# A WIRSING-TYPE APPROACH TO SOME CONTINUED FRACTION EXPANSION

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Chan (2004) considered a certain continued fraction expansion and the corresponding Gauss-Kuzmin-Lévy problem. A Wirsing-type approach to the Perron-Frobenius operator of the associated transformation under its invariant measure allows us to obtain a near-optimal solution to this problem.

## 1. Introduction

The Gauss 1812 problem gave rise to an extended literature. In modern times, the so-called Gauss-Kuzmin-Lévy theorem is still one of the most important results in the metrical theory of regular continued fractions (RCFs). A recent survey of this topic is to be found in [10]. From the time of Gauss, a great number of such theorems followed. See, for example, [2, 6, 7, 8, 18].

Apart from the RCF expansion there are many other continued fraction expansions: the continued fraction expansion to the nearest integer, grotesque expansion, Nakada's $\alpha$-expansions, Rosen expansions; in fact, there are too many to mention (see [4, 5, 11, 12, 13, 16, 17] for some background information). The Gauss-Kuzmin-Lévy problem has been generalized to the above continued fraction expansions (see [3, 14, 15, 19, 20, 21]).

Taking up a problem raised in [1], we consider another expansion of reals in the unit interval, different from the RCF expansion. In fact, in [1] Chan has studied the transformation related to this new continued fraction expansion and the asymptotic behaviour of its distribution function. Giving a solution to the Gauss-Kuzmin-Lévy problem, he showed in [1, Theorem 1] that the convergence rate involved is $O(q^n)$ as $n \to \infty$ with $0 < q < 1$. This unsurprising result can be easily obtained from well-known general results (see [9, pages 202 and 262–266] and [10, Section 2.1.2]) concerning the Perron-Frobenius operator of the transformation under the invariant measure induced by the limit distribution function.

Our aim here is to give a better estimation of the convergence rate discussed. First, in Section 2 we introduce equivalent, but much more concise and rigorous expressions than in [1] of the transformation involved and of the related incomplete quotients. Next, in Section 3, our strategy is to derive the Perron-Frobenius operator of this transformation


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under its invariant measure. In Section 4, we use a Wirsing-type approach (see [22]) to study the optimality of the convergence rate. Actually, in Theorem 4.3 of Section 4 we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

## 2. Another expansion of reals in the unit interval

In this section we describe another continued fraction expansion different from the regular continued fraction expansion for a number $x$ in the unit interval $I = [0,1]$, which has been actually considered in [1].

Define for any $x \in I$ the transformation

$$\tau(x) = 2^{\{(\log x^{-1})/\log 2\}} - 1, \quad x \neq 0;\ \tau(0) = 0, \tag{2.1}$$

where $\{u\}$ denotes the fractionary part of a real $u$ while log stands for natural logarithm. (Nevertheless, the definition of $\tau$ is independent of the base of the logarithm used.) Putting

$$a_n(x) = a_1(\tau^{n-1}(x)), \quad n \in \mathbb{N}_+ = \{1,2,\dots\}, \tag{2.2}$$

with $\tau^0(x) = x$ the identity map and

$$a_1(x) = \left[\frac{(\log x^{-1})}{\log 2}\right], \tag{2.3}$$

where $[u]$ denotes the integer part of a real $u$, one easily sees that every irrational $x \in (0,1)$ has a unique infinite expansion

$$x = \cfrac{2^{-a_1}}{1 + \cfrac{2^{-a_2}}{1 + \cdots}} = [a_1, a_2, \dots]. \tag{2.4}$$

Here, the incomplete quotients or digits $a_n(x)$, $n \in \mathbb{N}_+$ of $x \in (0,1)$ are natural numbers.

Let $\mathcal{B}_I$ be the $\sigma$-algebra of Borel subsets of $I$. There is a probability measure $\nu$ on $\mathcal{B}_I$ defined by

$$\nu(A) = \frac{1}{\log(4/3)} \int_A \frac{dx}{(x+1)(x+2)}, \quad A \in \mathcal{B}_I, \tag{2.5}$$

such that $\nu(\tau^{-1}(A)) = \nu(A)$ for any $A \in \mathcal{B}_I$, that is, $\nu$ is $\tau$-invariant.

## 3. An operator treatment

In the sequel we will derive the Perron-Frobenius operator of $\tau$ under the invariant measure $\nu$.

