ASYMPTOTIC BEHAVIOR FOR A NONLINEAR VISCOELASTIC PROBLEM WITH A VELOCITY-DEPENDENT MATERIAL DENSITY

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We consider a nonlinear viscoelastic problem and prove that the solutions are uniformly bounded and decay exponentially to zero as time goes to infinity. This is established under weaker conditions on the relaxation function than the usually used ones. In particular, we remove the assumptions on the derivative of the kernel. In fact, our kernels are not necessarily differentiable.

1. Introduction

The problem we would like to investigate is the following:

$$\begin{gathered} |u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds - \gamma \Delta u_t = 0 \quad \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 1$, with a smooth boundary Γ. The real number ρ is assumed to satisfy $0 < \rho \le 2/(n-2)$ if $n \ge 3$ or $\rho > 0$ if $n = 1,2$. The function $g(t)$ is positive and will be specified further below.

This model appears in viscoelasticity. We are in the case where the material density depends on u_t (see [5, 11]). In [1], Cavalcanti et al. studied this nonlinear problem ($\rho > 0$) and proved well posedness as well as a uniform decay result. It has been shown that solutions go to zero in an exponential manner provided that the kernel $g(t)$ is also exponentially decaying to zero. Namely, the following assumptions were assumed:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^1-function such that

$$1 - \int_0^{\infty} g(s)\,ds = l > 0, \tag{1.2}$$

(H2) there exist positive constants ξ_1, ξ_2 such that

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t) \tag{1.3}$$

for all $t \ge 0$.

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These two assumptions are in fact frequently used also in the linear case ($\rho = 0$) (see [3, 4, 5, 6, 7, 8] and also [10, 13]). In [9], the present author with Messaoudi have improved the result in [1] by showing that the same asymptotic behavior occurs also for the case $\gamma = 0$. This means that the convolution term produces a weak dissipation which is able to drive solutions to the equilibrium state in an exponential manner. We do not need the strong damping. In [6], the present author with Furati proved that for "sufficiently small" g and g', we also have exponential decay (in case $\rho > 0$). Namely, we need $e^{\alpha t}g(t)$ and $e^{\alpha t}g'(t)$ to have "small" L^1-norms for some $\alpha > 0$. The conditions in (H2) are not imposed. In particular, g' is not necessarily always negative.

Here in this work, we intend to improve further this latter result by removing the condition on g'. To this end, we combine the multiplier technique with some appropriate estimations and some new "Lyapunov-type" functionals. These functionals are somewhat similar in spirit to the one introduced by the author in [12].

The plan of the paper is as follows. In the next section, we state an existence theorem, introduce our functionals, and prove some useful propositions for our result. Section 3 is devoted to the exponential decay theorem.

2. Preliminaries

We start by stating an existence result due to Cavalcanti et al. [1] (see also [2]).

Theorem 2.1. *Assume that the kernel* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* $1 - \int_0^\infty g(s)ds = l > 0$. *Let* $u_0, u_1 \in H_0^1(\Omega)$ *and* $\gamma \geq 0$. *Then, problem (1.1) possesses at least one weak solution* $u : \Omega \times (0,\infty) \to \mathbb{R}$ *in the class*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)), \qquad u' \in L^\infty(0,\infty; H_0^1(\Omega)), \qquad u'' \in L^2(0,\infty; H_0^1(\Omega)). \tag{2.1}$$

We point out here that the differentiability of g is not needed to prove local existence. In this paper, we consider $\gamma \neq 0$. We may assume that $\gamma = 1$.

The (classical) energy associated to problem (1.1) is defined by

$$E(t) := \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx. \tag{2.2}$$

If we differentiate $E(t)$ with respect to t along solutions of (1.1), we get

$$E'(t) = \int_\Omega \nabla u_t \int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_\Omega |\nabla u_t|^2 dx. \tag{2.3}$$

