WEAKLY TIGHT FUNCTIONS AND THEIR DECOMPOSITION

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The present paper deals with the study of a weakly tight function and its relation to tight functions. We obtain a Jordan-decomposition-type theorem for a locally bounded weakly tight real-valued function defined on a sublattice of I^X, followed by the notion of a total variation.

1. Introduction

The notion of a signed measure arises if a measure is allowed to take on both positive and negative values. A set that is both positive and negative with respect to a signed measure is termed as a null set. Some concepts in measure theory can be generalized by means of classes of null sets. An abstract formulation and proof of the Lebesgue decomposition theorem using the concept of null sets is given by Ficker [5]. A real-valued function satisfying certain properties that can be expressed as a difference of two nonnegative functions possessing the same properties is called "decomposable." Several Jordan-decomposition-type theorems are exhibited in [3]. Faires and Morrison [4] exposed conditions on a vector-valued measure that ensure vector-valued Jordan-decomposition-type theorem to hold. For a signed null-additive fuzzy measure, a Jordan-decomposition-type theorem is investigated by Pap in [11].

The problem of generation of measures by tight functions defined on a lattice of sets has been taken up by several authors [1, 2, 6, 8, 9]. Nayak and Srinivasan [10] initiated a weaker form of tightness for a real-valued function μ defined on a lattice of sets to decompose μ as a difference $\mu^+ - \mu^-$ and then extended it to a countably additive measure.

In Section 2, we have defined and studied the notions of measuring envelopes, modular functions, and additive functions. The notions of superadditive and subadditive functions are also given with the help of pointwise addition of elements in I^X. The lower envelope β_* of a superadditive function β defined on a sublattice K of I^X turns out to be superadditive. In Section 3, we introduce the notion of a weakly tight function $\beta : K \to \mathbb{R}$, where K is a sublattice of I^X containing 0 and 1 (cf. [10]). The condition imposed on the $[0,1]$-valued function β to be a weakly tight function is less restrictive than that for being a tight function. It is proved that a superadditive, monotone, and weakly tight function

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β is tight (there K is taken to be closed under addition). In Section 4, our main result states that a locally bounded, weakly tight real-valued function β defined on K has a representation of the form $\beta^+ - \beta^-$; both β^+ and β^- are nonnegative monotone (and hence, locally bounded) functions defined on K. If, in addition, β is additive and modular, then the decomposed parts β^+ and β^- preserve superadditive and supermodular properties. The total variation $|\beta|$ of β is defined as the sum of β^+ and β^-, following the terminology in the classical measure theory (cf. [12]); few properties of $|\beta|$ are noted.

Notations. Throughout this paper, X is a nonempty set and $I \equiv [0,1]$ is the unit interval of the real line $\mathbb{R}$; C denotes a subfamily of I^X of all functions from X to I; K stands for a sublattice of I^X containing the least element 0 and the greatest element 1, where 0 and 1 are constant functions sending each $x \in X$ to 0 and 1, respectively. We will denote by β a function from K to I satisfying $\beta(0) = 0$.

2. Measuring envelopes

Let C be a sublattice of I^X and let $\xi : C \to I$ be a function. Then ξ is called *monotone* if f, $g \in C$, $g \le f \Rightarrow \xi(g) \le \xi(f)$. The mapping ξ is called *supermodular* (*submodular*, resp.) if for $f,g \in C$, $\xi(f)+\xi(g) \le \xi(f \vee g)+\xi(f \wedge g)(\xi(f)+\xi(g) \ge \xi(f \vee g)+\xi(f \wedge g)$, resp.); ξ is said to be *modular* if it is both supermodular and submodular. The mapping ξ is said to be *superadditive* (*subadditive*, resp.) if for $f_1, f_2 \in C$ such that $f_1 + f_2 \in C$, $\xi(f_1 + f_2) \ge \xi(f_1)+\xi(f_2)$ $(\xi(f_1 + f_2) \le \xi(f_1)+\xi(f_2)$, resp.); ξ is said to be *additive* if it is both superadditive and subadditive. If we restrict ourselves to disjoint crisp sets in C, then the condition of being additive for ξ coincides with that of [10].

