DOUBLE-DUAL TYPES OVER THE BANACH SPACE $C(K)$

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Let K be a compact Hausdorff space and $C(K)$ the Banach space of all real-valued continuous functions on K, with the sup-norm. Types over $C(K)$ (in the sense of Krivine and Maurey) can be uniquely represented by pairs (ℓ,u) of bounded real-valued functions on K, where ℓ is lower semicontinuous, u is upper semicontinuous, $\ell \leq u$, and $\ell(x) = u(x)$ for all isolated points x of K. A condition that characterizes the pairs (ℓ,u) that represent double-dual types over $C(K)$ is given.

1. Statement of the main theorem

The concept of *type over a Banach space E* was first introduced by Krivine and Maurey [7] in the context of separable Banach spaces. The reader is referred to Garling's monograph [4] for more details. We consider general, not necessarily separable Banach spaces. Let E be a Banach space. For every $x \in E$, we define a function $\tau_x : E \to \mathbb{R}$ by letting $\tau_x(y) = \|x+y\|$ for all $y \in E$.

Definition 1.1. A function $\tau : E \to \mathbb{R}$ is a *type over E* if τ is in the closure (with respect to the topology of pointwise convergence) of the set $\{\tau_x : x \in E\}$.

The definition given here is equivalent to the definition given in [1]. That is, τ is a type over E if and only if there exists an ultrafilter $\mathcal{U}$ over an infinite index set λ and a bounded family of elements $(x_\alpha)_{\alpha\in\lambda}$ in E such that $\tau(y) = \lim_{\alpha\in\mathcal{U}} \|x_\alpha + y\|$ for all $y \in E$. The reader is referred to [5] for more details regarding the choice of the ultrafilter.

Throughout, we let K be a compact Hausdorff topological space. The topology on K is denoted by Ω. We let $\ell_\infty(K)$ denote the Banach lattice of bounded real-valued functions on K equipped with the sup-norm. For $f,g \in \ell_\infty(K)$, the lattice ordering is defined pointwise.

An sc *pair (semicontinuous pair)* is a pair of functions (ℓ,u) from $\ell_\infty(K)$ such that ℓ is lower semicontinuous (lsc), u is upper semicontinuous (usc), $\ell \leq u$, and $\ell(x) = u(x)$ for all isolated points $x \in K$.

The Banach space of continuous real-valued functions on K with sup-norm is denoted by $C(K)$. The constant function with value 1 is denoted by $\mathbf{1}$.

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The following theorem gives a concrete representation of types over $C(K)$ in terms of sc pairs [9, 10].

THEOREM 1.2. *Let $\tau : C(K) \to \mathbb{R}$ be a function. Then the following are equivalent:*
(i) *τ is a type over $C(K)$;*
(ii) *there exists an sc pair (ℓ,u) such that $\tau(g) = \max\{\|\ell+g\|, \|u+g\|\}$ for all $g \in C(K)$.*

The correspondence between types over $C(K)$ and sc pairs (ℓ,u) is one-to-one.

The following proposition is immediate from Definition 1.1; see [9] for more equivalent conditions and a detailed proof.

PROPOSITION 1.3. *Let E be a Banach space and $\tau : E \to \mathbb{R}$ a function. Then the following are equivalent:*
(i) *τ is a type over E;*
(ii) *for every finite subset $\alpha \subseteq E$ and every $\varepsilon > 0$, there exists an element $x = x(\alpha,\varepsilon) \in E$ such that $|\tau(y) - \|x+y\|| < \varepsilon$ for all $y \in \alpha$;*
(iii) *there exists a bounded net $(x_\alpha)_{\alpha\in I}$ in E such that*

$$\lim_{\alpha,I} \|x_\alpha + y\| = \tau(y) \tag{1.1}$$

for all $y \in E$.

If τ is a type over E and $(x_\alpha)_{\alpha\in I}$ is as in (iii) above, we say that $(x_\alpha)_{\alpha\in I}$ *generates* the type τ. A net $(x_\alpha)_{\alpha\in I}$ in E *doubly generates* τ if for every $\lambda \in [0,1]$ and every $y \in E$,

$$\lim_{\beta,I}\lim_{\alpha,I} \|y + \lambda x_\alpha + (1-\lambda)x_\beta\| = \tau(y). \tag{1.2}$$

Let E be a Banach space and let E'' be its second dual. Throughout, we consider E as a subspace of E''. For every fixed $g'' \in E''$, define the function $\tau_{g''} : E \to \mathbb{R}$ by letting $\tau_{g''}(x) = \|x+g''\|$ for all $x \in E$. It is immediate from the principle of local reflexivity that $\tau_{g''}$ is a type over E.

If τ is a type over E that can be represented in this way, we call τ a *double-dual type over E*.

Maurey [8] and Rosenthal [11] have given a characterization of double-dual types over separable Banach spaces. The author [9] has generalized this characterization to not necessarily separable Banach spaces as follows.

THEOREM 1.4. *Let E be a Banach space and $\tau : E \to \mathbb{R}$ a type over E. Then the following are equivalent:*
(i) *τ is a double-dual type over E;*
(ii) *there exists a net $(x_\alpha)_{\alpha\in I}$ in E that doubly generates τ.*

This paper is devoted to proving the following characterization of double-dual types over $C(K)$ in terms of the representation using the sc pairs.

