ON A SUBCLASS OF n-STARLIKE FUNCTIONS

MUGUR ACU AND SHIGEYOSHI OWA

Received 21 February 2005

In 1999, Kanas and Rønning introduced the classes of starlike and convex functions, which are normalized with $f(w) = f'(w) - 1 = 0$ and w a fixed point in U. In 2005, the authors introduced the classes of functions close to convex and α-convex, which are normalized in the same way. All these definitions are somewhat similar to the ones for the uniform-type functions and it is easy to see that for $w = 0$, the well-known classes of starlike, convex, close-to-convex, and α-convex functions are obtained. In this paper, we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definitions of the well-known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$
$$S^c = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0,\ z \in U \right\}. \tag{1.1}$$

Let w be a fixed point in U and $A(w) = \{f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0\}$.

In [3], Kanas and Rønning introduced the following classes:

$$S(w) = \{ f \in A(w) : f \text{ is univalent in } U\},$$
$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0,\ z \in U \right\}, \tag{1.2}$$
$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \frac{(z-w)f''(z)}{f'(z)} > 0,\ z \in U \right\}.$$

International Journal of Mathematics and Mathematical Sciences 2005:17 (2005) 2841–2846
DOI: 10.1155/IJMMS.2005.2841

It is obvious that a natural "Alexander relation" exists between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \qquad \text{iff } f(z) = (z-w)g'(z) \in S^*(w). \tag{1.3}$$

Denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$ that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re}\ p(z) > 0$ for $z \in U$.

2. Preliminary results

If is easy to see that a function $f_{(z)} \in A(w)$ has the series of expansions:

$$f(z) = (z-w) + a_2(z-w)^2 + \dots. \tag{2.1}$$

In [8], Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$.

Theorem 2.1. *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$, then*

$$\left| B_n \right| \leq \frac{2}{(1+d)(1-d)^n}, \quad \text{where } d = |w|,\ n \geq 1. \tag{2.2}$$

Using the above result, Kanas and Rønning obtain the following theorem in [3].

Theorem 2.2. *Let $f \in S^*(w)$ and $f(z) = (z-w) + b_2(z-w)^2 + \dots$. Then*

$$\begin{gathered} \left| b_2 \right| \leq \frac{2}{1-d^2}, \qquad \left| b_3 \right| \leq \frac{3+d}{\left(1-d^2\right)^2}, \\ \left| b_4 \right| \leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{\left(1-d^2\right)^3}, \qquad \left| b_5 \right| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{\left(1-d^2\right)^4}, \end{gathered} \tag{2.3}$$

where $d = |w|$.

Remark 2.3. It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.

In [1], are also defined the following sets:

$$\begin{gathered} D(w) = \left\{ z \in U : \operatorname{Re}\left[\frac{w}{z}\right] < 1,\ \operatorname{Re}\left[\frac{z(1+z)}{(z-w)(1-z)}\right] > 0 \right\} \quad \text{for } w \neq 0,\ D(0) = U; \\ s(w) = \{ f : D(w) \longrightarrow \mathbb{C} \} \cap S(w); \qquad s^*(w) = S^*(w) \cap s(w), \end{gathered} \tag{2.4}$$

where w is a fixed point in U.

The authors consider the integral operator $L_a : A(w) \to A(w)$ defined by

$$f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \cdot \int_w^z F(t) \cdot (t-w)^{a-1} dt, \quad a \in \mathbb{R},\ a \geq 0. \tag{2.5}$$

The next theorem is a result of the so called "admissible functions method" introduced by Mocanu and Miller (see [3, 4, 6]).

Theorem 2.4. *Let h be convex in U and* $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. *If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{2.6}$$

then $p(z) \prec h(z)$.

3. Main results

Deffinition 3.1. Let w be a fixed point in U, $n \in \mathbb{N}$. D_w^n denotes the differential operator:

$$\begin{gathered} D_w^n : A(w) \longrightarrow A(w) \ \text{ with },\\ D_w^0 f(z) = f(z),\\ D_w^1 f(z) = D_w f(z) = (z-w) \cdot f'(z),\\ D_w^n f(z) = D_w\left(D_w^{n-1} f(z)\right). \end{gathered} \tag{3.1}$$

Remark 3.2. For $f \in A(w)$, $f(w) = (z-w) + \sum_{j=2}^{\infty} a_j (z-w)^j$, we have

$$D_w^n f(z) = (z-w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z-w)^j. \tag{3.2}$$

It easy to see that if we take $w = 0$, we obtain the Sălăgean differential operator (see [7]).

