PARALLEL SYNCHRONOUS ALGORITHM FOR NONLINEAR FIXED POINT PROBLEMS

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We give in this paper a convergence result concerning parallel synchronous algorithm for nonlinear fixed point problems with respect to the Euclidian norm in $\mathbb{R}^n$. We then apply this result to some problems related to convex analysis like minimization of functionals, calculus of saddle point, and convex programming.

1. Introduction

This study is motivated by the paper of Bahi [2], where he has given a convergence result concerning parallel synchronous algorithm for linear fixed point problems using nonexpansive linear mappings with respect to a weighted maximum norm. Our goal is to extend this result to a nonlinear fixed point problem

$$F(x^*) = x^*, \tag{1.1}$$

with respect to the Euclidian norm, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear operator. Section 2 is devoted to a brief description of asynchronous parallel algorithm. In Section 3, we prove the main result concerning the convergence of the general algorithm in the synchronous case to a fixed point of a nonlinear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$. A particular case of this algorithm (algorithm of Jacobi) is applied in Section 4 to the operator $F = (I+T)^{-1}$ which is called the proximal mapping associated with the maximal monotone operator T (see Rockafellar [10]).

2. Preliminaries on asynchronous algorithms

Asynchronous algorithms are used in the parallel treatment of problems taking in consideration the interaction of several processors. Write $\mathbb{R}^n$ as the product $\prod_{i=1}^{\alpha} \mathbb{R}^{n_i}$, where $\alpha \in \mathbb{N} - \{0\}$ and $n = \sum_{i=1}^{\alpha} n_i$. All vectors $x \in \mathbb{R}^n$ considered in this study are splitted in the form $x = (x_1,\dots,x_\alpha)$, where $x_i \in \mathbb{R}^{n_i}$. Let $\mathbb{R}^{n_i}$ be equipped with the inner product $\langle\cdot,\cdot\rangle_i$ and the associated norm $\|\cdot\|_i = \langle\cdot,\cdot\rangle_i^{1/2}$. $\mathbb{R}^n$ will be equipped with the inner

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product $\langle x,y\rangle = \sum_{i=1}^{\alpha}\langle x_i,y_i\rangle_i$, where $x,y \in \mathbb{R}^n$ and the associated norm $\|x\| = \langle x,x\rangle^{1/2} = (\sum_{i=1}^{\alpha}\|x_i\|_i^2)^{1/2}$. It will be equipped also with the maximum norm defined by

$$\|x\|_\infty = \max_{1\le i\le\alpha}\left\|x_i\right\|_i. \tag{2.1}$$

Define $J = \{J(p)\}_{p\in\mathbb{N}}$ as a sequence of nonempty subsets of $\{1,\ldots,\alpha\}$ and $S = \{(s_1(p),\ldots,$ and $s_\alpha(p))\}_{p\in\mathbb{N}}$ as a sequence of $\mathbb{N}^\alpha$ such that

(i) for all $i \in \{1,\ldots,\alpha\}$, the subset $\{p \in \mathbb{N},\ i \in J(p)\}$ is infinite;
(ii) for all $i \in \{1,\ldots,\alpha\}$, for all $p \in \mathbb{N}$, $s_i(p) \le p$;
(iii) for all $i \in \{1,\ldots,\alpha\}$, $\lim_{p\to\infty} s_i(p) = \infty$.

Consider an operator $F = (F_1,\ldots,F_\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ and define the asynchronous algorithm associated with F by

$$\begin{gathered} x^0 = (x_1^0,\ldots,x_\alpha^0) \in \mathbb{R}^n, \\ x_i^{p+1} = \begin{cases} x_i^p & \text{if } i \notin J(p), \\ F_i\left(x_1^{s_1(p)},\ldots,x_\alpha^{s_\alpha(p)}\right) & \text{if } i \in J(p), \end{cases} \\ i = 1,\ldots,\alpha, \\ p = 0,1,\ldots \end{gathered} \tag{2.2}$$

(see Bahi and et al. [3], El Tarazi [4]). It will be denoted by (F,x^0,J,S). This algorithm describes the behavior of iterative process executed asynchronously on a parallel computer with α processors. At each iteration $p+1$, the ith processor computes x_i^{p+1} by using (2.2) (see Bahi [1]).

$J(p)$ is the subset of the indexes of the components updated at the pth step. $p - s_i(p)$ is the delay due to the ith processor when it computes the ith block at the pth iteration.

If we take $s_i(p) = p$ for all $i \in \{1,\ldots,\alpha\}$, then (2.2) describes synchronous algorithm (without delay). During each iteration, every processor executes a number of computations that depend on the results of the computations of other processors in the previous iteration. Within an iteration, each processor does not interact with other processors, all interactions take place at the end of iterations (see Bahi [2]).

If we take

$$\begin{gathered} s_i(p) = p \quad \forall p \in \mathbb{N},\ \forall i \in \{1,\ldots,\alpha\}, \\ J(p) = \{1,\ldots,\alpha\} \quad \forall p \in \mathbb{N}, \end{gathered} \tag{2.3}$$

then (2.2) describes the algorithm of Jacobi.

