UNIQUENESS OF NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING SIMPLE AND DOUBLE 1-POINTS

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We prove two theorems on the uniqueness of nonlinear differential polynomials, one of which improves a result of Fang and Hong.

1. Introduction, definitions, and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{k)}(a; f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$, we say that f, g share the value a CM (counting multiplicities).

During the last few years, a considerable amount of work is being done on the uniqueness problem concerning differential polynomials (cf. [1, 3, 5, 8]). Recently, Fang and Hong [1] proved the following result.

Theorem 1.1 [1]. *Let f and g be two transcendental entire functions and let $n(\geq 11)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

In the paper, we prove the following two theorems, the first of which improves Theorem 1.1.

Theorem 1.2. *Let f and g be two transcendental entire functions and let $n(\geq 10)$ be an integer. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$, then $f \equiv g$.*

Theorem 1.3. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > 4/(n+1)$ and let $n(\geq 17)$ be an integer. If $E_{2)}(1; f^n(f-1)f') = E_{2)}(1; g^n(g-1)g')$, then $f \equiv g$.*

The following example shows that the condition $\Theta(\infty; f) + \Theta(\infty; g) > 4/(n+1)$ is sharp for Theorem 1.3.

Example 1.4. Let

$$f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad g = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}, \tag{1.1}$$

where $\alpha = \exp(2\pi i/(n+2))$ and n is a positive integer.

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Then, $T(r,f) = (n+1)T(r,h)+O(1)$ and $T(r,g) = (n+1)T(r,h)+O(1)$. Further, we see that $h \neq \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence, $\Theta(\infty; f) = \Theta(\infty; g) = 2/(n+1)$. Also $f^{n+1}(f/(n+2) - 1/(n+1)) \equiv g^{n+1}(g/(n+2) - 1/(n+1))$ and $f^n(f-1)f' \equiv g^n(g-1)g'$ but $f \not\equiv g$.

Though we do not explain the standard notations of the value distribution theory (see [2]), we give the following definitions.

Definition 1.5 [4]. For $a \in \mathbb{C} \cup \{\infty\}$, denote by $N(r,a;f \mid = 1)$ the counting functions of simple a-points of f.

For a positive integer m, denote by $N(r,a;f \mid \leq m)$ $(N(r,a;f \mid \geq m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid \leq m)$ and $\overline{N}(r,a;f \mid \geq m)$ are defined similarly, where in counting the a-points of f, the multiplicities are ignored.

Also $N(r,a;f \mid < m)$, $N(r,a;f \mid > m)$, $\overline{N}(r,a;f \mid < m)$ and $\overline{N}(r,a;f \mid > m)$ are defined analogously.

Definition 1.6 [12]. For $a \in \mathbb{C} \cup \{\infty\}$, put

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geq 2) + \overline{N}(r,a;f \mid \geq 3) + \cdots + \overline{N}(r,a;f \mid \geq k), \tag{1.2}$$

where k is a positive integer.

For a meromorphic function f, we denote by $S(r,f)$ any function satisfying $S(r,f)/T(r,f) \to 0$ as $r \to \infty$ possibly outside a set of finite linear measure.

2. Lemmas

In this section, we present some lemmas which are needed in the sequel. We denote by h the function

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.1}$$

Lemma 2.1. *If $E_{1)}(1;f) = E_{1)}(1;g)$ and $h \not\equiv 0$, then*

$$N(r,1;f \mid \leq 1) = N(r,1;g \mid \leq 1) \leq N(r,0;h) \leq N(r,\infty;h) + S(r,f) + S(r,g). \tag{2.2}$$

Proof. Since the functions f and g have the same simple one-points, there exists a meromorphic function α such that $\alpha \neq 0$ when $f-1$ has a simple zero and α has no simple zero where $f \neq 1$ and $g = \alpha(f-1)+1$. It is now easy to verify by direct computation that the function h is zero whenever $f-1$ has a simple zero. This proves the lemma. □