Deffinition 3.3. Let w be a fixed point in U, $n \in \mathbb{N}$ and $f \in S(w)$. f is said to be an n-w-starlike function if

$$\operatorname{Re} \frac{D_w^{n+1} f(z)}{D_w^n f(z)} > 0, \quad z \in U. \tag{3.3}$$

The class of all these functions is denoted by $S_n^*(w)$.

Remark 3.4. (1) $S_0^*(w) = S^*(w)$ and $S_n^*(0) = S_n^*$, where S_n^* is the class of n-starlike functions introduced by Sălăgean in [7].

(2) If $f(z) \in S_n^*(w)$ and we denote $D_w^n f(z) = g(z)$, we obtain $g(z) \in S^*(w)$.

(3) Using the class $s(w)$, we obtain $s_n^*(w) = S_n^*(w) \cap s(w)$.

Theorem 3.5. *Let w be a fixed point in U and $n \in \mathbb{N}$. If $f(z) \in s_{n+1}^*(w)$ then $f(z) \in s_n^*(w)$. This means*

$$s_{n+1}^*(w) \subset s_n^*(w). \tag{3.4}$$

Proof. From $f(z)\in s^*_{n+1}(w)$, we have $\operatorname{Re}(D^{n+2}_w f(z)/D^{n+1}_w f(z))>0, z\in U$. We denote $p(z)=(D^{n+1}_w f(z)/D^n_w f(z))$, where $p(0)=1$ and $p(z)\in\mathcal{H}(U)$. We obtain

$$\begin{aligned}\frac{D^{n+2}_w f(z)}{D^{n+1}_w f(z)} &= \frac{D_w(D^{n+1}_w f(z))}{D_w(D^n_w f(z))} = \frac{(z-w)(D^{n+1}_w f(z))'}{(z-w)(D^n_w f(z))'} = \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'},\\ p'(z) &= \frac{(D^{n+1}_w f(z))'\cdot(D^n_w f(z))-(D^{n+1}_w f(z))\cdot(D^n_w f(z))'}{(D^n_w f(z))^2}\\ &= \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'}\cdot\frac{(D^n_w f(z))'}{D^n_w f(z)} - p(z)\cdot\frac{(D^n_w f(z))'}{D^n_w f(z)}.\end{aligned}\tag{3.5}$$

Thus we have

$$\begin{aligned}(z-w)\cdot p'(z) &= \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'}\cdot\frac{(z-w)\cdot(D^n_w f(z))'}{D^n_w f(z)} - p(z)\cdot\frac{(z-w)\cdot(D^n_w f(z))'}{D^n_w f(z)},\\ (z-w)\cdot p'(z) &= \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'}\cdot p(z)-[p(z)]^2,\\ \frac{(D^{n+1}_w f(z))'}{(D^n_w f(z))'} &= p(z)+\frac{1}{p(z)}\cdot(z-w)\cdot p'(z).\end{aligned}\tag{3.6}$$

From $\operatorname{Re}(D^{n+2}_w f(z)/D^{n+1}_w f(z))>0$ we obtain $p(z)+(1/p(z))\cdot(z-w)\cdot p'(z)\prec((1+z)/(1-z))$ or

$$p(z)+\frac{zp'(z)}{1/(1-(w/z))\cdot p(z)}\prec\frac{1+z}{1-z}\equiv h(z),\quad \text{with } h(0)=1. \tag{3.7}$$

By hypothesis, we have $\operatorname{Re}[1/(1-(w/z))\cdot h(z)]>0$, and thus from Theorem 2.4 we obtain $p(z)\prec h(z)$ or $\operatorname{Re}p(z)>0$. This means $f\in s^*_n(w)$. □

Remark 3.6. From Theorem 3.5, we obtain $s^*_n(w)\subset s^*_0(w)\subset S^*(w)$, $n\in\mathbb{N}$.

Theorem 3.7. *If $F(z)\in s^*_n(w)$ then $f(z)=L_aF(z)\in S^*_n(w)$, where L_a is the integral operator defined by (2.5).*

Proof. From (2.5) we obtain

$$(1+a)\cdot F(z)=a\cdot f(z)+(z-w)\cdot f'(z). \tag{3.8}$$

By means of the application of the operator D^{n+1}_w we obtain

$$(1+a)\cdot D^{n+1}_w F(z)=a\cdot D^{n+1}_w f(z)+D^{n+1}_w[(z-w)\cdot f'(z)] \tag{3.9}$$

or

$$(1+a)\cdot D_w^{n+1}F(z) = a\cdot D_w^{n+1}f(z) + D_w^{n+2}f(z). \tag{3.10}$$

Similarly, by means of the application of the operator D_w^n, we obtain

$$(1+a)\cdot D_w^{n}F(z) = a\cdot D_w^{n}f(z) + D_w^{n+1}f(z). \tag{3.11}$$

Thus

$$\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{(D_w^{n+2}f(z)/D_w^{n+1}f(z))\cdot(D_w^{n+1}f(z)/D_w^nf(z)) + a\cdot(D_w^{n+1}f(z)/D_w^nf(z))}{(D_w^{n+1}f(z)/D_w^nf(z))+a}. \tag{3.12}$$