Theorem 1.5. *Let τ be a type over $C(K)$, represented by the sc pair (ℓ, u) as in Theorem 1.2. Let*

$$\begin{aligned} Y_\ell &= \Big\{x \in K : x \text{ is not isolated and } \ell(x) < \liminf_{y \to x} \ell(y)\Big\}, \\ Y_u &= \Big\{x \in K : x \text{ is not isolated and } u(x) > \limsup_{y \to x} u(y)\Big\}. \end{aligned} \tag{1.3}$$

The following are equivalent:

(i) *τ is a double-dual type over $C(K)$;*

(ii) $Y_\ell \cap Y_u = \varnothing$;

(iii) *there exists a net $(f_\alpha)_{\alpha \in I}$ which doubly generates τ.*

The next section will include a discussion of generating nets. In Section 3, several properties of singular points of sc pairs will be proved. The main Theorem 1.5 will then be proved in Section 4.

2. Generating nets in $C(K)$

In this section, we introduce concepts that are needed to prove the main theorem.

We use the standard notion for convergence of nets in topological spaces according to [3, Section 1.6]. We recall the basic definitions for the convenience of the reader.

Definition 2.1. (i) A partially ordered set $(I, \leq)$ is a *directed set* if for any $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. Such an element γ is called a *successor* of α (and β).

(ii) Let $(I, \leq)$ be a directed set. For every element $\alpha_0 \in I$, define $|\alpha_0| = \operatorname{card}(\{\alpha \in I : \alpha \leq \alpha_0\})$, the number of predecessors of α_0.

(iii) Let $(I, \leq)$ and $(J, \leq)$ be directed sets. A function $k : I \to J$ is *order-preserving* if $\alpha \leq \beta \in I$ implies $k(\alpha) \leq k(\beta)$. A function $k : I \to J$ is *cofinal* if for every $\gamma \in J$ there exists $\alpha \in I$ such that $\gamma \leq k(\alpha)$.

(iv) Let $(I, \leq)$ be a directed set and K a topological space. We say that $(x_\alpha)_{\alpha \in I}$ is a *net in K indexed by I* if $x_\alpha \in K$ for all $\alpha \in I$. If K is a normed space, then $(x_\alpha)_{\alpha \in I}$ is *bounded* if $\{\|x_\alpha\| : \alpha \in I\}$ is bounded in $\mathbb{R}$.

(v) Let $(I, \leq)$ be a directed set, K a topological space, and $(x_\alpha)_{\alpha \in I}$ a net in K indexed by I. If $j : I \to I$ is a cofinal order-preserving function, then $(x_{j(\alpha)})_{\alpha \in I}$ is a *subnet* of $(x_\alpha)_{\alpha \in I}$.

(vi) Let $(I, \leq)$ be a directed set, K a topological space, and $(x_\alpha)_{\alpha \in I}$ a net in K indexed by I. Let $x \in K$. Then $\lim_{\alpha, I} x_\alpha = x$ if and only if for every neighborhood U of x in K there exists $\alpha \in I$ such that $x_\beta \in U$ for all $\beta \geq \alpha$.

(vii) Let $(I, \leq)$ be a directed set and $(r_\alpha)_{\alpha \in I}$ a bounded net of real numbers. Then define

$$\begin{aligned} \limsup_{\alpha, I} r_\alpha &= \inf_{\alpha \in I} \sup \{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}, \\ \liminf_{\alpha, I} r_\alpha &= \sup_{\alpha \in I} \inf \{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}. \end{aligned} \tag{2.1}$$

Observe that $\limsup_{\alpha, I} r_\alpha$ and $\liminf_{\alpha, I} r_\alpha$ exist for every bounded net $(r_\alpha)_{\alpha \in I}$ in $\mathbb{R}$.

We now consider the Banach lattice $\ell_\infty(K)$ of bounded real-valued functions on K, equipped with the sup-norm.

A subset $H \subseteq \ell_\infty$ is called *bounded* if $\sup\{\|f\| : f \in H\} < \infty$. Let H be such a set. The pointwise supremum of H is the real-valued function L defined by $L(x) = \sup\{h(x) : h \in H\}$ for every $x \in K$. We write $L = \bigvee H$ for this function. Similarly, the pointwise infimum of H is the real-valued function U defined by $U(x) = \inf\{h(x) : h \in H\}$ for every $x \in K$. This function is denoted by $\bigwedge H$. Note that both $\bigvee H$ and $\bigwedge H$ are again in $\ell_\infty(K)$.

If $H \subseteq \ell_\infty(K)$ is a bounded set of usc functions, then the pointwise infimum $\bigwedge H$ is usc. Similarly, the pointwise supremum of a bounded set of lsc functions is lsc. Finally, it is clear that $f \in C(K)$ is continuous if and only if f is usc and lsc. Therefore, if H is a bounded set of continuous functions on K, then $\bigwedge H$ is usc and $\bigvee H$ is lsc.

Let τ be a type over $C(K)$ and let $(f_\alpha)_{\alpha\in I}$ generate τ as in Proposition 1.3(iii) above. We construct the sc pair (ℓ, u) of Theorem 1.2 as follows.