The family $C \subseteq I^X$ is said to be *closed under addition* if for $f,g \in C$ with $f+g \in I^X$, we have $f+g \in C$, and is said to be *closed under addition modulo* 1 (or *closed under* $\oplus$) if $f,g \in C$ implies that $f \oplus g \in C$, where $f \oplus g = (f+g) \wedge 1$.

A function $\xi : C \to I$ is called *subadditive modulo* 1 if for $f_1, f_2 \in C$ with $f_1 \oplus f_2 \in C$, we have $\xi(f_1 \oplus f_2) \le \xi(f_1)+\xi(f_2)$; here $C \subseteq I^X$.

If $f,g \in I^X$ and $f+g \in I^X$, then $f \oplus g = f+g$. Thus if C is closed under addition modulo 1, then C is closed under addition. Also, if ξ is subadditive modulo 1, then ξ is subadditive.

The definition for a function $\xi : C \to I$ to be supermodular (submodular, superadditive, subadditive, resp.) continues to hold for a real-valued function ξ defined on C.

Definition 2.1. Let $\beta : K \to I$ be a function satisfying $\beta(0) = 0$. Define $\beta_* : I^X \to I$ and $\beta^* : I^X \to I$ by

$$\begin{aligned} \beta_*(f) &= \sup\{\beta(g) : g \le f,\ g \in K\}, \\ \beta^*(f) &= \inf\{\beta(g) : g \ge f,\ g \in K\}, \quad f \in I^X. \end{aligned} \tag{2.1}$$

β_* and β^* are called the *lower envelope* and the *upper envelope* of β, respectively.

We obtain

(i) $\beta_*(0) = 0 = \beta^*(0)$;

(ii) both β_* and β^* are monotone;

(iii) $\beta^* \mid K \le \beta \le \beta_* \mid K$;
(iv) β is monotone if and only if $\beta_* \mid K = \beta = \beta^* \mid K$.

Proposition 2.2. (i) *If β is supermodular, then β_* is supermodular.*
(ii) *If β is submodular, then β^* is submodular.*

Proof. (i) Let $f_1, f_2 \in I^X$. Let $\varepsilon > 0$. Then there exist $g_1, g_2 \in K$, $f_1 \ge g_1$, $f_2 \ge g_2$ such that

$$\begin{aligned} \beta_*(f_1) - \frac{\varepsilon}{2} &< \beta(g_1), \\ \beta_*(f_2) - \frac{\varepsilon}{2} &< \beta(g_2). \end{aligned} \tag{2.2}$$

It follows that

$$\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta(g_1 \vee g_2) + \beta(g_1 \wedge g_2). \tag{2.3}$$

Consequently,

$$\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta_*(f_1 \vee f_2) + \beta_*(f_1 \wedge f_2). \tag{2.4}$$

Since ε is arbitrary, we get

$$\beta_*(f_1) + \beta_*(f_2) \le \beta_*(f_1 \vee f_2) + \beta_*(f_1 \wedge f_2). \tag{2.5}$$

Proof of (ii) follows analogously. □

Proposition 2.3. (i) *If K is closed under addition and β is superadditive, then β_* is superadditive.*
(ii) *If K is closed under addition modulo 1 and β is subadditive modulo 1, then β^* is subadditive modulo 1, and hence β^* is subadditive.*

Proof. (i) Let f_1 and f_2 be in I^X such that $f_1 + f_2 \in I^X$. Let $\varepsilon > 0$. Then there exist $g_1, g_2 \in K$ with $g_1 \le f_1$ and $g_2 \le f_2$ such that $\beta(g_1) > \beta_*(f_1) - \varepsilon/2$ and $\beta(g_2) > \beta_*(f_2) - \varepsilon/2$. Hence, $\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta(g_1) + \beta(g_2)$. Since, for any $x \in X$, $0 \le (g_1 + g_2)(x) \le (f_1 + f_2)(x) \le 1$, we get $g_1 + g_2 \in I^X$ and so $g_1 + g_2 \in K$. Since β is superadditive, $\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta(g_1 + g_2) \le \beta_*(f_1 + f_2)$ and hence, we obtain $\beta_*(f_1) + \beta_*(f_2) \le \beta_*(f_1 + f_2)$.