This expression is of an undefined sign, and therefore the boundedness (and the dissipativity) of the energy functional $E(t)$ is not clear. In the prior works, the authors defined

$$(g\Box\nabla u)(t) := \int_\Omega \int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx \tag{2.4}$$

and observed that

$$\int_\Omega \nabla u_t \int_0^t g(t-s)\nabla u(s)ds\,dx = \frac{1}{2}(g'\Box\nabla u)(t) - \frac{1}{2}(g\Box\nabla u)'(t) + \frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t g(s)ds\right)\int_\Omega |\nabla u_t|^2 dx\right\} - \frac{1}{2}g(t)\int_\Omega |\nabla u_t|^2 dx. \tag{2.5}$$

Then, considering the modified energy functional

$$\begin{aligned}\mathscr{E}(t) = {} & \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\left(1-\int_0^t g\,ds\right)\int_\Omega |\nabla u|^2 dx \\ & + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(g\Box\nabla u)(t),\end{aligned} \tag{2.6}$$

it appears that

$$\mathscr{E}'(t) = -\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(g'\Box\nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx. \tag{2.7}$$

At this point, they use the fact that $g'(t) \le 0$ to obtain uniform boundedness. In our case, we do not have this assumption. To overcome this, a new functional has been proposed in [6]. An exponential decay result has been obtained under some "smallness" condition on $g(t)$ and $g'(t)$. It is our objective here to remove the smallness condition on $g'(t)$. In fact, even the differentiability of g is not required. We will need the assumptions

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded continuous function such that

$$1 - \int_0^\infty g(s)ds = l > 0, \tag{2.8}$$

(G2) $g(t)e^{\alpha t} \in L^1(0,\infty)$ for some $\alpha > 0$.

We will use repeatedly the following inequality.

Lemma 2.2. *For any $a, b \in \mathbb{R}$ and $\delta > 0$,*

$$ab \le \delta a^2 + \frac{1}{4\delta}b^2. \tag{2.9}$$

We denote

$$\bar{g} := \int_0^{+\infty} g(s)ds, \qquad \bar{g}_\alpha := \int_0^{+\infty} e^{\alpha s}g(s)ds. \tag{2.10}$$

Next, we prove the uniform boundedness of the classical energy.

Proposition 2.3. *Assume that* (G1) and (G2) hold. If g is such that $\bar{g}_\alpha \le \alpha/2$, then

$$E(t) \le E(0) \tag{2.11}$$

for all $t \ge 0$.

Proof. We have

$$\begin{aligned}\int_\Omega \nabla u_t \int_0^t g(t-s)\nabla u(s)\,ds\,dx &= \left(\int_0^t g(s)ds\right)\int_\Omega \nabla u_t \nabla u\,dx \\ &\quad + \int_\Omega \nabla u_t \int_0^t g(t-s)[\nabla u(s)-\nabla u(t)]\,ds\,dx,\end{aligned} \tag{2.12}$$

$$\left(\int_0^t g\,ds\right)\int_\Omega \nabla u_t \nabla u\,dx = \frac{1}{2}\frac{d}{dt}\left[\left(\int_0^t g\,ds\right)\int_\Omega |\nabla u|^2 dx\right] - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx. \tag{2.13}$$

From (2.12) and Lemma 2.2 with $\delta = 1/4$, we find

$$\begin{aligned}\int_\Omega \nabla u_t \int_0^t g(t-s)\nabla u(s)\,ds\,dx &\le \frac{1}{4}\int_\Omega |\nabla u_t|^2 dx - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx \\ &\quad + \bar{g}(g\Box\nabla u)(t) + \frac{1}{2}\frac{d}{dt}\left[\left(\int_0^t g(s)ds\right)\int_\Omega |\nabla u|^2 dx\right].\end{aligned} \tag{2.14}$$

Considering

$$e(t) := \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2} dx + \frac{1}{2}\left(1 - \int_0^t g\,ds\right)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx, \tag{2.15}$$

a simple computation shows, with the help of (2.14), that

$$e'(t) \le -\frac{3}{4}\int_\Omega |\nabla u|^2 dx + \bar{g}(g\Box\nabla u)(t). \tag{2.16}$$

