NONEXTENDIBILITY OF $D(-1)$-TRIPLES OF THE FORM $\{1,10,c\}$

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We prove that there do not exist different positive integers $c,d>1$ such that the product of any two distinct elements of the set $\{1,10,c,d\}$ diminished by 1 is a perfect square.

1. Introduction

Let n be an integer. A set of positive integers $\{a_1,a_2,\dots,a_m\}$ is said to have the property $D(n)$ if a_ia_j+n is a perfect square for all $1\le i<j\le m$. Such a set is called a Diophantine m-tuple (with the property $D(n)$) or a $D(n)$-m-tuple.

In fact, this problem was first studied by Diophantus for the case $n=1$ and he found a set of four positive rationals with the above property: $\{1/16,33/16,17/4,105/16\}$ (see [4]). The first set of four positive integers with the same property was found by Fermat, and it was $\{1,3,8,120\}$. The conjecture is that there does not exist a $D(1)$-quintuple. In 1969, Baker and Davenport proved that Fermat's set cannot be extended to a $D(1)$-quintuple, (see [2]). In 2004, Dujella proved that there exists no $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples (see [9]). In the case $n=-1$, the conjecture is that there does not exist a $D(-1)$-quadruple. This conjecture, for the first time, appeared in [8].

We assume that the $D(-1)$-triple $\{a,b,c\}$ can be extended to a $D(-1)$-quadruple. Then there exist positive integers d, x, y, z such that

$$ad-1=x^2,\qquad bd-1=y^2,\qquad cd-1=z^2. \tag{1.1}$$

Eliminating d, we obtain the following system of Pellian equations:

$$ay^2-bx^2=b-a,\qquad az^2-cx^2=c-a,\qquad bz^2-cy^2=c-b. \tag{1.2}$$

The conjecture can thus be written in terms of Pellian equations (see [7]).

Conjecture 1.1. *Let $\{a,b,c\}$ be the set of distinct positive integers with the property that there exist integers r, s, t such that*

$$ab-1=r^2,\qquad ac-1=s^2,\qquad bc-1=t^2. \tag{1.3}$$

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If $1 \notin \{a,b,c\}$, *then the system of Pellian equations*

$$az^2 - cx^2 = c - a, \qquad bz^2 - cy^2 = c - b \tag{1.4}$$

has no solution. If $a = 1$, *then all solutions of system (1.4) are given by* $(x,y,z) = (0,\pm r,\pm s)$.

For certain triples $\{a,b,c\}$, with $1 \notin \{a,b,c\}$ the validity of Conjecture 1.1 can be verified by simple use of congruences (see [6]). On the other hand, the triples of the form $\{1,b,c\}$ have one "extension," $d = 1$. Although we do not count it as a proper extension, its existence implies that such triples extendibility cannot be proved by simple congruence consideration. Recently, the original conjecture if $1 \notin \{a,b,c\}$ was completely proved by Dujella and Fuchs in [10]. There they proved that there exists no $D(-1)$-quintuple, and if there exists $D(-1)$-quadruple $\{a,b,c,d\}$ with $a < b < c < d$, then $a = 1$ and $b \geq 5$.

In the case $a = 1$, Conjecture 1.1 was verified for particular $D(-1)$-triples, namely $\{1,2,5\}$ (by Brown in [6]), $\{1,5,10\}$ (by Mohanty and Ramasamy in [14]), $\{1,2,145\}$, $\{1,2,4901\}$, $\{1,5,65\}$, $\{1,5,20737\}$, $\{1,10,17\}$, $\{1,26,37\}$ (by Kedlaya in [12]), and $\{17,26,85\}$ (again by Brown in [6]). Moreover, Brown proved the conjecture for the following infinite families of $D(-1)$-triples:

$$\begin{aligned} &\{n^2+1,(n+1)^2+1,(2n+1)^2+4\} \quad \text{if } n \not\equiv 0 \pmod 4, \\ &\{2,2n^2+2n+1,2n^2+6n+5\} \quad \text{if } n \equiv 1 \pmod 4. \end{aligned} \tag{1.5}$$

Dujella proved the conjecture in [7] for all triples of the form $\{1,2,c\}$. Recently, Abu Muriefah and Al-Rashed studied extendibility of triples of the form $\{1,5,c\}$ in [1]. So the consideration of triples of the form $\{1,10,c\}$ seems to be the next natural step.