For every $\alpha \in I$, define a lower semicontinuous functions ℓ_α and an upper semicontinuous function u_α on K by setting

$$\begin{aligned} \ell_\alpha &= \bigvee \left\{ f \in C(K) : f \le f_\beta \ \forall \beta \ge \alpha \right\}, \\ u_\alpha &= \bigwedge \left\{ f \in C(K) : f \ge f_\beta \ \forall \beta \ge \alpha \right\}. \end{aligned} \tag{2.2}$$

Then set

$$u = \bigwedge_\alpha u_\alpha, \qquad \ell = \bigvee_\alpha \ell_\alpha. \tag{2.3}$$

Here are some basic properties of the functions ℓ and u defined in (2.3). See [10] for details.

Remark 2.2. Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of functions and let ℓ_α, ℓ, u_α, and u be as in (2.3) above.

(i) If $\alpha_1, \alpha_2 \in I$ and $\alpha_1 \le \alpha_2$, then $\ell_{\alpha_1} \le \ell_{\alpha_2} \le \ell$ and $u_{\alpha_1} \ge u_{\alpha_2} \ge u$.

(ii) If $x \in K$ and $\varepsilon > 0$, then there exists an $\alpha_0 = \alpha(x,\varepsilon) \in I$ such that for all indices $\alpha > \alpha_0$,

$$\ell_\alpha(x) \ge \ell(x) - \varepsilon, \qquad u_\alpha(x) \le u(x) + \varepsilon. \tag{2.4}$$

(iii) For every $\beta \in I$, every $x \in K$, every $\delta > 0$, and every neighborhood U of x, there exists $y \in U$ and $\gamma \ge \beta$ such that $f_\gamma(y) \le \ell_\beta(x) + \delta$.

(iv) For every $\beta \in I$, every $x \in K$, every $\delta > 0$, and every neighborhood U of x, there exists $y \in U$ and $\gamma \ge \beta$ such that $f_\gamma(y) \ge u_\beta(x) - \delta$.

Proof. (i) and (ii) are trivial. To prove (iii) let $\beta \in I$, let $x \in K$, $\delta > 0$, and U a neighborhood of x. Suppose that for every $y \in U$ and all $\gamma \ge \beta$ we have $f_\gamma(y) > \ell_\beta(x) + \delta$. Then we may choose a function $g \in C(K)$ such that $g \le f_\gamma$ for all $\gamma \ge \beta$ and $g(x) = \ell_\beta(x) + \delta$. This would imply that $\ell_\beta(x) = \bigvee\{f \in C(K) : f \le f_\gamma \text{ for all } \gamma \ge \beta\} \ge g(x) = \ell_\beta(x) + \delta$. This is a contradiction. The proof of (iv) is dual to the proof of (iii). □

Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of functions in $C(K)$ that generates a type τ over $C(K)$. Choose ℓ and u as in (2.3) and assume $x \in K$ and $u(x) = r$. It can be shown that for every neighborhood U of x and for every $\varepsilon > 0$, there exists an index α_0 such that for every

$\alpha \geq \alpha_0$, there exists $y \in U$ such that $f_\alpha(y) > r - \varepsilon$. If U, ε, and r are fixed, then we define for every $\alpha \in I$,

$$V_\alpha := \{y \in U : f_\alpha(y) > r - \varepsilon\}. \tag{2.5}$$

Hence, for every $x \in K$, every neighborhood U of x, and every $\varepsilon > 0$, there exists an index α_0 such that $V_\alpha \neq \emptyset$ for all $\alpha > \alpha_0$.

The following definition introduces stronger conditions.

Definition 2.3. Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of functions in $C(K)$. Let ℓ and u be as in (2.3).

(i) $(f_\alpha)_{\alpha\in I}$ *generates* u *at* x *within* Ω if for every $\varepsilon > 0$ and every neighborhood U of x, there exists an index α_0 such that for all $\alpha > \alpha_0$, there exists β_0 such that $V_\alpha \cap V_\beta \neq \emptyset$ for all $\beta > \beta_0$.

(ii) The net $(f_\alpha)_{\alpha\in I}$ *generates* u *within* Ω if it generates u at x within Ω for every $x \in K$.

(iii) $(f_\alpha)_{\alpha\in I}$ generates ℓ at x within Ω if $(-f_\alpha)_{\alpha\in I}$ generates $-\ell$ at x within Ω.

(iv) The net $(f_\alpha)_{\alpha\in I}$ generates ℓ within Ω if it generates ℓ at x within Ω for every $x \in K$.

Proposition 2.4. *Let $(f_\alpha)_{\alpha\in I}$ be a bounded net of functions in $C(K)$ that generates a type τ. Let u be as in (2.3).*

(i) *If u is continuous at x, then $(f_\alpha)_{\alpha\in I}$ generates u at x within Ω.*

(ii) *If $\lim_{\alpha,I} f_\alpha(x) = u(x)$, then $(f_\alpha)_{\alpha\in I}$ generates u at x within Ω.*

(iii) *If $(x_\beta)_{\beta\in I}$ is a net in K that converges to x and if $\lim_{\beta,I} u(x_\beta) = u(x)$ and $\lim_{\alpha,I} f_\alpha(x_\beta) = u(x_\beta)$ for all β, then $(f_\alpha)_{\alpha\in I}$ generates u at x within Ω.*

The statement is also true if u is replaced with ℓ.