(ii) Let $f_1, f_2 \in I^X$. Let $\varepsilon > 0$. Then there exist $g_1, g_2 \in K$ with $f_1 \le g_1$ and $f_2 \le g_2$ such that

$$\beta^*(f_1) + \beta^*(f_2) + \varepsilon > \beta(g_1) + \beta(g_2). \tag{2.6}$$

Since $g_1, g_2 \in K$ and K is closed under $\oplus$, we get $g_1 \oplus g_2 \in K$. Also, $f_1 \oplus f_2 \le g_1 \oplus g_2$. Hence (2.6) gives

$$\beta^*(f_1) + \beta^*(f_2) + \varepsilon > \beta(g_1 \oplus g_2) \ge \beta^*(f_1 \oplus f_2). \tag{2.7}$$

Since ε is arbitrary, we obtain that β^* is subadditive modulo 1. □

Definitions 2.4. Let β_1 and β_2 be real-valued functions defined on K. Define $(\beta_1+\beta_2)(f)=\beta_1(f)+\beta_2(f)$, $f\in K$, and $(\beta_1-\beta_2)(f)=\beta_1(f)-\beta_2(f)$, $f\in K$. Likewise, for $\beta: K\to\mathbb{R}$ and $\lambda\in\mathbb{R}$, define $(\lambda\beta)(f)=\lambda\beta(f)$, $f\in K$.

Proposition 2.5. (i) *If β_1 and β_2 are supermodular (submodular, resp.), then $\beta_1+\beta_2$ is supermodular (submodular, resp.).*

If both β_1 and β_2 are modular, then so are $\beta_1+\beta_2$ and $\beta_1-\beta_2$.

(ii) *If β is supermodular (submodular, resp.) then $\lambda\beta$ is supermodular (submodular, resp.), where λ is a nonnegative real number.*

(iii) *If K is closed under addition, and β_1,β_2 are superadditive (subadditive, resp.), then $\beta_1+\beta_2$ is superadditive (subadditive, resp.). If both β_1 and β_2 are additive, then so are $\beta_1+\beta_2$ and $\beta_1-\beta_2$.*

(iv) *If K is closed under addition, and β is superadditive (subadditive, resp.), then $\lambda\beta$ is superadditive (subadditive, resp.), where λ is a nonnegative real number.*

Proof. We will prove only (i) and (iii).

(i) Let β_1 and β_2 be supermodular. Let $f,g\in K$. Then,

$$\begin{aligned}(\beta_1+\beta_2)(f)+(\beta_1+\beta_2)(g)&=\beta_1(f)+\beta_2(f)+\beta_1(g)+\beta_2(g)\\&\leq\beta_1(f\vee g)+\beta_1(f\wedge g)+\beta_2(f\vee g)+\beta_2(f\wedge g)\\&=(\beta_1+\beta_2)(f\vee g)+(\beta_1+\beta_2)(f\wedge g).\end{aligned}\tag{2.8}$$

If β_1 and β_2 are submodular, then, by similar arguments, $\beta_1+\beta_2$ is submodular.

(iii) Let K be closed under addition. Let β_1 and β_2 be superadditive. Let $f,g\in K$. Then,

$$\begin{aligned}(\beta_1+\beta_2)(f)+(\beta_1+\beta_2)(g)&=\beta_1(f)+\beta_2(f)+\beta_1(g)+\beta_2(g)\\&\leq\beta_1(f+g)+\beta_2(f+g)=(\beta_1+\beta_2)(f+g).\end{aligned}\tag{2.9}$$

□

3. Weakly tight functions

Definition 3.1. Let $\beta: K\to I$ with $\beta(0)=0$. Then β is called *tight* (*cotight*, resp.) if

$$\begin{gathered}\beta(f_2)=\beta(f_1)+\beta_*(f_2-f_1),\quad f_1,f_2\in K,\ f_1\leq f_2,\\(\beta(f_2)=\beta(f_1)+\beta^*(f_2-f_1),\ f_1,f_2\in K,\ f_1\leq f_2,\ \text{resp.}).\end{gathered}\tag{3.1}$$

If β is tight (or cotight), then β is modular and monotone. Furthermore, if β is tight (cotight, resp.), then β_* (β^*, resp.) is an extension of β. A detailed study of tight and cotight functions is made in [7, 13].