In the present paper, we will verify Conjecture 1.1 for all triples of the form $\{1,10,c\}$, and in the end, we will finish the work started by Abu Muriefah and Al-Rashed in [1], proving the same conjecture for all the triples of the form $\{1,5,c\}$. In the first part of our proof we will follow the strategy from [1, 7, 11]. Some of the lemmas are already proved in [10] in a more general context, but in order to keep the paper self-contained, we prefer to give complete proofs everywhere.

First we see that the conditions $c - 1 = s^2$ and $10c - 1 = t^2$ imply that

$$t^2 - 10s^2 = 9. \tag{1.6}$$

We have three fundamental solutions of Pellian equation (1.6), namely $(t_0,s_0) = (3,0)$, $(7,2)$, $(13,4)$. Then all solutions in nonnegative integers (t_k,s_k) of (1.6) belong to three classes.

Since numbers s_k satisfy recursive relations $s_{k+1} = 19s_k + 6t_k$ and $s_{k+2} = 38s_{k+1} - s_k$, we find that integers s_k belong to the following sequences:

$$\begin{aligned} s_k &= \frac{3}{2\sqrt{10}}\left(19+6\sqrt{10}\right)^k - \frac{3}{2\sqrt{10}}\left(19-6\sqrt{10}\right)^k, \\ s_k &= \left(1+\frac{7}{2\sqrt{10}}\right)\left(19+6\sqrt{10}\right)^k + \left(1-\frac{7}{2\sqrt{10}}\right)\left(19-6\sqrt{10}\right)^k, \\ s_k &= \left(2+\frac{13}{2\sqrt{10}}\right)\left(19+6\sqrt{10}\right)^k + \left(1-\frac{13}{2\sqrt{10}}\right)\left(19-6\sqrt{10}\right)^k. \end{aligned} \tag{1.7}$$

Now from the relation $c_k = s_k^2 + 1$, we can form the sequence $(c_k)_{k\geq 1}$,

$$c_k = \frac{1}{40} \times \begin{cases} 9\left(19+6\sqrt{10}\right)^{2m} + 9\left(19-6\sqrt{10}\right)^{2m} + 22, & k = 3m+1, \\ \left(89+28\sqrt{10}\right)\left(19+6\sqrt{10}\right)^{2m} \\ \quad + \left(89-28\sqrt{10}\right)\left(19-6\sqrt{10}\right)^{2m} + 22, & k = 3m+2, \\ \left(329+104\sqrt{10}\right)\left(19+6\sqrt{10}\right)^{2m} \\ \quad + \left(329-104\sqrt{10}\right)\left(19-6\sqrt{10}\right)^{2m} + 22, & k = 3m+3. \end{cases} \tag{1.8}$$

It is easy to check that $c_k < c_l$ for $k < l$.

Hence if the triple $\{1,10,c\}$ satisfies the property $D(-1)$, then there exists a positive integer k, such that $c = c_k$.

Now we can formulate our main result.

Theorem 1.2. *Let k be a positive integer and c_k as above. All solutions of the system of simultaneous Pellian equations*

$$z^2 - c_k x^2 = c_k - 1, \tag{1.9}$$

$$10z^2 - c_k y^2 = c_k - 10 \tag{1.10}$$

are given by $(x,y,z) = (0,\pm 3,\pm\sqrt{c_k - 1})$.

From this statement, we get immediately the result on the extension of $\{1,10,c\}$.

2. Preliminaries

Let k be the minimal positive integer, if such exists, for which the statement of Theorem 1.2 is not valid. Then results of Mohanty and Ramasamy (see [14]), and Kedlaya (see [12]) imply that $k \geq 4$. For simplicity, we will omit the index k. Now we have $s^2 = c - 1$, $t^2 = 10c - 1$, and $c \geq c_4 = 325$.