Lemma 2.2. *If $E_{2)}(1;f) = E_{2)}(1;g)$ and $h \not\equiv 0$, then*

$$\begin{aligned} N(r,\infty;h) \leq{} & \overline{N}(r,\infty;f \mid \geq 2) + \overline{N}(r,0;f \mid \geq 2) + \overline{N}(r,\infty;g \mid \geq 2) \\ & + \overline{N}(r,0;g \mid \geq 2) + \overline{N}(r,1;f \mid \geq 3) + \overline{N}(r,1;g \mid \geq 3) \\ & + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned} \tag{2.3}$$

where $\overline{N}_0(r,0;f')$ *and* $\overline{N}_0(r,0;g')$ *are the reduced counting functions of the zeros of* f' *and* g' *which are not the zeros of* $f(f-1)$ *and* $g(g-1)$, *respectively.*

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f, g; (ii) multiple poles of f, g; (iii) zeros of $f-1$, $g-1$ with multiplicities greater than or equal to 3; (iv) zeros of f' which are not the zeros of $f(f-1)$; (v) zeros of g' which are not the zeros of $g(g-1)$.

Since all the poles of h are simple, the lemma follows from above. This proves the lemma. □

Lemma 2.3 [6]. *If* $N(r,0;f^{(k)} \mid f \neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid < k) + k\overline{N}(r,0;f \mid \geq k) + S(r,f). \tag{2.4}$$

Lemma 2.4. *If* $E_{2)}(1;f) = E_{2)}(1;g)$ *and* $h \not\equiv 0$, *then*

$$\begin{aligned} T(r,f)+T(r,g) \leq{} & \{3\overline{N}(r,0;f)+2\overline{N}(r,0;f \mid \geq 2)\} + \{3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;f \mid \geq 2)\} \\ & + \{3\overline{N}(r,0;g)+2\overline{N}(r,0;g \mid \geq 2)\} + \{3\overline{N}(r,\infty;g)+2\overline{N}(r,\infty;g \mid \geq 2)\} \\ & + S(r,f)+S(r,g). \end{aligned} \tag{2.5}$$

Proof. By Nevanlinna's second fundamental theorem and Lemmas 2.1 and 2.2, we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f)-N_0(r,0;f')+S(r,f) \\ &\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f \mid \geq 2)+\overline{N}(r,0;f \mid \geq 0) \\ &\quad+\overline{N}(r,\infty;f \mid \geq 2)+\overline{N}(r,0;g \mid \geq 2)+\overline{N}(r,1;f \mid \geq 3) \\ &\quad+\overline{N}(r,1;g \mid \geq 3)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \end{aligned} \tag{2.6}$$

Also we get

$$\begin{aligned} &\overline{N}(r,1;f \mid \geq 2)+\overline{N}(r,1;f \mid \geq 3) \leq N(r,0;f' \mid f \neq 0), \\ &\overline{N}(r,1;g \mid \geq 3)+\overline{N}_0(r,0;g') \leq N(r,0;g' \mid g \neq 0). \end{aligned} \tag{2.7}$$

So from (2.6), we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,0;f \mid \geq 2)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;f \mid \geq 2) \\ &\quad+\overline{N}(r,0;g \mid \geq 2)+\overline{N}(r,\infty;g \mid \geq 2)+N(r,0;f' \mid f \neq 0) \\ &\quad+N(r,0;g' \mid g \neq 0)+S(r,f)+S(r,g). \end{aligned} \tag{2.8}$$

Similarly,

$$\begin{aligned} T(r,g) \le{} & \overline{N}(r,0;g) + \overline{N}(r,0;g \mid\ge 2) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;g \mid\ge 2) \\ & + \overline{N}(r,0;f \mid\ge 2) + \overline{N}(r,\infty;f \mid\ge 2) + N(r,0;f' \mid f \ne 0) \\ & + N(r,0;g' \mid g \ne 0) + S(r,f) + S(r,g). \end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9) and using Lemma 2.3, we obtain the following lemma. □

Lemma 2.5 [10]. *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, \ldots, a_n$ are constants and $a_n \ne 0$. Then*

$$T(r,P(f)) = nT(r,f) + O(1). \tag{2.10}$$

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n+1}, \tag{2.11}$$

where $n(\ge 2)$ is an integer. Then

$$f^{n+1}(af+b) \equiv g^{n+1}(ag+b) \tag{2.12}$$

implies that $f \equiv g$, where a, b are finite nonzero constants.

Proof. Let

$$f^{n+1}(af+b) \equiv g^{n+1}(ag+b) \tag{2.13}$$

and $f \not\equiv g$. We consider the following two cases.