Let $\mu$ be a probability measure on $\mathcal{B}_I$ such that $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0$, $A \in \mathcal{B}_I$, where $\tau$ is the continued fraction transformation defined in Section 2. In particular, this condition is satisfied if $\tau$ is $\mu$-preserving, that is, $\mu\tau^{-1} = \mu$. It is known from [10, Section 2.1] that the Perron-Frobenius operator $P_\mu$ of $\tau$ under $\mu$ is defined as the bounded linear operator on $L^1_\mu = \{f : I \to \mathbb{C} \mid \int_I |f| d\mu < \infty\}$ which takes $f \in L^1_\mu$ into $P_\mu f \in L^1_\mu$ with

$$\int_A P_\mu f \, d\mu = \int_{\tau^{-1}(A)} f \, d\mu, \quad A \in \mathcal{B}_I. \tag{3.1}$$

In particular the Perron-Frobenius operator $P_\lambda$ of $\tau$ under the Lebesgue measure $\lambda$ is

$$P_\lambda(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f \, d\lambda \quad \text{a.e. in } I. \tag{3.2}$$

Proposition 3.1. *The Perron-Frobenius operator $P_\nu = U$ of $\tau$ under $\nu$ is given a.e. in $I$ by the equation*

$$U f(x) = \sum_{k \in \mathbb{N}} p_k(x) f(u_k(x)), \quad f \in L^1_\nu, \tag{3.3}$$

*where*

$$\begin{aligned} p_k(x) &= \frac{\gamma^{k+1}(x+1)(x+2)}{(\gamma^k + x + 1)(\gamma^{k+1} + x + 1)}, \quad x \in I, \\ u_k(x) &= \frac{\gamma^k}{x+1}, \quad x \in I, \end{aligned} \tag{3.4}$$

*with $\gamma = 1/2$.*

The *proof* is entirely similar to that of [10, Proposition 2.1.2].

An analogous result to [10, Proposition 2.1.5] is shown as follows.

Proposition 3.2. *Let $\mu$ be a probability measure on $\mathcal{B}_I$. Assume that $\mu \ll \lambda$ and let $h = d\mu/d\lambda$. Then*

$$\mu(\tau^{-n}(A)) = \int_A \frac{U^n f(x)}{(x+1)(x+2)} dx \tag{3.5}$$

*for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_I$, where $f(x) = (x+1)(x+2)h(x)$, $x \in I$.*

## 4. A Wirsing-type approach

Let $\mu$ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}$, put

$$F_n(x) = \mu(\tau^n < x), \quad x \in I, \tag{4.1}$$

where $\tau^0$ is the identity map. As $(\tau^n < x) = \tau^{-n}((0,x))$, by Proposition 3.2 we have

$$F_n(x) = \int_0^x \frac{U^n f_0(u)}{(u+1)(u+2)} du, \quad n \in \mathbb{N}, \, x \in I, \tag{4.2}$$

with $f_0(x) = (x+1)(x+2)F_0'(x)$, $x \in I$, where $F_0' = d\mu/d\lambda$.

In this section we will assume that $F_0' \in C^1(I)$. So, we study the behaviour of $U^n$ as $n \to \infty$, assuming that the domain of $U$ is $C^1(I)$, the collection of all functions $f : I \to \mathbb{C}$ which have a continuous derivative.

Let $f \in C^1(I)$. Then the series (3.3) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. Putting $\Delta_k = \gamma^k - \gamma^{2k}$, $k \in \mathbb{N}$ we get

$$p_k(x) = \gamma^{k+1} + \frac{\Delta_k}{\gamma^k + x + 1} - \frac{\Delta_{k+1}}{\gamma^{k+1} + x + 1},$$

$$\begin{aligned}(Uf)'(x) &= \sum_{k\in\mathbb{N}} \left[ p_k'(x) f\left(\frac{\gamma^k}{x+1}\right) - p_k(x)\frac{\gamma^k}{(x+1)^2} f'\left(\frac{\gamma^k}{x+1}\right)\right] \\ &= \sum_{k\in\mathbb{N}} \left[\left(\frac{\Delta_{k+1}}{(\gamma^{k+1}+x+1)^2} - \frac{\Delta_k}{(\gamma^k+x+1)^2}\right) f\left(\frac{\gamma^k}{x+1}\right) - p_k(x)\frac{\gamma^k}{(x+1)^2} f'\left(\frac{\gamma^k}{x+1}\right)\right] \\ &= -\sum_{k\in\mathbb{N}} \left[\frac{\Delta_{k+1}}{(\gamma^{k+1}+x+1)^2}\left(f\left(\frac{\gamma^{k+1}}{x+1}\right) - f\left(\frac{\gamma^k}{x+1}\right)\right) + p_k(x)\frac{\gamma^k}{(x+1)^2} f'\left(\frac{\gamma^k}{x+1}\right)\right],\end{aligned} \tag{4.3}$$

$x \in I$. Thus, we can write

$$(Uf)' = -Vf', \quad f \in C^1(I), \tag{4.4}$$

where $V : C(I) \to C(I)$ is defined by

$$Vg(x) = \sum_{k\in\mathbb{N}} \left(\frac{\Delta_{k+1}}{(\gamma^{k+1}+x+1)^2} \int_{\gamma^k/(x+1)}^{\gamma^{k+1}/(x+1)} g(u)du + p_k(x)\frac{\gamma^k}{(x+1)^2} g\left(\frac{\gamma^k}{x+1}\right)\right), \tag{4.5}$$