Since neither c nor $10c$ is a square, $Q(\sqrt{c})$ and $Q(\sqrt{10c})$ are real quadratic number fields. Moreover, $2c - 1 + 2s\sqrt{c} = (s + \sqrt{c})^2$ and $20c - 1 + 2t\sqrt{10c} = (t + \sqrt{10c})^2$ are nontrivial units in the rings $Z\lfloor\sqrt{c}\rfloor$ and $Z\lfloor\sqrt{10c}\rfloor$. Thus we know that there are finite sets $\{z_0{}^{(i)} + x_0{}^{(i)}\sqrt{c} : i = 1,\dots,i_0\}$ and $\{z_1{}^{(j)} + y_1{}^{(j)}\sqrt{10c} : j = 1,\dots,j_0\}$ of elements of $Z\lfloor\sqrt{c}\rfloor$ and $Z\lfloor\sqrt{10c}\rfloor$, respectively, such that all solutions of (1.9) and (1.10) are given by

$$z + x\sqrt{c} = \left(z_0{}^{(i)} + x_0{}^{(i)}\sqrt{c}\right)\left(2c - 1 + 2s\sqrt{c}\right)^m, \quad i = 1,\dots,i_0,\ m \geq 0, \tag{2.1}$$

$$z\sqrt{10} + y\sqrt{c} = \left(z_1{}^{(j)}\sqrt{10} + y_1{}^{(j)}\sqrt{c}\right)\left(20c - 1 + 2t\sqrt{10c}\right)^n, \quad j = 1,\dots,j_0,\ n \geq 0. \tag{2.2}$$

From (2.1), we conclude that $z = v_m{}^{(i)}$ for some index i and positive integer m, where

$$v_0{}^{(i)} = z_0{}^{(i)}, \qquad v_1{}^{(i)} = (2c-1)z_0{}^{(i)} + 2scx_0{}^{(i)}, \qquad v_{m+2}{}^{(i)} = (4c-2)v_{m+1}{}^{(i)} - v_m{}^{(i)}, \tag{2.3}$$

and from (2.2), we conclude that $z = w_n^{(j)}$ for some index j and positive integer n, where

$$w_0^{(j)} = z_1^{(j)}, \qquad w_1^{(j)} = (20c-1)z_1^{(j)} + 2tcy_1^{(j)}, \qquad w_{m+2}^{(j)} = (40c-2)w_{m+1}^{(j)} - w_m^{(j)}. \tag{2.4}$$

Our system of equations (1.9) and (1.10) is thus transformed to finitely many equations of the form

$$v_m^{(i)} = w_n^{(j)}. \tag{2.5}$$

If we choose representatives $z_0^{(i)} + x_0^{(i)}\sqrt{c}$ and $z_1^{(j)}\sqrt{10} + y_1^{(j)}\sqrt{c}$ such that $|z_0^{(i)}|$ and $|z_1^{(j)}|$ are minimal, then by [15, Theorem 108], we have estimates $0 < |z_0^{(i)}|, |z_1^{(j)}| < c$, (see also [10, Lemma 1]).

3. Application of congruence relations

From (2.3) and (2.4), it follows by induction that

$$\begin{aligned} v_{2m}^{(i)} &\equiv z_0^{(i)} \pmod{2c}, & v_{2m+1}^{(i)} &\equiv -z_0^{(i)} \pmod{2c}, \\ w_{2n}^{(j)} &\equiv z_1^{(j)} \pmod{2c}, & v_{2n+1}^{(j)} &\equiv -z_1^{(j)} \pmod{2c}. \end{aligned} \tag{3.1}$$

So if (2.5) has a solution, we must have $|z_0^{(i)}| = |z_1^{(j)}|$.

Let $d_0 = ((z_0^{(i)})^2 + 1)/c$. Then we have $d_0 - 1 = (x_0^{(i)})^2$, $10d_0 - 1 = (y_1^{(j)})^2$, $cd_0 - 1 = (z_0^{(i)})^2$, and

$$d_0 \le \frac{(c-1)^2 + 1}{c} < c. \tag{3.2}$$

Now if $d_0 > 1$, then there exists positive integer l, $l < k$, such that $d_0 = c_l$. But then the system

$$z^2 - c_l x^2 = c_l - 1, \qquad 10z^2 - c_l y^2 = c_l - 10 \tag{3.3}$$

has nontrivial solution $(x, y, z) = (s_k, t_k, z_0^{(i)})$, contradicting the minimality of k. So $d_0 = 1$, $|z_0^{(i)}| = |z_1^{(j)}| = s$. Thus we proved the following lemma.