Using the notation $D_w^{n+1}f(z)/D_w^nf(z) = p(z)$, with $p(0)=1$, we have

$$\frac{(z-w)\cdot p'(z)}{p(z)} = \frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} - p(z) \tag{3.13}$$

or

$$\frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} = p(z) + \frac{(z-w)\cdot p'(z)}{p(z)}. \tag{3.14}$$

Thus

$$\begin{aligned}\frac{D_w^{n+1}F(z)}{D_w^nF(z)} &= \frac{p(z)\left[p(z)+((z-w)p'(z)/p(z))+a\right]}{p(z)+a}\\ &= p(z) + \frac{zp'(z)}{(1/(1-(w/z)))p(z)+(a/(1-(w/z)))}.\end{aligned} \tag{3.15}$$

From $F(z)\in s_n^*(w)$ we obtain $(D_w^{n+1}F(z)/D_w^nF(z)) \prec ((1+z)/(1-z)) \equiv h(z)$ or

$$p(z) + \frac{zp'(z)}{(1/(1-(w/z)))p(z)+(a/(1-(w/z)))} \prec h(z). \tag{3.16}$$

By hypothesis, we have $\mathrm{Re}[(1/(1-(w/z)))\cdot h(z)+(a/(1-(w/z)))]>0$ and from Theorem 2.4 we obtain $p(z)\prec h(z)$ or $\mathrm{Re}\{D_w^{n+1}f(z)/D_w^nf(z)\}>0$, $z\in U$. This means $f(z) = L_aF(z)\in S_n^*(w)$. □

Remark 3.8. If we consider $w=0$ in Theorem 3.7 we obtain that the integral operator defined by (2.5) preserves the class of n-starlike functions, and if we consider $w=0$ and $n=0$ in the above theorem we obtain that the integral operator defined by (2.5) preserves the well-known class of starlike functions.

THEOREM 3.9. *Let w be a fixed point in U and $f \in S_n^*(w)$ with $f(z) = (z-w) + \sum_{j=2}^{\infty} a_j \cdot (z-w)^j$. Then*

$$\begin{aligned} |a_2| &\le \frac{1}{2^{n-1}\cdot(1-d^2)}, \\ |a_3| &\le \frac{3+d}{3^n\cdot(1-d^2)^2}, \\ |a_4| &\le \frac{(2+d)(3+d)}{2^{2n-1}\cdot 3\cdot(1-d^2)^3}, \\ |a_5| &\le \frac{(2+d)(3+d)(3d+5)}{5^n\cdot 6\cdot(1-d^2)^4}, \end{aligned} \tag{3.17}$$

where $d = |w|$.

Proof. From Remark 3.4 for $f \in S_n^*(w)$ we obtain

$$D_w^n f(z) = g(z) \in S^*(w). \tag{3.18}$$

If we consider $g(z) = (z-w) + \sum_{j=2}^{\infty} b_j \cdot (z-w)^j$, using Remark 3.2, from (3.18) we obtain $j^n \cdot a_j = b_j$, $j = 2,3,\ldots$.

Thus we have $a_j = 1/j^n \cdot b_j$, $j = 2,3,\ldots$, and from the estimates (2.3) we get the result. □

References

[1] M. Acu and S. Owa, *On some subclasses of univalent functions*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 3, article 70, 1–6.

[2] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), no. 1, 87–92.

[3] S. Kanas and F. Rønning, *Uniformly starlike and convex functions and other related classes of univalent functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 95–105.

[4] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), no. 2, 157–172.

[5] ———, *On some classes of first-order differential subordinations*, Michigan Math. J. **32** (1985), no. 2, 185–195.

[6] ———, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations **56** (1985), no. 3, 297–309.

[7] G. Ş. Sălăgean, *Subclasses of univalent functions*, Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362–372.

[8] J. K. Wald, *On starlike functions*, Ph.D. thesis, University of Delaware, Delaware, 1978.

Mugur Acu: Department of Mathematics, "Lucian Blaga" University of Sibiu, 5–7 Ion Raţiu Street, 550012 Sibiu, Romania

E-mail address: acu_mugur@yahoo.com

Shigeyoshi Owa: Department of Mathematics, School of Science and Engineering, Kinki University, Higashiosaka, Osaka 577-8502, Japan

E-mail address: owa@math.kindai.ac.jp