Proof. To show (i) let $\varepsilon > 0$ and U a neighborhood of x. We may assume that $|u(y) - u(x)| < \varepsilon/2$ for all $y \in U$. By Remark 2.2(ii) there exists $\alpha_0 \in I$ such that for all $\alpha > \alpha_0$,

$$V_\alpha = \{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \neq \emptyset. \tag{2.6}$$

Now fix such an α and choose $y \in V_\alpha$. Then (using Remark 2.2(iii)) there exists β_0 such that for every $\beta \geq \beta_0$,

$$f_\beta(z_\beta) > u(y) - \frac{\varepsilon}{2} \tag{2.7}$$

for some $z_\beta \in V_\alpha$. Therefore, $z_\beta \in V_\alpha \cap V_\beta$, which shows that the net $(f_\alpha)_{\alpha\in I}$ generates u at x within Ω. Statement (ii) is immediate from the definition.

To show (iii) let U be a neighborhood of x and $\varepsilon > 0$. There exists $\beta \in I$ such that $x_\beta \in U$ and $|u(x) - u(x_\beta)| < \varepsilon/2$. Fix such a $\beta \in I$ and choose $\alpha_0 \in I$ such that $|f_\alpha(x_\beta) - u(x_\beta)| < \varepsilon/2$ for all $\alpha > \alpha_0$. Then $x_\beta \in V_\alpha = \{y \in U : f_\alpha(y) > u(x) - \varepsilon\}$ for all $\alpha > \alpha_0$; that is, $V_\alpha \cap V_{\alpha'} \neq \emptyset$ for all $\alpha, \alpha' > \alpha_0$. □

3. Singular points of semicontinuous pairs

Our next goal is to find necessary and sufficient conditions on ℓ and u for the existence of a single net that generates both ℓ and u within Ω.

Definition 3.1. Let u be a usc function and $x \in K$. We call x a *singular* point of u, if x is not an isolated point of K and

$$u(x) > \limsup_{y \to x} u(y). \tag{3.1}$$

Similarly, we call x a *singular* point of an lsc function ℓ if x is not isolated and

$$\ell(x) < \liminf_{y \to x} \ell(y). \tag{3.2}$$

We call x a *regular point* of u (resp., ℓ) if x is not isolated and not a singular point of u (resp., ℓ).

It is immediate from the definition that x is a singular point of u if and only if there exists an open neighborhood U of x such that

$$u(x) > \sup\{u(y) : y \in U \setminus \{x\}\}. \tag{3.3}$$

If U is such a neighborhood and $V \subseteq U$ is another neighborhood of x, then

$$u(x) > \sup\{u(y) : y \in V \setminus \{x\}\}. \tag{3.4}$$

If x is a regular point of u, then there exists a net $(x_\beta)_{\beta \in I}$ in K which converges to x such that $u(x) = \lim_{\beta,I} u(x_\beta)$ and $x_\beta \neq x$ for all $\beta \in I$.

Proposition 3.2. *Let (ℓ, u) be an sc pair in $\ell_\infty(K)$. Let $x \in K$ be a nonisolated point and $(f_\alpha)_{\alpha \in I}$ a net which generates both ℓ and u within Ω at x.*

(i) *If x is a singular point of u, then x is a regular point of ℓ and* $\lim_{\alpha,I} f_\alpha(x) = u(x)$.

(ii) *If x is a singular point of ℓ, then x is a regular point of u and* $\lim_{\alpha,I} f_\alpha(x) = \ell(x)$.

Proof. First we prove the following claim, which is the second statement of (i).

$$\text{If } x \text{ is a singular point of } u, \text{ then } \lim_{\alpha,I} f_\alpha(x) = u(x). \tag{3.5}$$

Proof of the claim. Let x be a singular point of u and suppose $\lim_{\alpha,I} f_\alpha(x) \neq u(x)$. Choose $\varepsilon > 0$ and an open neighborhood U' of x such that

$$u(x) - \varepsilon > \sup\{u(y) : y \in U' \setminus \{x\}\} \tag{3.6}$$

and such that

$$\limsup_{\alpha,I} f_\alpha(x) < u(x) - 2\varepsilon. \tag{3.7}$$

There exists a further open neighborhood U of x such that $x \in U \subseteq \overline{U} \subseteq U'$ and $\overline{U}$ is compact. We may fix α_0 such that for all $\alpha > \alpha_0$,

$$f_\alpha(x) < u(x) - \varepsilon, \qquad V_\alpha = \left\{y \in U : f_\alpha(y) > u(x) - \frac{\varepsilon}{3}\right\} \neq \varnothing. \tag{3.8}$$

Let $\alpha > \alpha_0$. Then

$$W_\alpha = \left\{ y \in U : f_\alpha(y) < u(x) - \frac{2\varepsilon}{3} \right\} \tag{3.9}$$

is an open neighborhood of x which is disjoint from V_α. Since $(f_\alpha)_{\alpha \in I}$ generates u at x within Ω, there exists β_0 such that for all $\beta > \beta_0$, we may choose

$$y_\beta \in \left\{ y \in V_\alpha : f_\beta(y) > u(x) - \frac{\varepsilon}{3} \right\}. \tag{3.10}$$

By passing to a subnet if necessary, we may assume that $\lim_{\beta,I} y_\beta = y$ for some $y \in \overline{U}$. We obtain $u_\beta(y) \geq u(x) - \varepsilon/3$ for all $\beta \in I$ with $\beta \geq \beta_0$ and hence

$$u(y) \geq u(x) - \frac{\varepsilon}{3}, \tag{3.11}$$

which contradicts (3.6). So $\lim_{\alpha,I} f_\alpha(x) = u(x)$ and the claim is established. □

The dual statement of claim (3.5) reads as follows:

$$\text{if } x \text{ is a singular point of } \ell, \text{ then } \lim_{\alpha,I} f_\alpha(x) = \ell(x). \tag{3.12}$$

It is proved using an argument dual to the proof of claim (3.5). This shows the second part of (ii).