Lemma 3.1. *If the equation $v_m^{(i)} = w_n^{(j)}$ has a solution, then $|z_0^{(i)}| = |z_1^{(j)}| = s$, $x_0^{(i)} = 0$, $y_1^{(j)} = \pm 3$.*

For the simplicity, from now on, the superscripts (i) and (j) will be omitted.

The following lemma can be proved easily by induction.

Lemma 3.2.

$$\begin{aligned} v_m &\equiv (-1)^m (z_0 - 2cm^2 z_0 - 2csmx_0) \pmod{8c^2}, \\ w_n &\equiv (-1)^n (z_1 - 20cn^2 z_1 - 2ctny_1) \pmod{8c^2}. \end{aligned} \tag{3.4}$$

Since we may restrict ourselves to positive solutions of the system of equations (1.9) and (1.10), we may assume that $z_0 = z_1 = s$. If $y_1 = 3$, then it is obvious that $v_l < w_l$ for

$l > 0$, and $v_m = w_n$, $n \neq 0$, implies that $m > n$. If $y_1 = -3$, then it is easy to check that $v_0 < w_1$. Then $v_l < w_{l+1}$ for $l \geq 0$, and thus $v_m = w_n$ implies that $m \geq n$ in all cases.

LEMMA 3.3. *If $v_m = w_n$, then m and n are even.*

Proof. From Lemma 3.2 and relations $z_0 = z_1 = s$, we have $m \equiv n \pmod 2$. If $v_{2m+1} = w_{2n+1}$, Lemma 3.2 implies that

$$(2m+1)^2 s \equiv (2n+1)\left[(20n+10)s \pm 3t\right] \pmod{4c}, \tag{3.5}$$

which contradicts the fact that s is even and t is odd. □

LEMMA 3.4. *If $v_{2m} = w_{2n}$, then $n \leq m \leq 1.66n$.*

Proof. We have already proved that $m \geq n$. Now from (2.3) and (2.4), we have

$$\begin{aligned} v_m &= \frac{s}{2}\left[\left(2c-1+2s\sqrt{c}\right)^m + \left(2c-1-2s\sqrt{c}\right)^m\right] > \frac{1}{2}\left(2c-1+2s\sqrt{c}\right)^m, \\ w_n &= \frac{1}{2\sqrt{10}}\left[\left(s\sqrt{10} \pm 3\sqrt{c}\right)\left(20c-1+2t\sqrt{10c}\right)^n + \left(s\sqrt{10} \mp 3\sqrt{c}\right)\left(20c-1-2t\sqrt{10c}\right)^n\right] \\ &< \frac{s\sqrt{10}+3\sqrt{c}+1}{2\sqrt{10}} \cdot \left(20c-1+2t\sqrt{10c}\right)^n < \frac{1}{2}\left(20c-1+2t\sqrt{10c}\right)^{n+1/2}. \end{aligned} \tag{3.6}$$

Since $c \geq 325$, $v_{2m} = w_{2n}$ implies that

$$\frac{2m}{2n+1/2} < \frac{\log\left(20c-1+2t\sqrt{10c}\right)}{\log\left(2c-1+2s\sqrt{c}\right)} < 1.3214. \tag{3.7}$$

If $n = 0$, then $m = 0$; and if $n \geq 1$, then we have $m < 1.3214n + 0.33035 < 1.66n$. □

LEMMA 3.5. *If $v_{2m} = w_{2n}$ and $n \neq 0$, then $m \geq n > (1/3\sqrt{2})\sqrt[4]{c}$.*