Case 1. Let $y = g/f$ be a constant. Then $y \ne 1$ and from (2.13), we get

$$af(1-y^{n+2}) \equiv -b(1-y^{n+1}), \tag{2.14}$$

from which it follows that $y^{n+1} \ne 1$, $y^{n+2} \ne 1$, and

$$f \equiv -\frac{b(1-y^{n+1})}{a(1-y^{n+2})}. \tag{2.15}$$

This is a contradiction because f is nonconstant.

Case 2. Let $y = g/f$ be not a constant. Then from (2.13), we get

$$f \equiv \frac{b}{a}\left(\frac{y^{n+1}}{1+y+y^2+\cdots+y^{n+1}} - 1\right). \tag{2.16}$$

From (2.16), we obtain by Nevanlinna's first fundamental theorem and Lemma 2.5

$$\begin{aligned} T(r,f) &= T\left(r,\sum_{j=0}^{n+1}\frac{1}{y^j}\right)+S(r,y) \\ &= (n+1)T\left(r,\frac{1}{y}\right)+S(r,y) \\ &= (n+1)T(r,y)+S(r,y). \end{aligned} \tag{2.17}$$

Now we note that a pole of y is not a pole of $(b/a)(y^{n+1}/(1+y+y^2+\cdots+y^{n+1})-1)$. So from (2.16), we get

$$\sum_{k=1}^{n+1}\overline{N}(r,u_k;y) \le \overline{N}(r,\infty;f), \tag{2.18}$$

where $u_k = \exp(2k\pi i/(n+2))$ for $k = 1,2,\ldots,n+1$.

So by Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned} (n-1)T(r,y) &\le \sum_{k=1}^{n+1}\overline{N}(r,u_k;y)+S(r,y) \\ &\le \overline{N}(r,\infty;f)+S(r,y) \\ &< (1-\Theta(\infty;f)+\varepsilon)T(r,f)+S(r,y) \\ &= (n+1)(1-\Theta(\infty;f)+\varepsilon)T(r,y)+S(r,y), \end{aligned} \tag{2.19}$$

where $\varepsilon(>0)$.

Again putting $y_1 = 1/y$, noting that $T(r,y) = T(r,y_1)+O(1)$, and proceeding as above we get

$$(n-1)T(r,y) \le (n+1)(1-\Theta(\infty;g)+\varepsilon)T(r,y)+S(r,y). \tag{2.20}$$

Since $\Theta(\infty;f)+\Theta(\infty;g) > 4/(n+1)$, there exists a $\delta(>0)$ such that $\Theta(\infty;f)+\Theta(\infty;g) > \delta+4/(n+1)$. Now adding (2.19) and (2.20), we obtain

$$\begin{aligned} 2(n-1)T(r,y) &\le (n+1)(2-\Theta(\infty;f)-\Theta(\infty;g)+2\varepsilon)T(r,y)+S(r,y) \\ &\le (n+1)\left(2-\frac{4}{n+1}-\delta+2\varepsilon\right)T(r,y)+S(r,y), \end{aligned} \tag{2.21}$$

and so $(\delta-2\varepsilon)T(r,y) \le S(r,y)$, which is a contradiction for any $\varepsilon(0<2\varepsilon<\delta)$. Therefore, $f \equiv g$ and the proof of the lemma is complete. □

Lemma 2.7. *Let f and g be nonconstant meromorphic functions. Then*

$$f^n(f-1)f'g^n(g-1)g' \not\equiv 1, \tag{2.22}$$

where $n(\ge 5)$ is an integer.

Proof. Let

$$f^n(f-1)f'g^n(g-1)g' \equiv 1. \tag{2.23}$$

Let z_0 be a 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$ such that

$$2p-1=(n+2)q+1, \tag{2.24}$$

that is,

$$2p=(n+2)q+2 \geq n+4, \tag{2.25}$$

that is,

$$p \geq \frac{n+4}{2}. \tag{2.26}$$

Let z_0 be a zero of f with multiplicity $p(\geq 1)$ and let it be a pole of g with multiplicity $q(\geq 1)$. Then

$$(n+1)p-1=(n+2)q+1. \tag{2.27}$$

From (2.27), we get

$$q+2=(n+1)(p-q) \geq n+1, \tag{2.28}$$

that is,

$$q \geq n-1. \tag{2.29}$$

Again from (2.27), we get

$$(n+1)p=(n+2)q+2 \geq (n+2)(n-1)+2, \tag{2.30}$$

that is,

$$p \geq \frac{(n+2)(n-1)+2}{n+1}=n. \tag{2.31}$$

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we see that

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}_0(r,0;g')\\ &\leq \frac{1}{n}N(r,0;g)+\frac{2}{n+4}N(r,1;g)+\overline{N}_0(r,0;g')\\ &\leq \left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,g)+\overline{N}_0(r,0;g').\end{aligned} \tag{2.32}$$