Definition 3.2. Let $\beta: K\to\mathbb{R}$ be a function with $\beta(0)=0$. Then β is called *weakly tight* if for every pair $f_1,f_2\in K$ with $f_1\leq f_2$ and for any $\varepsilon>0$, there exists $f\in K$ such that $f\leq f_2-f_1$ and

$$|\beta(f_2)-\beta(f_1)-\beta(f)|<\varepsilon.\tag{3.2}$$

Proposition 3.3. *Let $\beta : K \to I$ be a function satisfying $\beta(0) = 0$. If β is tight, then β is weakly tight.*

Proof. Let $f_1, f_2 \in K$ with $f_1 \le f_2$. Let $\varepsilon > 0$. Since β is tight, there exists $f \in K$ with $f \le f_2 - f_1$ such that

$$\beta_*(f_2 - f_1) - \varepsilon = \beta(f_2) - \beta(f_1) - \varepsilon < \beta(f) \tag{3.3}$$

or

$$\beta(f_2) - \beta(f_1) - \beta(f) < \varepsilon. \tag{3.4}$$

Since β is monotone and $f \le f_2 - f_1$, we obtain

$$\beta(f) = \beta_*(f) \le \beta_*(f_2 - f_1) = \beta(f_2) - \beta(f_1), \tag{3.5}$$

and so

$$\beta(f_2) - \beta(f_1) - \beta(f) \ge 0. \tag{3.6}$$

Thus, $|\beta(f_2) - \beta(f_1) - \beta(f)| < \varepsilon$. □

Proposition 3.4. *Let K be closed under addition. Let $\beta : K \to I$ be superadditive, monotone, and weakly tight. Then β is tight.*

Proof. Let $f_1, f_2 \in K$ with $f_1 \le f_2$. Let $\varepsilon > 0$. Since β is weakly tight, there exists $f \in K$ such that $f \le f_2 - f_1$ and

$$|\beta(f_2) - \beta(f_1) - \beta(f)| < \varepsilon. \tag{3.7}$$

Consequently,

$$\beta(f_2) - \beta(f_1) < \beta(f) + \varepsilon \le \beta_*(f_2 - f_1) + \varepsilon. \tag{3.8}$$

Since ε is arbitrary, we get $\beta(f_2) - \beta(f_1) \le \beta_*(f_2 - f_1)$.

Since, by Proposition 2.3(i), β_* is superadditive, we get $\beta(f_2) = \beta_*(f_2) \ge \beta_*(f_2 - f_1) + \beta_*(f_1)$, which yields that $\beta_*(f_2 - f_1) \le \beta_*(f_2) - \beta_*(f_1) = \beta(f_2) - \beta(f_1)$.

Thus $\beta_*(f_2 - f_1) = \beta(f_2) - \beta(f_1)$, that is, β is tight. □

4. A Jordan-decomposition-type theorem

In this section, β is a real-valued function defined on a sublattice K of I^X containing 0 and 1. Also, it is assumed throughout this section that β is *locally bounded*, that is, for any f in K, $\sup\{\beta(g) : g \le f,\ g \in K\}$ exists. For a locally bounded real-valued function β, the definitions of lower and upper envelopes, β_* and β^*, of β may be given in the same way.

Definition 4.1. For $f \in K$, define $\beta^+(f)$ and $\beta^-(f)$ as follows:

$$\begin{aligned}\beta^+(f) &= \sup\{\beta(g) : g \le f,\ g \in K\},\\ \beta^-(f) &= -\inf\{\beta(g) : g \le f,\ g \in K\}.\end{aligned} \tag{4.1}$$

Remarks 4.2. (i) $\beta^+ = \beta_* \mid K$.

(ii) $\beta^- = (-\beta)^+$; $\beta^+ = (-\beta)^-$.

(iii) Both β^+ and β^- are nonnegative, monotone (and hence, locally bounded).

(iv) $-\beta^- \le \beta \le \beta^+$.