Proof. If $v_{2m} = w_{2n}$, Lemma 3.2 implies that $2s(m^2 - 10n^2) \equiv \pm 3tn \pmod{2c}$. From $s^2 \equiv t^2 \pmod c$, we get $4(m^2 - 10n^2)^2 \equiv 9n^2 \pmod c$. Assume that $n \neq 0$ and $n \leq (1/3\sqrt{2})\sqrt[4]{c}$. We thus have the following:

$$\begin{gathered} \left|2s(m^2-10n^2)\right| < 2\sqrt{c} \cdot 9n^2 \leq 18\sqrt{c} \cdot \frac{1}{18}\sqrt{c} = c, \\ 4(m^2-10n^2)^2 \leq 4 \cdot 9^2 \cdot n^4 \leq c, \\ 9n^2 \leq 9 \cdot \frac{1}{18}\sqrt{c} < c, \\ 3tn < 3 \cdot \sqrt{10c} \cdot \frac{1}{3\sqrt{2}} \cdot \sqrt[4]{c} < c. \end{gathered} \tag{3.8}$$

In the first two inequalities, we have used the relations $n \leq m \leq 1.66n$ that are already proved in Lemma 3.4. Now we can conclude that $4(m^2 - 10n^2)^2 = 9n^2$ and $2s(m^2 - 10n^2) = -3tn$, which imply that $s^2 = t^2$, a contradiction. □

4. Linear form in logarithms

Lemma 4.1. *If $v_m = w_n$, $n \neq 0$, then*

$$0 < n\log\left(20c-1+2t\sqrt{10c}\right) - m\log\left(2c-1+2s\sqrt{c}\right) + \log\frac{s\sqrt{10}\pm 3\sqrt{c}}{s\sqrt{10}} < (20c)^{-n}. \tag{4.1}$$

Proof. Let $p = s(2c-1+2s\sqrt{c})^m$ and $q = (1/\sqrt{10})(s\sqrt{10}\pm 3\sqrt{c})(20c-1+2t\sqrt{10c})^n$.
Then $v_m = w_n$ implies that

$$p + s^2p^{-1} = q + \frac{c-10}{10}q^{-1}. \tag{4.2}$$

It is clear that $p,\ q > 1$. Moreover,

$$p - q = \frac{c-10}{10}q^{-1} - s^2p^{-1} < (c-1)q^{-1} - (c-1)p^{-1} = (c-1)(p-q)p^{-1}q^{-1}. \tag{4.3}$$

If $p > q$ from (4.3), we get that $pq < (c-1)$, which is a contradiction because $q > 1$ and $p > (4s\sqrt{c})s = 4s^2\sqrt{c} = 4(c-1)\sqrt{c} > c-1$. So we have $q > p$.
Furthermore,

$$0 < \log\left(\frac{p}{q}\right)^{-1} = -\log\left(\frac{p}{q}\right) = -\log\left(1-\frac{q-p}{q}\right). \tag{4.4}$$

From $-\log(1-x) < x + x^2$ for $x \in \langle 0, 1/2\rangle$, we get $0 < \log(q/p) < (q-p)/q + ((q-p)/q)^2$.
Relation (4.2) implies that $p > q - (c-1)p^{-1} > q-1$, thus $q - p < 1$ and $(q-p)/q < q^{-1}$.
Then we can conclude that

$$0 < \log\left(\frac{q}{p}\right) < q^{-1}(1+q^{-1}) < 2q^{-1} = \frac{2\sqrt{10}}{s\sqrt{10}\pm 3\sqrt{c}}\left(20c-1+2t\sqrt{10c}\right)^{-n} < (20c)^{-n}, \tag{4.5}$$

which proves our lemma. □

The last inequality follows easily from the facts that $2\sqrt{10}/(s\sqrt{10}\pm 3\sqrt{c}) < 2.23$ for $c \geq 325$ and $n \geq 2$ (n is even and $n \neq 0$).
Now, we apply the following theorem of Baker and Wüstholz (see [3]).

Theorem 4.2. *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1,\dots,\alpha_l$ with rational coefficients $b_1,\dots,b_l$,*

$$\log\Lambda \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdot\cdots\cdot h'(\alpha_l)\log(2ld)\log B, \tag{4.6}$$

where $B = \max|b_i|$, d is the degree of the number field generated by $\alpha_1,\dots,\alpha_l$, and $h'(\alpha) = 1/r\max\{h(\alpha), |\log\alpha|, 1\}$, where $h(\alpha)$ denotes the standard logarithmic Weil height of α, and r is the degree of α.

We consider our equation $v_{2m} = w_{2n}$ with $n \neq 0$. We will apply Theorem 4.2 to the linear form from Lemma 4.1.