Next, we introduce the functional

$$\Phi(t) := \int_\Omega \int_0^t G_\alpha(t-s)|\nabla u(t) - \nabla u(s)|^2 ds\,dx =: (G_\alpha \Box \nabla u)(t) \tag{2.17}$$

with

$$G_\alpha(t) := e^{-\alpha t}\int_t^{+\infty} e^{\alpha s} g(s)ds \tag{2.18}$$

for some $\alpha > 0$. A differentiation of (2.17) yields

$$\Phi'(t) = -\alpha\Phi(t) - (g\Box\nabla u)(t) + 2\int_\Omega \nabla u_t \int_0^t G_\alpha(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx. \tag{2.19}$$

By Lemma 2.2 with $\delta = 1/8\lambda$, for some $\lambda > 0$ to be determined, we have

$$\begin{aligned}&\int_\Omega \nabla u_t \int_0^t G_\alpha(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\quad \le \frac{1}{8\lambda}\int_\Omega |\nabla u_t|^2 dx + 2\lambda\left(\int_0^t G_\alpha(s)ds\right)(G_\alpha\Box\nabla u)(t).\end{aligned} \tag{2.20}$$

Notice that

$$\int_0^t G_\alpha(s)ds \le \frac{1}{\alpha}\int_0^\infty e^{\alpha s}g(s)ds = \frac{\bar{g}_\alpha}{\alpha}. \tag{2.21}$$

Therefore,

$$\begin{aligned}\Phi'(t) &\le -\alpha\Phi(t) - (g\Box\nabla u)(t) + \frac{1}{4\lambda}\int_\Omega |\nabla u_t|^2 dx + \frac{4\lambda\bar{g}_\alpha}{\alpha}\Phi(t)\\ &\le -\left(\alpha - \frac{4\lambda\bar{g}_\alpha}{\alpha}\right)\Phi(t) - (g\Box\nabla u)(t) + \frac{1}{4\lambda}\int_\Omega |\nabla u_t|^2 dx.\end{aligned} \tag{2.22}$$

We define

$$V(t) := e(t) + \lambda\Phi(t). \tag{2.23}$$

Clearly, by (2.16) and (2.22), we have

$$V'(t) \le -\frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + (\bar{g} - \lambda)(g\Box\nabla u)(t) - \lambda\left(\alpha - \frac{4\lambda\bar{g}_\alpha}{\alpha}\right)\Phi(t). \tag{2.24}$$

If $\bar{g}_\alpha \le \alpha/2$, then it is possible to choose λ so that $\lambda \ge \bar{g}_\alpha$ (notice that $\bar{g}_\alpha > \bar{g}$) and $\lambda \le \alpha^2/4\bar{g}_\alpha$. Hence, $V'(t) \le 0$. Consequently, $e(t)$ and thereafter $E(t)$ are uniformly bounded for all $t \ge 0$ by $e(0)$. □

This proposition will be used in a crucial manner in our main result. However, the functional $V(t)$ is still not suitable to work with. We introduce

$$\begin{aligned}\Psi(t) &:= \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\, dx + \int_\Omega \nabla u \nabla u_t dx,\\ \chi(t) &:= \int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t G_\alpha(t-s)(u(t) - u(s))\,ds\,dx.\end{aligned} \tag{2.25}$$

Then, we form the expression

$$W(t) := V(t) + \varepsilon\big(\Psi(t) + \chi(t)\big), \quad t \ge 0, \tag{2.26}$$

for some $\varepsilon > 0$ to be determined later.

The next proposition will show, in particular, that the result we will derive for $W(t)$ will also hold for the classical energy.

Proposition 2.4. *There exist an ε_0, m, and $M > 0$ such that*

$$mE(t) \le W(t) \le M\big(E(t) + \Phi(t)\big), \quad t \ge 0, \tag{2.27}$$

for all $\lambda \ge \bar{g}_\alpha$ and $0 < \varepsilon \le \varepsilon_0$.