To prove the first part of (i) observe that x singular for u implies $\ell(x) < u(x)$, and therefore $\lim_{\alpha,I} f_\alpha(x) = u(x) \neq \ell(x)$. Using the contrapositive of statement (3.12) above shows that x is not a singular point of ℓ; that is, x is a regular point of ℓ.

Likewise, (3.5) can be used to show that if x a singular point of ℓ, then x is a regular point of u. □

Let (ℓ, u) be an sc pair and Y_ℓ and Y_u the sets of singular points of ℓ and u, respectively. If $(f_\alpha)_{\alpha \in I}$ is a net that generates both ℓ and u within Ω, then Y_ℓ and Y_u are disjoint by Proposition 3.2. The following proposition proves the existence of such a net, provided that Y_ℓ and Y_u are disjoint.

Proposition 3.3. *Let K be a compact Hausdorff space and (ℓ, u) an sc pair in $\ell_\infty(K)$. Consider the sets Y_ℓ, Y_u of singular points of ℓ, u, respectively. Suppose that $Y_\ell \cap Y_u = \varnothing$. Then there exists a net $(f_\alpha)_{\alpha \in I}$ of continuous functions which generates ℓ and u within Ω.*

The proof of this proposition requires the following theorem.

Theorem 3.4 (Edwards [2]). *Let U be a usc function and L an lsc function on a compact Hausdorff space K such that $U \leq L$. Then there exists a continuous function F such that $U \leq F \leq L$.*

A proof of this theorem can be found in Kaplan [6, (48.5)].

Proof of Proposition 3.3. Let $\mathcal{U}$ be a base for the topology Ω such that $\mathcal{U}$ does not contain the empty set and the only finite sets in $\mathcal{U}$ are singletons. Let $I = \mathfrak{P}_{<\infty}(\mathcal{U}) \setminus \{\varnothing\}$, the set of finite subsets of $\mathcal{U}$, be partially ordered by inclusion.

By induction on $|\alpha|$ construct an increasing net of integers $(k_\alpha)_{\alpha\in I}$ and for every $1 \le k \le k_\alpha$ construct functions $g_\alpha^{(1)}$, $g_\alpha^{(2)}$ and $f_\alpha \in C(K)$ and finite collections of nonempty open sets $\mathfrak{B}_\alpha = \{V_{\alpha,1},\dots,V_{\alpha,k_\alpha} \supseteq \alpha$ and elements $z_{i,\alpha,k} \in V_{\alpha,k}$ for $i = 1,2$ and all $1 \le k \le k_\alpha$, such that the following conditions hold for every $\alpha \in I$ and every $k = 1,\dots,k_\alpha$:

$$u(z_{1,\alpha,k}) \ge \sup\{u(y) : y \in V_{\alpha,k}\} - \frac{1}{|\alpha|}, \tag{3.13}$$

$$\ell(z_{2,\alpha,k}) \le \inf\{\ell(y) : y \in V_{\alpha,k}\} + \frac{1}{|\alpha|}, \tag{3.14}$$

$$g_\alpha^{(1)}(z_{j,\alpha,k}) = u(z_{j,\alpha,k}) \quad \text{for } j = 1,2, \tag{3.15}$$

$$g_\alpha^{(2)}(z_{j,\alpha,k}) = \ell(z_{j,\alpha,k}) \quad \text{for } j = 1,2, \tag{3.16}$$

$$u \le g_\alpha^{(1)} \le \bigwedge_{\beta<\alpha} g_\beta^{(1)} \le \|u\|\mathbf{1}, \tag{3.17}$$

$$\ell \ge g_\alpha^{(2)} \ge \bigvee_{\beta<\alpha} g_\beta^{(2)} \ge -\|\ell\|\mathbf{1}, \tag{3.18}$$

$$f_\alpha(z_{1,\alpha,k}) = u(z_{1,\alpha,k}), \qquad f_\alpha(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k}). \tag{3.19}$$

Furthermore, for every $\beta < \alpha$ and every $1 \le k \le k_\beta$, the following nonempty open sets are required to be among the elements of $\mathfrak{B}_\alpha$:

$$V_{1,\beta,k}^{(\alpha)} = \left\{y \in V_{\beta,k} : f_\beta(y) > u(z_{1,\beta,k}) - \frac{1}{|\alpha|}\right\}, \tag{3.20}$$

$$V_{2,\beta,k}^{(\alpha)} = \left\{y \in V_{\beta,k} : f_\beta(y) < \ell(z_{2,\beta,k}) + \frac{1}{|\alpha|}\right\}. \tag{3.21}$$

We use induction on $|\alpha|$. If $\alpha = \varnothing$, let $f_\varnothing = g_\varnothing^{(1)} = \|u\|\mathbf{1}$ and $g_\varnothing^{(2)} = -\|\ell\|\mathbf{1}$ and set $\mathfrak{B}_\varnothing = \varnothing$. With this choice, conditions (3.13)–(3.21) are either trivial or vacuously true.