We have $l = 3$, $d = 4$, $B = 2m$, and numbers $\alpha_1 = 20c - 1 + 2t\sqrt{10c}$, $\alpha_2 = 2c - 1 + 2s\sqrt{c}$, $\alpha_3 = (s\sqrt{10} \pm 3\sqrt{c})/s\sqrt{10}$.

Minimal polynomials of α_1, α_2, and α_3 are

$$\begin{gathered} {\alpha_1}^2 - (40c - 2)\alpha_1 + 1 = 0, \qquad {\alpha_2}^2 - (4c - 2)\alpha_2 + 1 = 0, \\ (10c - 10){\alpha_3}^2 - (20c - 20)\alpha_3 + c - 10 = 0. \end{gathered} \tag{4.7}$$

Hence,

$$\begin{gathered} h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 40c, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log 4c, \\ h'(\alpha_3) = \frac{1}{2}\log\left[\frac{s\sqrt{10} + 3\sqrt{c}}{s\sqrt{10}} \cdot (10c - 10)\right] < \frac{1}{2}\log 20c. \end{gathered} \tag{4.8}$$

From Lemma 4.1, if $v_{2m} = w_{2n}$, we get

$$2n\log 20c \leq 18 \cdot 4! \cdot 3^4 \cdot (32 \cdot 4)^5 \cdot \frac{1}{2}\log 40c \cdot \frac{1}{2}\log 4c \cdot \frac{1}{2}\log 20c \cdot \log 24 \cdot \log 2m. \tag{4.9}$$

Now using Lemmas 3.4 and 3.5, relation (4.9) implies that

$$2n < 4.78 \cdot 10^{14} \cdot \log 12960n^4 \cdot \log 1296n^4 \cdot \log 3.32n, \tag{4.10}$$

hence $n < 5 \cdot 10^{20}$, and finally $c < 2.025 \cdot 10^{85}$. Therefore, we get $k \leq 81$.

5. Reduction

For completing the proof of Theorem 1.2 for all positive integers k, we must check that for $4 \leq k \leq 81$, $v_{2m} = w_{2n}$ implies that $n = m = 0$.

First, we will prove that

$$\begin{aligned} 0 &< 2n\log\left(20c_k - 1 + 2t_k\sqrt{10c_k}\right) - 2m\log\left(2c_k - 1 + 2s_k\sqrt{c_k}\right) \\ &\quad + \log\frac{s_k\sqrt{10} \pm 3\sqrt{c_k}}{s_k\sqrt{10}} < (20c_k)^{-2n}, \end{aligned} \tag{5.1}$$

where $n < 5 \cdot 10^{20}$, implies $n \leq m \leq 1$.

Having proved that, only one possibility will remain for the solution of (2.5), and that is $v_2 = w_2$. However, it is easy to prove that this is not possible. First, we see that $v_2 = w_2$ is only possible when $y_1 = -3$.

In that case, we get $(4c - 2)(2c - 1)s - s = (40c - 2)[(20c - 1)s - 6tc] - s$, which implies that $356c^3 - 4348c^2 + 778c - 35 = 0$; and this equation does not have any solution modulo 2.

To finish the proof, we use the reduction method described in [16, Section VI.3] and [17].

If we have the inequality of the type

$$|\Lambda| = \left| a_0 + \sum_{i=1}^{r} x_i a_i \right| \le k_2 e^{-k_3 X}, \tag{5.2}$$

where $a_i \in C$ are fixed numbers, k_2, k_3 are positive real constants, and x_i are unknown integers with $X = \max |x_i| \le X_0$, we can try to reduce the upper bound for X, using the following method. In our case, there is $r = 2$.

Let L be a lattice generated by the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 \\ [Ka_1] & [Ka_2] \end{pmatrix} \in M_2(Z), \tag{5.3}$$

where K is some large constant and $[\cdot]$ rounds number to the nearest integer. Now using the LLL algorithm (see [13]), we can find reduced basis of L. We do that by using the integral version of the algorithm in the package GP/Pari (see [5]). Let $y = (0 \quad -[Ka_0])^t \in Z^2$. Using the properties of reduced basis, it is easy to compute the lower bound k_4 of $\|x - y\|$, where $x \in L$ and $\|\cdot\|$ denotes the usual (Euclidean) norm of a vector. We can compute that using GP/Pari again. In most cases, if we choose the constant K big enough, we can use the following lemma.