Proof. We begin by the left inequality. Observe first that by the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ for $0 < \rho \le 2/(n-2)$ if $n \ge 3$ or $\rho > 0$ if $n = 1,2$, we can write

$$\int_\Omega |u_t|^{2(\rho+1)} dx \le C_e \left(\int_\Omega |\nabla u_t|^2 dx \right)^{\rho+1}, \tag{2.28}$$

where $C_e > 0$ is the embedding constant (the subscript "e" is for embedding). Further, in virtue of Proposition 2.3, we get

$$\int_\Omega |u_t|^{2(\rho+1)} dx \le C_e (2e(0))^\rho \int_\Omega |\nabla u_t|^2 dx = \widetilde{C} \int_\Omega |\nabla u_t|^2 dx, \tag{2.29}$$

where $\widetilde{C} = C_e(2e(0))^\rho$. This relation, together with Lemma 2.2, implies that

$$\begin{aligned} \int_\Omega |u_t|^\rho u_t u \, dx &\le \delta_2 \int_\Omega |u_t|^{2(\rho+1)} dx + \frac{C_p}{4\delta_2} \int_\Omega |\nabla u|^2 dx \\ &\le \widetilde{C}\delta_2 \int_\Omega |\nabla u_t|^2 dx + \frac{C_p}{4\delta_2} \int_\Omega |\nabla u|^2 dx, \quad \delta_2 > 0, \end{aligned} \tag{2.30}$$

where C_p is the Poincaré constant (the subscript "p" is for Poincaré), and

$$\begin{aligned} &\int_\Omega |u_t|^\rho u_t \int_0^t G_\alpha(t-s)(u(t) - u(s)) \, ds \, dx \\ &\quad \le \delta_3 \widetilde{C} \int_\Omega |\nabla u_t|^2 dx + \frac{C_p \bar{g}_\alpha}{4\delta_3 \alpha} (G_\alpha \Box \nabla u)(t), \quad \delta_3 > 0. \end{aligned} \tag{2.31}$$

Gathering (2.30), (2.31), (2.20) with δ_1 and

$$\int_\Omega \nabla u \nabla u_t dx \le \delta_4 \int_\Omega |\nabla u_t|^2 dx + \frac{1}{4\delta_4} \int_\Omega |\nabla u|^2 dx, \quad \delta_4 > 0, \tag{2.32}$$

we obtain from (2.25) and (2.26) that

$$\begin{aligned} W(t) \ge{}& \frac{1}{\rho+2} \int_\Omega |u_t|^{\rho+2} dx + \frac{1}{2} \left[l - \frac{\varepsilon}{2} \left(\frac{C_p}{\delta_2(\rho+1)} + \frac{1}{\delta_4} \right) \right] \int_\Omega |\nabla u|^2 dx \\ &+ \left[\frac{1}{2} - \varepsilon \left(\delta_1 + \delta_4 + \frac{(\delta_2 + \delta_3)\widetilde{C}}{\rho+1} \right) \right] \int_\Omega |\nabla u_t|^2 dx \\ &+ \left[\lambda - \frac{\varepsilon \bar{g}_\alpha}{4\alpha} \left(\frac{1}{\delta_1} + \frac{C_p}{\delta_3(\rho+1)} \right) \right] \Phi(t). \end{aligned} \tag{2.33}$$

Taking for instance $\delta_1 = 1/5$, $\delta_2 = 3C_p/4(\rho+1)l$, $\delta_3 = (\rho+1)/5\widetilde{C}$, $\delta_4 = 3/4l$, and ε sufficiently small,

$$\varepsilon \le \varepsilon_0 := \frac{2}{5} \min \left\{ \frac{l}{3}, \frac{(\rho+1)^2 l}{3C_p\widetilde{C}}, \frac{2\alpha(\rho+1)^2}{(\rho+1)^2 + C_p\widetilde{C}} \right\}, \tag{2.34}$$

we find that $W(t) \ge mE(t)$, for all $t \ge 0$, for some positive constant m. The right-hand side inequality may be proved easily by taking for instance all the δ_i, $i = 1,2,3,4$, equal to $1/2$ and summing up the inequalities in (2.30), (2.31), (2.34), and (2.20) with their respective coefficients in the expression of $W(t)$. $\square$

3. Long-time behavior

In this section, we state and prove our main result. Observe that assuming the hypotheses in Proposition 2.3, we have uniqueness of the weak solution. The solution corresponding to $E(0) = 0$ is the trivial one and is included in our next result.