Theorem 4.3. *Let β be a weakly tight real-valued function defined on K. Then $\beta = \beta^+ - \beta^-$.*

Proof. Let $\varepsilon > 0$. Let $f \in K$. Then there exists $f_1 \in K$ such that $f_1 \le f$ and

$$\beta^+(f) < \beta(f_1) + \frac{\varepsilon}{2}. \tag{4.2}$$

Since β is weakly tight, there exists $f_2 \in K$ such that $f_2 \le f - f_1$ and

$$|\beta(f) - (\beta(f_1) + \beta(f_2))| < \frac{\varepsilon}{2}. \tag{4.3}$$

This implies that

$$\beta(f_1) + \beta(f_2) < \beta(f) + \frac{\varepsilon}{2}. \tag{4.4}$$

Also, $f_2 \le f - f_1 \le f$ yields that $-\beta^-(f) \le \beta(f_2)$. Hence, using (4.2) and (4.4), we get $\beta^+(f) - \beta^-(f) < \beta(f_1) + \varepsilon/2 + \beta(f_2) < \beta(f) + \varepsilon$. Since ε is arbitrary, we get

$$\beta^+ - \beta^- \le \beta. \tag{4.5}$$

Replacing β in (4.5) by $-\beta$, we get

$$(-\beta)^+ - (-\beta)^- \le -\beta, \tag{4.6}$$

or

$$\beta^- - \beta^+ \le -\beta, \tag{4.7}$$

or

$$\beta^+ - \beta^- \ge \beta. \tag{4.8}$$

Thus, $\beta^+ - \beta^- = \beta$. □

Proposition 4.4. *Let K be closed under addition. If β is additive, then both β^+ and β^- are superadditive.*

Proof. The proof follows from Proposition 2.3(i). □

Proposition 4.5. *If β is modular, then both β^+ and β^- are supermodular.*

Proof. The proof follows from Proposition 2.2(i). □

The results obtained in this section may be summarized as follows.

Theorem 4.6 (Jordan-decomposition-type theorem). *Let K be a sublattice of I^X containing* 0 *and* 1. *If $\beta : K \to \mathbb{R}$ is locally bounded and weakly tight, then β can be written as*

$$\beta = \beta^+ - \beta^-, \tag{4.9}$$

where both β^+ and β^- are nonnegative and monotone (and hence, locally bounded) functions defined on K. Furthermore, if β is modular (β is additive and K is closed under addition), then the decomposed parts β^+ and β^- are supermodular (superadditive).

Definition 4.7. For a function $\beta : K \to \mathbb{R}$, define the *total variation* of β, written as $|\beta|$, by

$$|\beta| = \beta^+ + \beta^-. \tag{4.10}$$

Theorem 4.8. *Let $\beta : K \to \mathbb{R}$ be a locally bounded function.*

(i) *If β is weakly tight, then $\beta = 0 \Leftrightarrow |\beta| = 0$.*

(ii) *For each $f \in K$, $|\beta(f)| \le |\beta|(f)$.*

(iii) *If β is modular, then $|\beta|$ is supermodular.*

(iv) *Let K be closed under addition. Then β being additive implies that $|\beta|$ is superadditive.*

Proof. (i) If $\beta = 0$, then $\beta^+ = 0 = \beta^-$ and so $|\beta| = 0$. Conversely, if $|\beta| = 0$, then both β^+ and β^- vanish. Since β is weakly tight, by Theorem 4.3, $\beta = \beta^+ - \beta^-$. Hence $\beta = 0$.

(ii) Let $f \in K$. Then by Remark 4.2(iv), $\beta(f) \le \beta^+(f)$, and $-\beta(f) \le \beta^-(f)$.

If $\beta(f) > 0$, then

$$|\beta|(f) = \beta^+(f) + \beta^-(f) \ge \beta(f) = \left|\beta(f)\right|. \tag{4.11}$$

If $\beta(f) < 0$, then

$$\left|\beta(f)\right| = -\beta(f) \le \beta^-(f) \le |\beta|(f). \tag{4.12}$$

(iii) Follows from Proposition 4.5 and Proposition 2.5(i).

(iv) Follows from Proposition 4.4 and Proposition 2.5(iii). □

Remark 4.9. If we restrict ourselves to $\{0,1\}$-valued functions in K, then K may be viewed as a sublattice of $P(X)$. For a $[0,\infty]$-valued function β defined on this restricted K over $P(X)$, the definitions of lower and upper envelopes, β_* and β^*, reduce to the corresponding definitions in classical theory given by Adamski [1]. In this manner, the present study generalizes the theory in [1].

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