Lemma 5.1. *Let $S = {X_0}^2$ and $T = 0.5 \cdot (1 + 2X_0)$. If ${k_4}^2 \ge T^2 + S$, then*

$$X \le \frac{\log(Kk_2) - \log\left(\sqrt{k_4^2 - S} - T\right)}{k_3} \tag{5.4}$$

or $x_1 = 0$ and $x_2 = -[Ka_0]/[Ka_2]$.

Proof of the lemma can be found in [16].

In our case, there is an inequality of the type (5.2) with

$$\begin{gathered} \Lambda = a_0 + na_1 + ma_2, \\ a_0 = \log \frac{s_k\sqrt{10} \pm 3\sqrt{c_k}}{s_k\sqrt{10}}, \qquad a_1 = 2\log\left(20c_k - 1 + 2t_k\sqrt{10c_k}\right), \\ a_2 = -2\log\left(2c_k - 1 + 2s_k\sqrt{c_k}\right), \quad k_2 = 1,\ k_3 = 1.2 \cdot \log 20c_k, \\ X = m \le 8.3 \cdot 10^{20}. \end{gathered} \tag{5.5}$$

Using the described method, after two steps of reduction, in all 78 remaining cases, for $4 \le k \le 81$, we get $m \le 1$, which finishes the proof of Theorem 1.2.

6. Concluding remarks

We can prove Conjecture 1.1 for all the triples of the form $\{1,5,c\}$, in exactly the same way. This gives a much shorter and neat proof of the result in [1]. Since Abu Muriefah and Al-Rashed in [1] have already prepared everything for the use of the described reduction method, we will only give some details here that are different from their paper.

First, there is no need for three sequences (c_k), because it is well known that all nonnegative solutions of the Pellian equation $t^2 - 5s^2 = 4$ are given by $t = L_{2k}$, $s = F_{2k}$ (Lucas and Fibonacci numbers). Thus, we only have one sequence $c_k = {F_{2k}}^2 + 1$, which leads to significant simplification. Later, when we work with a linear form in logarithms, we get the following.

If $v_n = w_m$, $n \neq 0$, then

$$0 < n\log\left(10c - 1 + 2t\sqrt{5c}\right) - m\log\left(2c - 1 + 2s\sqrt{c}\right) + \log\frac{s\sqrt{5} \pm 2\sqrt{c}}{s\sqrt{5}} < (10c)^{-n}. \tag{6.1}$$

And when we apply the Baker-Wüstholz theorem, we have

$$h'(\alpha_3) = \frac{1}{2}\log\left[\frac{s\sqrt{5} + 2\sqrt{c}}{s\sqrt{5}} \cdot (5c - 5)\right] < \frac{1}{2}\log 10c. \tag{6.2}$$

This slightly changes the constants which appeared in [1]. We get that $n < 5 \cdot 10^{20}$, and finally $c = c_k < 10^{84}$, which implies that $k \leq 101$.

Now we can use the same reduction method, because in this case there is an inequality of type (5.2) with

$$\begin{gathered} \Lambda = a_0 + na_1 + ma_2, \\ a_0 = \log\frac{s_k\sqrt{5} \pm 2\sqrt{c_k}}{s_k\sqrt{5}}, \qquad a_1 = \log\left(10c_k - 1 + 2t_k\sqrt{5c_k}\right), \\ a_2 = -\log\left(2c_k - 1 + 2s_k\sqrt{c_k}\right), \quad k_2 = 1,\ k_3 = \frac{\log 10c_k}{\sqrt{5}}, \\ X = m \leq 1.12 \cdot 10^{21}. \end{gathered} \tag{6.3}$$

By using the reduction method in all cases, for $k \leq 101$, after three steps of reduction, we get $m \leq 3$. Then to finish our proof, we only need to show that the remaining possibilities, $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$, are impossible because we have the relations $n \leq m \leq n\sqrt{5}$ and $m \equiv n \pmod 2$. However, this is easy to check.

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