If $\alpha \in I$ and $\alpha \ne \varnothing$, suppose as inductive hypothesis that the construction has been completed for every $\beta \in I$ with $\beta < \alpha$. Let

$$\mathfrak{B}_\alpha = \left\{V_{i,\beta,k}^{(\alpha)} : i = 1,2;\ \beta < \alpha;\ 1 \le k \le k_\beta\right\} \cup \alpha \cup \bigcup_{\beta<\alpha} \mathfrak{B}_\beta, \tag{3.22}$$

where $V_{1,\beta,k}$ and $V_{2,\beta,k}$ are as in (3.20) for all $1 \le k \le k_\alpha$. Observe that $\mathfrak{B}_\alpha$ is a finite collection of nonempty open sets. Say

$$\mathfrak{B}_\alpha = \{V_{\alpha,1},\dots,V_{\alpha,k_\alpha}\}, \tag{3.23}$$

where $(V_{\alpha,k})_{k=1}^{k_\alpha}$ are pairwise distinct. For $i = 1,2$ and $1 \le k \le k_\alpha$, we choose $z_{i,\alpha,k} \in V_{\alpha,k}$ satisfying (3.13) and (3.14), and such that for all $1 \le k, j \le k_\alpha$, and $i_1, i_2 \in \{1,2\}$, we have $z_{i_1,\alpha,k} = z_{i_2,\alpha,j}$ if and only if either $j = k$ and $i_1 = i_2$ or $j = k$ and $V_{\alpha,k}$ is a singleton.

Note that such a choice is possible, since the singular points of ℓ and u are disjoint and the only finite sets in $\mathcal{U}$ are singletons.

We now construct $g_\alpha^{(1)}$ and $g_\alpha^{(2)}$ satisfying (3.15) through (3.18).

By inductive hypothesis in (3.17), $u \le \bigwedge_{\beta<\alpha} g_\beta^{(1)} \le \|u\|\mathbf{1}$. We define an lsc function L on K by setting

$$L(x) = \begin{cases} u(x) & \text{if } x = z_{j,\alpha,k} \quad \text{for some } j = 1,2;\ 1 \le k \le k_\alpha, \\ \bigwedge_{\beta<\alpha} g_\beta^{(1)}(x) & \text{otherwise.} \end{cases} \tag{3.24}$$

Because $u \le L$, we may apply Theorem 3.4 and obtain $g_\alpha^{(1)} \in C(K)$ with $u \le g_\alpha^{(1)} \le L$. This choice of $g_\alpha^{(1)}$ satisfies (3.15) and (3.17). We use a dual construction to define $g_\alpha^{(2)}$ satisfying conditions (3.16) and (3.18).

In order to construct f_α define a usc function U and an lsc function L on K by setting for every $x \in K$,

$$\begin{aligned} U(x) &= \begin{cases} g_\alpha^{(1)}(x) & \text{if } x = z_{1,\alpha,k} \quad \text{for some } 1 \le k \le k(\alpha), \\ g_\alpha^{(2)}(x) & \text{otherwise,} \end{cases} \\ L(x) &= \begin{cases} g_\alpha^{(2)}(x) & \text{if } x = z_{2,\alpha,k} \quad \text{for some } 1 \le k \le k(\alpha), \\ g_\alpha^{(1)}(x) & \text{otherwise.} \end{cases} \end{aligned} \tag{3.25}$$

Observe that $U \le L$; by Theorem 3.4 there exists a continuous function f_α with $U \le f_\alpha \le L$. By construction of U and L and (3.15), we have

$$U(x_{1,\alpha,k}) = L(x_{1,\alpha,k}) = g_\alpha^{(1)}(z_{1,\alpha,k}) = u(z_{1,\alpha,k}) \tag{3.26}$$

for all $1 \le k \le k_\alpha$. Hence, $f_\alpha(z_{1,\alpha,k}) = u(z_{1,\alpha,k})$. Furthermore,

$$U(x_{2,\alpha,k}) = L(x_{2,\alpha,k}) = g_\alpha^{(2)}(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k}) \tag{3.27}$$

for all $1 \le k \le k_\alpha$. Thus, $f_\alpha(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k})$. Condition (3.19) follows from these last two observations.

This completes the construction and we now proceed to show that the net $(f_\alpha)_{\alpha\in I}$ generates u and ℓ within Ω.

Fix $x \in K$, $\varepsilon > 0$, and $U \in \Omega$. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$. Fix $\beta \in I$ with $|\beta| > n$, such that for some $V \in \beta$ we have $x \in V \subseteq U$. Choose $1 \le k \le k_\beta$ such that $V = V_{\beta,k} \in \mathfrak{B}_\beta$. Applying (3.13) yields

$$u(z_{1,\beta,k}) \ge \sup\{u(y) : y \in V_{\beta,k}\} - \frac{1}{|\beta|} \ge u(x) - \frac{1}{|\beta|}. \tag{3.28}$$

So by (3.19), $f_\beta(z_{1,\beta,k}) = u(z_{1,\beta,k})$. Now let $\alpha > \beta$. By (3.20) there exists $1 \le j \le k_\alpha$ such that

$$V_{\alpha,j} = V_{1,\beta,k}^{(\alpha)} = \left\{ y \in V_{\beta,k} : f_\beta(y) > u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \right\}. \tag{3.29}$$

