrm{OA}}, \\ \overrightarrow{\mathrm{OV}}\left(-\frac{\pi}{2}\right) = -\overrightarrow{\mathrm{OA}} = \overrightarrow{\mathrm{OB}}. \end{gathered} \tag{3.6}$$

Since the image of this unit circle lies in T, it follows that the image of the whole S under f lies in T.

It is also obvious that the x_1-coordinate of any point in T is nonpositive. (Note that the center of any unit circle passing through a and b has coordinate

$$\sqrt{1-c^2}\,y-\left(1+\sqrt{1-c^2},0,\ldots,0\right) \quad \text{for some } y \in \mathrm{Y}, \tag{3.7}$$

(see (2.4)) and the distance between this center and any $x=(x_1,\ldots,x_n)$ is

$$\sqrt{\left(x_1+1+\sqrt{1-c^2}(1-y_1)\right)^2+\cdots} \tag{3.8}$$

and because

$$\sqrt{1-c^2}(1-y_1)\geq 0, \tag{3.9}$$

positive x_1 makes the distance larger than 1, which means that if $x_1>0$, we have $x\notin \mathrm{T}$.)

Now, if d = ẽ, then the image of any unit circle passing through E and Ẽ lies in both T and S_1. However, by Lemma 2.1, $\mathrm{T}\cap\mathrm{S}_1=\mathrm{C}_1$ and this fact contradicts the injectivity of f.

On the other hand, if d ≠ ẽ, the image of any unit circle, except the circle C, passing through E and Ẽ is a unit circle passing through e and d. This unit circle is not C_1 since f is one-to-one, and by Lemma 2.2 it cannot stay completely in T, a contradiction.

Consequently, f preserves the distance 2. According to the well-known theorem of Beckman and Quarles, f is a linear isometry up to translation. □

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