Now by Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f)\\ &\leq \frac{1}{n}N(r,0;f)+\frac{2}{n+4}N(r,1;f)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f),\end{aligned} \tag{2.33}$$

that is,

$$\left(1-\frac{1}{n}-\frac{2}{n+4}\right)T(r,f)\le\left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,g)+\overline{N}_0(r,0;g')-\overline{N}_0(r,0;f')+S(r,f). \tag{2.34}$$

Similarly, we get

$$\left(1-\frac{1}{n}-\frac{2}{n+4}\right)T(r,g)\le\left(\frac{1}{n}+\frac{2}{n+4}\right)T(r,f)+\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r,g). \tag{2.35}$$

Adding (2.34) and (2.35), we get

$$\left(1-\frac{2}{n}-\frac{4}{n+4}\right)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g), \tag{2.36}$$

which is a contradiction because $1-(2/n)-4/(n+4)>0$. This proves the lemma. □

Lemma 2.8. *Let f and g be two nonconstant meromorphic functions and*

$$F=f^{n+1}\left(\frac{f}{n+2}-\frac{1}{n+1}\right),\qquad G=g^{n+1}\left(\frac{g}{n+2}-\frac{1}{n+1}\right), \tag{2.37}$$

where $n(\ge 4)$ is an integer. Then $F'\equiv G'$ implies that $F\equiv G$.

Proof. If $F'\equiv G'$, then $F\equiv G+c$, where c is a constant. Let $c\ne 0$. Then by Nevanlinna's second fundamental theorem and Lemma 2.5, we get

$$\begin{aligned}(n+2)T(r,f)&\le\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,c;F)+S(r,F)\\&=\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}\left(r,\frac{n+2}{n+1};f\right)+\overline{N}(r,0;g)\\&\quad+\overline{N}\left(r,\frac{n+2}{n+1};g\right)+S(r,f)\le 3T(r,f)+2T(r,g)+S(r,f),\end{aligned} \tag{2.38}$$

that is,

$$(n-1)T(r,f)\le 2T(r,g)+S(r,f). \tag{2.39}$$

Similarly, we get

$$(n-1)T(r,g)\le 2T(r,f)+S(r,g). \tag{2.40}$$

This shows that

$$(n-3)T(r,f)+(n-3)T(r,g)\le S(r,f)+S(r,g), \tag{2.41}$$

which is a contradiction. Therefore $c=0$ and so $F\equiv G$. This proves the lemma. □

Lemma 2.9. *If F and G are defined as in Lemma 2.8, then*

(i) $T(r,F) \le T(r,F') + N(r,0;f) + N(r,(n+2)/(n+1);f) - N(r,1;f) - N(r,0;f') + S(r,f)$,

(ii) $T(r,G) \le T(r,G') + N(r,0;g) + N(r,(n+2)/(n+1);g) - N(r,1;g) - N(r,0;g') + S(r,g)$.

Proof. We prove (i) because (ii) is similar. Now in view of Nevanlinna's first fundamental theorem and Lemma 2.5, we get

$$\begin{aligned} T(r,F) &= T\left(r,\frac{1}{F}\right) + O(1) \\ &= N(r,0;F) + m\left(r,\frac{1}{F}\right) + O(1) \le N(r,0;F) + m\left(r,\frac{F'}{F}\right) + m\left(r,\frac{1}{F'}\right) \\ &= T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F) \\ &= T(r,F') + (n+1)N(r,0;f) + N\left(r,\frac{n+2}{n+1};f\right) - nN(r,0;f) \\ &\quad - N(r,1;f) - N(r,0;f') + S(r,f) \\ &= T(r,F') + N(r,0;f) + N\left(r,\frac{n+2}{n+1};f\right) - N(r,1;f) - N(r,0;f') + S(r,f). \end{aligned} \tag{2.42}$$