Theorem 3.1. *Assume that the kernel g satisfies* (G1) *and* (G2). *Then, the weak solution of (1.1) decays exponentially to zero, in the energy norm, provided that $\bar{g}_\alpha \leq \alpha\sqrt{2}/4$.*

Proof. We differentiate $W(t)$ (see (2.26)) along solutions of (1.1), we obtain from (2.26) and (2.24) that

$$W'(t) \leq -\frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + (\bar{g}-\lambda)(g\Box\nabla u)(t) - \lambda\left(\alpha - \frac{4\lambda\bar{g}_\alpha}{\alpha}\right)\Phi(t) + \varepsilon\Psi'(t) + \varepsilon\chi'(t), \tag{3.1}$$

with

$$\begin{aligned}\Psi'(t) = &-\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)ds\,dx - \int_\Omega \nabla u\nabla u_t dx\\ &+\int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\chi'(t) = &\int_\Omega \nabla u\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))ds\,dx\\ &+(1+\alpha)\int_\Omega \nabla u_t\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))ds\,dx\\ &-\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)ds\right)\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))ds\,dx\\ &+\int_\Omega \nabla u_t\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\,dx - \left(\int_0^t G_\alpha(s)ds\right)\int_\Omega |\nabla u_t|^2 dx\\ &+\frac{\alpha}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t G_\alpha(t-s)(u(t)-u(s))ds\,dx\\ &+\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)(u(t)-u(s))ds\,dx\\ &-\frac{1}{\rho+1}\left(\int_0^t G_\alpha(s)ds\right)\int_\Omega |u_t|^{\rho+2}dx.\end{aligned} \tag{3.3}$$

We estimate some terms in the expressions of $\Psi'(t)$ and $\chi'(t)$ separately. We denote by G_0 the value $G_0 := \int_0^{t_0} G_\alpha(s)ds$ for some $t_0 > 0$ (selected so that $E(t_0) > 0$, and thereafter by Proposition 2.4, $W(t_0) > 0$). Applying Lemma 2.2 with $\delta = l/4$, $\delta = l/4(1+\bar{g})$, $\delta = G_0/4(1+\alpha)$, and $\delta = G_0/4$, we obtain

$$\begin{aligned}&\int_\Omega \nabla u\int_0^t g(t-s)\nabla u(s)ds\,dx\\ &\quad\leq \bar{g}\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{l}{4}+\bar{g}\right)\int_\Omega |\nabla u|^2 dx + \frac{\bar{g}}{l}(g\square\nabla u)(t),\\
&\int_\Omega \nabla u \int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\leq \frac{l}{4(1+\bar{g})}\int_\Omega |\nabla u|^2 dx + \frac{1+\bar{g}}{l\alpha}\bar{g}_\alpha(G_\alpha\square\nabla u)(t),\\
&\int_\Omega \nabla u_t \int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\leq \frac{G_0}{4(1+\alpha)}\int_\Omega |\nabla u_t|^2 dx + \frac{(1+\alpha)\bar{g}_\alpha}{\alpha G_0}(G_\alpha\square\nabla u)(t),\\
&\int_\Omega \nabla u_t \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\leq \frac{G_0}{4}\int_\Omega |\nabla u_t|^2 dx + \frac{\bar{g}}{G_0}(g\square\nabla u)(t),
\end{aligned}
\tag{3.4}
$$

respectively.

We also have

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\leq \frac{1}{2}\int_\Omega \left|\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))ds\right|^2 dx\\
&\qquad\quad+\frac{1}{2}\int_\Omega \left|\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2 dx\\
&\qquad\quad+\left(\int_0^t g(s)ds\right)\int_\Omega \nabla u\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx.
\end{aligned}
\tag{3.5}
$$

Therefore,

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\leq \frac{\bar{g}}{2}(g\square\nabla u)(t)+\bar{g}\int_\Omega \nabla u\int_0^t G_\alpha(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\qquad\quad+\frac{\bar{g}_\alpha}{2\alpha}(G_\alpha\square\nabla u)(t).
\end{aligned}
\tag{3.6}
$$

By Lemma 2.2 again with $\delta = l/4$, we find

$$
\int_\Omega \nabla u\nabla u_t dx \leq \frac{l}{4}\int_\Omega |\nabla u|^2 dx + \frac{1}{l}\int_\Omega |\nabla u_t|^2 dx. \tag{3.7}
$$