In particular,

$$z_{1,\beta,k} \in V_{\alpha,j}. \tag{3.30}$$

Observe that by (3.13)

$$u(z_{1,\alpha,j}) \geq \sup\{u(y) : y \in V_{\alpha,j}\} - \frac{1}{|\alpha|} \tag{3.31}$$

and $z_{1,\alpha,j} \in V_{\alpha,j}$. Thus,

$$\begin{aligned} f_\beta(z_{1,\alpha,j}) &> u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \quad \text{by (3.29)} \\ &\geq u(x) - \frac{1}{|\beta|} - \frac{1}{|\alpha|} \quad \text{by (3.28)} \\ &> u(x) - \varepsilon. \end{aligned} \tag{3.32}$$

On the other hand,

$$\begin{aligned} f_\alpha(z_{1,\alpha,j}) &= u(z_{1,\alpha,j}) \quad \text{by (3.19)} \\ &\geq \sup\{u(y) : y \in V_{\alpha,j}\} - \frac{1}{|\alpha|} \quad \text{by (3.31)} \\ &\geq u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \quad \text{by (3.30)} \\ &\geq u(x) - \frac{1}{|\beta|} - \frac{1}{|\alpha|} \quad \text{by (3.28)} \\ &> u(x) - \varepsilon. \end{aligned} \tag{3.33}$$

Therefore,

$$z_{1,\alpha,j} \in \{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \cap \{y \in U : f_\beta(y) > u(x) - \varepsilon\} \neq \emptyset \tag{3.34}$$

for all $\alpha \geq \beta$. This shows that the net $(f_\alpha)_{\alpha\in I}$ generates u within Ω. The proof that it generates ℓ within Ω follows from a similar argument. □

Let τ be a type over $C(K)$ that is represented by the sc pair (ℓ, u) as in Theorem 1.2. Propositions 3.2 and 3.3 prove that the sets Y_ℓ and Y_u of singular points of ℓ and u are disjoint if and only if there exists a net $(f_\alpha)_{\alpha\in I}$ that generates both ℓ and u within Ω.

4. Proof of the main theorem

We now consider a net $(f_\alpha)_{\alpha\in I}$ that generates a type τ over $C(K)$. As before, let this type be represented by the sc pair (ℓ, u).

To establish the main theorem, we will now prove that the net doubly generates the type τ if and only if the net generates both ℓ and u within Ω. This is accomplished in the following two lemmas.

Lemma 4.1. *Let K be a compact Hausdorff space and τ a type over $C(K)$. Let $(f_\alpha)_{\alpha\in I}$ be a net that doubly generates τ. Let (ℓ, u) be the sc pair such that $\tau(g) = \max\{\|\ell + g\|, \|u + g\|\}$ for all $g \in C(K)$. Then $(f_\alpha)_{\alpha\in I}$ generates ℓ and u within Ω.*

Proof. Assume the conclusion does not hold. Then either $(f_\alpha)_{\alpha\in I}$ does not generate u within Ω at some $x \in K$, or it does not generate ℓ within Ω at some $x \in K$. We distinguish between these two cases.

Case 1. $(f_\alpha)_{\alpha\in I}$ does not generate u at x within Ω. Let $\lambda = 1/2$. There exists an open neighborhood U of x and $\varepsilon > 0$ such that for all $\alpha_0 \in I$ and $\beta_0 \in I$ there exist $\alpha > \alpha_0$ and $\beta > \beta_0$, for which

$$\{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \cap \{y \in U : f_\beta(y) > u(x) - \varepsilon\} = \varnothing. \tag{4.1}$$

Let $U_0 = \{y \in U : u(y) < u(x) + \varepsilon/2\}$ and choose an open neighborhood U_1 of x such that $U_1 \subseteq \overline{U}_1 \subseteq U_0 \subseteq U$. We claim that there exists $\alpha_0 \in I$ such that $\|f_\alpha\| \le \|\tau\| + \varepsilon/2$ and $f_\alpha\restriction_{\overline{U}_1} \le u(x) + \varepsilon/2$ for all $\alpha \ge \alpha_0$. (Here, $\|\tau\| = \tau(0)$.)

First observe that there exists α_1 such that for all $\alpha \ge \alpha_1$ we have $\|f_\alpha\| \le \|\tau\| + \varepsilon/2$. Suppose there does not exist $\alpha_0 \ge \alpha_1$ such that $f_\alpha\restriction_{\overline{U}_1} \le u(x) + \varepsilon/2$ for all $\alpha \ge \alpha_0$. Then there exist a cofinal order-preserving map $i : I \to I$ such that $f_{i(\alpha)}(y_{i(\alpha)}) > u(x) + \varepsilon/2$, where $y_{i(\alpha)} \in \overline{U}_1$ for all $\alpha \in I$. We may assume that $(y_{i(\alpha)})_{\alpha\in I}$ converges to $y_0 \in \overline{U}_1$. Thus, $u(y_0) \ge u(x) + \varepsilon/2$, which contradicts the choice of U_0 and establishes the claim.

Fix a function $g \in C(K)$ such that $g\restriction_{K\setminus U_1} = 0$ and $g(x) = 3\|\tau\|$ and $0 \le g \le 3\|\tau\|$. Observe that $\|u + g\| \ge g(x) + u(x) = 3\|\tau\| + u(x)$.