$g \in C(I)$, $x \in I$. Clearly,

$$(U^n f)' = (-1)^n V^n f', \quad n \in \mathbb{N}_+,\ f \in C^1(I). \tag{4.6}$$

We are going to show that $V^n$ takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

Proposition 4.1. *There are positive constants $v > 0.206968896$ and $w < 0.209364308$, and a real-valued function $\varphi \in C(I)$ such that $v\varphi \le V\varphi \le w\varphi$.*

*Proof.* Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous bounded function such that $\lim_{x\to\infty} h(x) < \infty$. We look for a function $g : (0,1] \to \mathbb{R}$ such that $Ug = h$, assuming that the equation

$$Ug(x) = \sum_{k\in\mathbb{N}} p_k(x) g\left(\frac{\gamma^k}{x+1}\right) = h(x) \tag{4.7}$$

holds for $x \in \mathbb{R}_+$. Then (4.7) yields

$$\frac{h(x)}{x+2} - \frac{h(2x+1)}{2x+3} = \frac{x+1}{(x+2)(2x+3)} g\left(\frac{1}{x+1}\right), \quad x \in \mathbb{R}_+. \tag{4.8}$$

Hence

$$g(u)=(u+2)h\left(\frac{1}{u}-1\right)-(u+1)h\left(\frac{2}{u}-1\right),\quad u\in(0,1],\tag{4.9}$$

and we indeed have $Ug=h$ since

$$\begin{aligned}Ug(x)&=\sum_{k\in\mathbb{N}}p_k(x)\left[\left(\frac{y^k}{x+1}+2\right)h\left(\frac{x+1}{y^k}-1\right)-\left(\frac{y^k}{x+1}+1\right)h\left(\frac{2(x+1)}{y^k}-1\right)\right]\\&=\frac{x+2}{2}\sum_{k\in\mathbb{N}}\frac{y^{2k}}{(y^k+x+1)(y^{k+1}+x+1)}\\&\qquad\times\left[\left(\frac{x+1}{y^{k+1}}+1\right)h\left(\frac{x+1}{y^k}-1\right)-\left(\frac{x+1}{y^k}+1\right)h\left(\frac{x+1}{y^{k+1}}-1\right)\right]\\&=h(x),\quad x\in\mathbb{R}_+.\end{aligned}\tag{4.10}$$

In particular, for any fixed $a\in I$ we consider the function $h_a:\mathbb{R}_+\to\mathbb{R}$ defined by $h_a(x)=1/(x+a+1)$, $x\in\mathbb{R}_+$. By the above, the function $g_a:(0,1]\to\mathbb{R}$ defined as

$$\begin{aligned}g_a(x)&=(x+2)h_a\left(\frac{1}{x}-1\right)-(x+1)h_a\left(\frac{2}{x}-1\right)\\&=\frac{x(x+2)}{ax+1}-\frac{x(x+1)}{ax+2},\quad x\in(0,1],\end{aligned}\tag{4.11}$$

satisfies $Ug_a(x)=h_a(x)$, $x\in I$. Setting

$$\varphi_a(x)=g_a'(x)=\frac{3ax^2+4(a+1)x+6}{(ax+2)^2(ax+1)^2},\tag{4.12}$$

we have

$$V\varphi_a(x)=-\left(Ug_a\right)'(x)=\frac{1}{(x+a+1)^2},\quad x\in I.\tag{4.13}$$

We choose $a$ by asking that $(\varphi_a/V\varphi_a)(0)=(\varphi_a/V\varphi_a)(1)$. This amounts to $3a^4+12a^3+18a^2-2a-17=0$ which yields as unique acceptable solution $a=0.794741181\ldots$. For this value of $a$, the function $\varphi_a/V\varphi_a$ attains its maximum equal to $(3/2)(a+1)^2=4.83164386\ldots$ at $x=0$ and $x=1$, and has a minimum $m(a)\simeq(\varphi_a/V\varphi_a)(0.39)=4.776363306\ldots$. It follows that for $\varphi=\varphi_a$ with $a=0.794741181\ldots$, we have

$$\frac{2\varphi}{3(a+1)^2}\le V\varphi\le\frac{\varphi}{m(a)},\tag{4.14}$$

that is, $v\varphi\le V\varphi\le w\varphi$, where $v=2/3(a+1)^2>0.206968896$, and $w=1/m(a)<0.209364308$. □