Finally, by virtue of the embedding stated at the beginning of the proof of Proposition 2.4 (see (2.29)), we can deduce that

$$\begin{aligned}\int_\Omega |u_t|^\rho u_t \int_0^t G_\alpha(t-s)(u(t)-u(s))\,ds\,dx \\ \leq \frac{\widetilde{C}C_p}{(\rho+1)G_0}\bar{g}_\alpha(G_\alpha\Box\nabla u)(t) + \frac{(\rho+1)G_0}{4\alpha}\int_\Omega |\nabla u_t|^2 dx.\end{aligned} \tag{3.8}$$

Here we have used Lemma 2.2 with $\delta = \alpha\widetilde{C}/(\rho+1)G_0$. With $\delta = \widetilde{C}/(\rho+1)G_0$, we find that

$$\begin{aligned}\int_\Omega |u_t|^\rho u_t \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\ \leq \frac{\widetilde{C}C_p\bar{g}}{(\rho+1)G_0}(g\Box\nabla u)(t) + \frac{(\rho+1)G_0}{4}\int_\Omega |\nabla u_t|^2 dx.\end{aligned} \tag{3.9}$$

Taking into account all the above estimates (3.4), (3.6), (3.7), (3.8), (3.9), (3.2), and (3.3) in (3.1), we entail that for $t \geq t_0$,

$$\begin{aligned}W'(t) \leq & -\left[\frac{1}{2} - \varepsilon\left(1 + \frac{1}{l} + \frac{\widetilde{C}+C_p}{2(\rho+1)}\right)\right]\int_\Omega |\nabla u_t|^2 dx - \frac{\varepsilon l}{4}\int_\Omega |\nabla u|^2 dx \\ & - \left[\lambda - \bar{g} - \varepsilon\bar{g}\left(\frac{1}{2} + \frac{1}{l} + \frac{1}{G_0} + \frac{C_p\widetilde{C}}{(\rho+1)^2 G_0}\right)\right](g\Box\nabla u)(t) - \frac{\varepsilon G_0}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx \\ & - \left[\lambda\left(\alpha - \frac{4\lambda\bar{g}_\alpha}{\alpha}\right) - \frac{\varepsilon\bar{g}_\alpha}{\alpha}\left(\frac{(1+\bar{g})^2}{l} + \frac{(1+\alpha)^2}{G_0} + \frac{1}{2} + \frac{\alpha^2\widetilde{C}C_p}{G_0}\right)\right](G_\alpha\Box\nabla u)(t).\end{aligned} \tag{3.10}$$

We must point out here that, to avoid a contradiction, the term in $\int_\Omega |u_t|^{\rho+2} dx$ which appears in the derivative of $\Psi(t)$ (see (3.2)) has been estimated by

$$\begin{aligned}\frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx &\leq \frac{\varepsilon}{\rho+1}\left[\frac{1}{2}\int_\Omega |u_t|^{2(\rho+1)} dx + \frac{C_p}{2}\int_\Omega |\nabla u_t|^2 dx\right] \\ &\leq \frac{\varepsilon}{2(\rho+1)}(\widetilde{C}+C_p)\int_\Omega |\nabla u_t|^2 dx.\end{aligned} \tag{3.11}$$

From (3.10), it is clear that for sufficiently small ε and $\bar{g}_\alpha \leq \alpha^2/8\lambda$, there exists $C_1 > 0$ such that

$$W'(t) \leq -C_1(E(t) + \Phi(t)). \tag{3.12}$$

The right-hand side inequality in Proposition 2.4 implies that

$$W'(t) \leq -\frac{C_1}{M}W(t). \tag{3.13}$$

From this, we infer that

$$W(t) \leq W(t_0)e^{-C_1(t-t_0)/M}, \quad t \geq t_0. \tag{3.14}$$

Then, the left-hand side inequality in Proposition 2.4 allows us to conclude that

$$E(t) \leq \frac{W(t_0)}{m} e^{-C_1(t-t_0)/M}, \quad t \geq t_0. \tag{3.15}$$

This completes the proof of the theorem. □

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