Further, for each $\alpha \ge \alpha_0$, there exists $\alpha_2 \ge \alpha$ and a cofinal order-preserving function $j = j_{\alpha_2} : I \to I$ such that

$$\left\{y \in U_1 : \frac{1}{2} f_{\alpha_2}(y) > \frac{1}{2}u(x) - \frac{1}{2}\varepsilon\right\} \cap \left\{y \in U_1 : \frac{1}{2} f_{j(\beta)}(y) > \frac{1}{2}u(x) - \frac{1}{2}\varepsilon\right\} = \varnothing. \tag{4.2}$$

Fix such α_2 and $j = j_{\alpha_2}$. If $y \in \overline{U}_1$,

$$g(y) + \frac{1}{2} f_{\alpha_2}(y) + \frac{1}{2} f_{j(\beta)}(y) \le 3\|\tau\| + u(x) - \frac{\varepsilon}{4} \tag{4.3}$$

for all $\beta \in I$. If $y \in K \setminus U_1$, we have

$$g(y) + \frac{1}{2} f_{\alpha_2}(y) + \frac{1}{2} f_{j(\beta)}(y) \le \|\tau\| + \frac{\varepsilon}{2}. \tag{4.4}$$

Observe that $\lim_{\beta,I} \|g + 1/2 f_{\alpha_2} + 1/2 f_\beta\|$ exists. Thus,

$$\lim_{\beta,I}\left\|g + \frac{1}{2} f_{\alpha_2} + \frac{1}{2} f_\beta\right\| = \lim_{\beta,I}\left\|g + \frac{1}{2} f_{\alpha_2} + \frac{1}{2} f_{j(\beta)}\right\| \le 3\|\tau\| + u(x) - \frac{\varepsilon}{4}. \tag{4.5}$$

Hence,

$$\liminf_{\alpha,I} \lim_{\beta,I}\left\|g + \frac{1}{2} f_\alpha + \frac{1}{2} f_\beta\right\| \le 3\|\tau\| + u(x) - \frac{\varepsilon}{4} < \lim_{\alpha,I}\|g + f_\alpha\|. \tag{4.6}$$

This contradicts the assumption that $(f_\alpha)_{\alpha\in I}$ doubly generates τ.

Case 2. $(f_\alpha)_{\alpha\in I}$ does not generate ℓ at x within Ω. This case is handled with an argument dual to the one in Case 1. □

Lemma 4.2. *Let K be a compact Hausdorff space and τ a type over $C(K)$. Let (ℓ,u) be the sc pair such that $\tau(g) = \max\{\|\ell+g\|,\|u+g\|\}$ for all $g \in C(K)$. Assume that $(f_\alpha)_{\alpha\in I}$ generates ℓ and u within Ω. Then $(f_\alpha)_{\alpha\in I}$ doubly generates τ.*

Proof. Fix $g \in C(K)$. Because $\tau(g) = \max\{\|\ell+g\|,\|u+g\|\}$, we distinguish between two cases.

Case 1. Suppose that $\tau(g) = \|u+g\|$. Choose $x \in K$ such that $\|u+g\| = u(x)+g(x)$. Let $\varepsilon > 0$ and choose a neighborhood U of x such that $|g(y)-g(x)| < \varepsilon/2$ for all $y \in U$. Choose $\alpha_0 \in I$ such that for all $\alpha > \alpha_0$, there exists $\beta_0 \in I$ such that for all $\beta > \beta_0$, we have $f_\alpha(z) > u(x)-\varepsilon/2$ and $f_\beta(z) > u(x)-\varepsilon/2$ for some $z \in U$. Then

$$\|g+\lambda f_\alpha+(1-\lambda)f_\beta\| \geq |g(z)+\lambda f_\alpha(z)+(1-\lambda)f_\beta(z)| > u(x)+g(x)-\varepsilon = \|u+g\|-\varepsilon. \tag{4.7}$$

Therefore,

$$\liminf_{\alpha,I}\lim_{\beta,I}\|g+\lambda f_\beta+(1-\lambda)f_\alpha\| \geq \|u+g\|-\varepsilon. \tag{4.8}$$

On the other hand,

$$\begin{aligned}&\limsup_{\alpha,I}\lim_{\beta,I}\|g+\lambda f_\alpha+(1-\lambda)f_\beta\|\\ &\leq \limsup_{\alpha,I}\lambda\|g+f_\alpha\|+\lim_{\beta,I}(1-\lambda)\|f_\beta+g\|\\ &\leq \|u+g\|.\end{aligned} \tag{4.9}$$

Because ε was arbitrary, this shows that

$$\lim_{\alpha,I}\lim_{\beta,I}\|g+\lambda f_\alpha+(1-\lambda)f_\beta\| \tag{4.10}$$

exists and equals $\tau(g)$.

Case 2. If $\tau(g) = \|\ell+g\|$, consider the net $(-f_\alpha)_{\alpha\in I}$, which generates $-u$ and $-\ell$ within Ω and the function $-g \in C(K)$. We infer from Case 1 that

$$\lim_{\beta,I}\lim_{\alpha,I}\|g+\lambda f_\alpha+(1-\lambda)f_\beta\| = \lim_{\beta,I}\lim_{\alpha,I}\|-g+\lambda(-f_\alpha)+(1-\lambda)(-f_\beta)\| = \|\ell+g\|. \tag{4.11}$$

Therefore, $\lim_{\beta,I}\lim_{\alpha,I}\|g+\lambda f_\alpha+(1-\lambda)f_\beta\| = \tau(g)$ for all $g \in C(K)$. □

Proof of Theorem 1.5. The equivalence between (i) and (iii) is Theorem 1.4 above. The implication (ii) ⇒ (iii) follows from Proposition 3.2 and Lemma 4.1 and (iii)⇒(ii) follows from Proposition 3.3 and Lemma 4.2. □

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