Corollary 4.2. *Let $f_0\in C^1(I)$ such that $f_0'>0$. Put $\alpha=\min_{x\in I}\varphi(x)/f_0'(x)$ and $\beta=\max_{x\in I}\varphi(x)/f_0'(x)$. Then*

$$\frac{\alpha}{\beta}v^nf_0'\le V^nf_0'\le\frac{\beta}{\alpha}w^nf_0',\quad n\in\mathbb{N}_+.\tag{4.15}$$

*Proof.* Since $V$ is a positive operator, we have

$$v^n\varphi \le V^n\varphi \le w^n\varphi, \quad n \in \mathbb{N}_+. \tag{4.16}$$

Noting that $\alpha f_0' \le \varphi \le \beta f_0'$, we can write

$$\frac{\alpha}{\beta}v^n f_0' \le \frac{1}{\beta}v^n\varphi \le \frac{1}{\beta}V^n\varphi \le V^n f_0' \le \frac{1}{\alpha}V^n\varphi \le \frac{1}{\alpha}w^n\varphi \le \frac{\beta}{\alpha}w^n f_0', \tag{4.17}$$

$n \in \mathbb{N}_+$, which shows that (4.15) holds. □

Theorem 4.3 (near-optimal solution to Gauss-Kuzmin-Lévy problem). *Let $f_0 \in C^1(I)$ such that $f_0' > 0$. For any $n \in \mathbb{N}_+$ and $x \in I$,*

$$\begin{aligned} &\frac{(\log(4/3))^2\alpha\min_{x\in I} f_0'(x)}{2\beta}v^n F(x)(1-F(x)) \\ &\quad \le |\mu(\tau^n < x) - F(x)| \le \frac{(\log(4/3))^2\beta\max_{x\in I} f_0'(x)}{\alpha}w^n F(x)(1-F(x)), \end{aligned} \tag{4.18}$$

*where $\alpha$, $\beta$, $v$ and $w$ are defined in Proposition 4.1 and Corollary 4.2 and $F(x) = (1/\log(4/3))\log(2(x+1))/x+2$. In particular, for any $n \in \mathbb{N}_+$ and $x \in I$,*

$$\begin{aligned} 0.01023923v^nF(x)(1-F(x)) &\le |\lambda(\tau^n < x) - F(x)| \\ &\le 0.334467468w^nF(x)(1-F(x)). \end{aligned} \tag{4.19}$$

*Proof.* For any $n \in \mathbb{N}$ and $x \in I$, set $d_n(F(x)) = \mu(\tau^n < x) - F(x)$. Then by (4.2) we have

$$d_n(F(x)) = \int_0^x \frac{U^n f_0(u)}{(u+1)(u+2)}du - F(x). \tag{4.20}$$

Differentiating twice with respect to $x$ yields

$$\begin{aligned} d_n'(F(x))\frac{1}{(\log(4/3))(x+1)(x+2)} &= \frac{U^n f_0(x)}{(x+1)(x+2)} - \frac{1}{(\log(4/3))(x+1)(x+2)}, \\ (U^n f_0(x))' &= \frac{1}{(\log(4/3))^2}\frac{d_n''(F(x))}{(x+1)(x+2)}, \quad n \in \mathbb{N},\ x \in I. \end{aligned} \tag{4.21}$$

Hence by (4.6) we have

$$d_n''(F(x)) = (-1)^n\left(\log\left(\frac{4}{3}\right)\right)^2(x+1)(x+2)V^n f_0'(x), \quad n \in \mathbb{N},\ x \in I. \tag{4.22}$$

Since $d_n(0) = d_n(1) = 0$, it follows from a well-known interpolation formula that

$$d_n(x) = -\frac{x(1-x)}{2}d_n''(\theta), \quad n \in \mathbb{N},\ x \in I \tag{4.23}$$

for a suitable $\theta = \theta(n,x) \in I$. Therefore

$$\mu(\tau^n < x) - F(x) = (-1)^{n+1}\left(\log\left(\frac{4}{3}\right)\right)^2 \frac{\theta+1}{2} V^n f_0'(\theta) F(x)(1-F(x)) \tag{4.24}$$

for any $n \in \mathbb{N}$ and $x \in I$, and another suitable $\theta = \theta(n,x) \in I$. The result stated follows now from Corollary 4.2. In the special case $\mu = \lambda$, we have $f_0(x) = (x+1)(x+2)$, $x \in I$. Then with $a = 0.794741181\ldots$, we have

$$\begin{gathered} \alpha = \min_{x\in I} \frac{\varphi(x)}{f_0'(x)} = \frac{7a+10}{5(a+2)^2(a+1)^2} = 0.123720515\ldots, \\ \beta = \max_{x\in I} \frac{\varphi(x)}{f_0'(x)} = 0.5, \end{gathered} \tag{4.25}$$

so that $(\log(4/3))^2\alpha/2\beta = 0.01023923\ldots$ and $(\log 4/3)^2\beta/\alpha = 0.334467468\ldots$. The proof is complete. □

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