This proves the lemma. □

Lemma 2.10 [7]. *Let f be a nonconstant meromorphic function and let k be a positive integer. Then*

$$N_2(r,0;f^{(k)}) \le k\overline{N}(r,\infty;f) + N_{2+k}(r,0;f) + S(r,f). \tag{2.43}$$

Lemma 2.11 [13]. *If $h \equiv 0$ then f, g share 1 CM.*

Proof. Since $h \equiv 0$, integrating, we get $f'/(f-1)^2 \equiv Ag'/(g-1)^2$, where A is a nonzero constant. From this, the lemma follows. □

Lemma 2.12 [9, 11]. *If f and g share* 1 *CM, then one of the following cases holds:*

(i) $T(r,f) + T(r,g) \le 2\{N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g)\} + S(r,f) + S(r,g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv 1$.

3. Proof of theorems

We prove Theorem 1.3 only because Theorem 1.2 can be proved similarly noting that in this case, $\overline{N}(r,\infty;f) \equiv \overline{N}(r,\infty;g) \equiv 0$.

Proof of Theorem 1.3. Let F and G be defined as in Lemma 2.8 and $F_1 = F' = f^n(f-1)f'$, $G_1 = G' = g^n(g-1)g'$. Also we put

$$H = \left(\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1 - 1}\right) - \left(\frac{G_1''}{G_1'} - \frac{2G_1'}{G_1 - 1}\right). \tag{3.1}$$

If possible, let $H \not\equiv 0$. Then by Lemmas 2.4 and 2.9, we get

$$\begin{aligned}
T(r,F)+T(r,G) &\leq T(r,F')+T(r,G')+N(r,0;f)+N\left(r,\frac{n+2}{n+1};f\right)\\
&\quad -N(r,1;f)-N(r,0;f')+N(r,0;g)+N\left(r,\frac{n+2}{n+1};g\right)\\
&\quad -N(r,1;g)-N(r,0;g')+S(r,f)+S(r,g)\\
&\leq \{3\overline{N}(r,0;f^n(f-1)f')+2\overline{N}(r,0;f^n(f-1)f' \mid \geq 2)\}\\
&\quad +\{3\overline{N}(r,0;g^n(g-1)g')+2\overline{N}(r,0;g^n(g-1)g' \mid \geq 2)\}\\
&\quad +5\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;g)+N(r,0;f)+N\left(r,\frac{n+2}{n+1};f\right)\\
&\quad -N(r,1;f)-N(r,0;f')+N(r,0;g)+N\left(r,\frac{n+2}{n+1};g\right)\\
&\quad -N(r,1;g)-N(r,0;g')+S(r,f)+S(r,g)\\
&\leq 6N(r,0;f)+2N_2(r,1;f)+2N_2(r,0;f')+5\overline{N}(r,\infty;f)\\
&\quad +N\left(r,\frac{n+2}{n+1};f\right)+6N(r,0;g)+2N_2(r,1;g)+2N_2(r,0;g')\\
&\quad +5\overline{N}(r,\infty;g)+N\left(r,\frac{n+2}{n+1};g\right)+S(r,f)+S(r,g).
\end{aligned} \tag{3.2}$$

So by Lemmas 2.5 and 2.10, we get

$$\begin{aligned}
(n+2)T(r,f)+(n+2)T(r,g) &\leq 9T(r,f)+7\overline{N}(r,\infty;f)+2N_3(r,0;f)+9T(r,g)\\
&\quad +7\overline{N}(r,\infty;g)+2N_3(r,0;g)+S(r,f)+S(r,g)\\
&\leq 18T(r,f)+18T(r,g)+S(r,f)+S(r,g),
\end{aligned} \tag{3.3}$$

that is,

$$(n-16)T(r,f)+(n-16)T(r,g) \leq S(r,f)+S(r,g), \tag{3.4}$$

which is a contradiction.

Therefore $H \equiv 0$ and so by Lemma 2.11, F_1 and G_1 share 1 CM. In a similar manner as above, we can verify that the following inequality does not hold:

$$\begin{aligned}
T(r,F_1)+T(r,G_1) &\leq 2\{N_2(r,0;F_1)+N_2(r,\infty;F_1)+N_2(r,0;G_1)+N_2(r,\infty;G_1)\}\\
&\quad +S(r,F_1)+S(r,G_1).
\end{aligned} \tag{3.5}$$

So by Lemmas 2.12, 2.7, 2.8, and 2.6, we get $f \equiv g$. This proves the theorem. □

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