If we take

$$\begin{gathered} s_i(p) = p \quad \forall p \in \mathbb{N},\ \forall i \in \{1,\ldots,\alpha\}, \\ J(p) = p+1 \ (\mathrm{mod}\ \alpha) \quad \forall p \in \mathbb{N}, \end{gathered} \tag{2.4}$$

then (2.2) describes the Gauss-Seidel algorithm. For more details about asynchronous algorithms, see [1, 2, 3, 4]. In the following theorem, Bahi [2] has shown the convergence of the sequence $\{x^p\}$ defined by (2.2) in the synchronous linear case, that is, F is a linear operator and $s_i(p) = p$, for all $p \in \{1,\ldots,\alpha\}$.

Theorem 2.1. *Consider $\{T^p\}_{p\in\mathbb{N}}$ a sequence of matrices in $\mathbb{R}^{n\times n}$. Suppose the following hold.*

(h$_0$) *There exists a subsequence $\{p_k\}_{k\in\mathbb{N}}$ such that $J(p_k)=\{1,\dots,\alpha\}$.*

(h$_1$) *There exists $\gamma \gg 0$($\gamma \gg 0$ means that $\gamma_i>0$ $\forall i\in\{1,\dots,\alpha\}$), for all $p\in\mathbb{N}$, T^p is nonexpansive (a matrice $A\in\mathbb{R}^{n\times n}$ is said to be nonexpansive with respect to the norm $\|\cdot\|$ if for all $x\in\mathbb{R}^n$, $\|Ax\|\leq \|x\|$. A is said to be paracontracting if for all $x\in\mathbb{R}^n$, $x\neq Ax \iff \|Ax\|<\|x\|$), with respect to a weighted maximum norm $\|\cdot\|_{\infty,\gamma}$ defined by*

$$x\in\mathbb{R}^n,\quad \|x\|_{\infty,\gamma}=\max_{1\leq i\leq\alpha}\frac{\|x_i\|_i}{\gamma_i}. \tag{2.5}$$

(h$_2$) *$\{T^p\}_{p\in\mathbb{N}}$ converges to a matrix Q which is paracontracting with respect to the norm $\|\cdot\|_{\infty,\gamma}$.*

(h$_3$) *For all $p\in\mathbb{N}$, $\mathcal{N}(I-Q)\subseteq\mathcal{N}(I-T^p)$ ($\mathcal{N}$ denotes the null space), then*
(1) *for all $x^0\in\mathbb{R}^n$, the sequence $\{x^p\}_{p\in\mathbb{N}}$ is convergent in $\mathbb{R}^n$;*
(2) $\lim_{p\to\infty}x^p=x^*\in\mathcal{N}(I-Q)$.

For the proof, see Bahi [2].

Remark 2.2. The hypothesis (h_0) means that the processors are synchronized and all the components are infinitely updated at the same iteration. This subsequence can be chosen by the programmer.

3. Convergence of the general algorithm

We establish in this section the convergence of the general parallel synchronous algorithm to a fixed point of a nonlinear operator $F:\mathbb{R}^n\to\mathbb{R}^n$ with respect to the Euclidian norm defined in Section 2. We recall that an operator F from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be nonexpansive with respect to a norm $\|\cdot\|$ if

$$\forall x,y\in\mathbb{R}^n,\quad \|F(x)-F(y)\|\leq\|x-y\|. \tag{3.1}$$

Theorem 3.1. *Suppose*

(h$_0$) *there exists a subsequence $\{p_k\}_{k\in\mathbb{N}}$ such that $J(p_k)=\{1,\dots,\alpha\}$;*

(h$_1$) *there exist $u\in\mathbb{R}^n$, $F(u)=u$;*

(h$_2$) *for all $x,y\in\mathbb{R}^n$, $\|F(x)-F(y)\|_\infty\leq\|x-y\|_\infty$;*

(h$_3$) *for all $x,y\in\mathbb{R}^n$, $\|F(x)-F(y)\|^2\leq\langle F(x)-F(y),x-y\rangle$.*

Then any parallel synchronous (in this case, $s_i(p)=p\,\forall i\in\{1,\dots,\alpha\}$ $\forall p\in\mathbb{N}$) algorithm defined by (2.2) associated with the operator F converges to a fixed point x^* of F.

Proof. (i) We prove first that the sequence $\{x^p\}_{p\in\mathbb{N}}$ is bounded. For all $i\in\{1,\dots,\alpha\}$, we have either $i\notin J(p)$ so

$$\|x_i^{p+1}-u_i\|_i=\|x_i^p-u_i\|_i\leq\|x^p-u\|_\infty, \tag{3.2}$$

or $i \in J(p)$ so

$$\begin{aligned} \|x_i^{p+1} - u_i\|_i &= \|F_i(x^p) - F_i(u)\|_i \\ &\le \|F(x^p) - F(u)\|_\infty \\ &\le \|x^p - u\|_\infty \quad (\text{by } (h_2)), \end{aligned} \tag{3.3}$$

so

$$\forall i \in \{1,\dots,\alpha\}, \quad \|x_i^{p+1} - u_i\|_i \le \|x^p - u\|_\infty, \tag{3.4}$$

then

$$\forall p \in \mathbb{N}, \quad \|x^{p+1} - u\|_\infty \le \|x^p - u\|_\infty, \tag{3.5}$$

hence

$$\forall p \in \mathbb{N}, \quad \|x^p - u\|_\infty \le \|x^0 - u\|_\infty. \tag{3.6}$$

This proves that the sequence $\{x^p\}_{p\in\mathbb{N}}$ is bounded with respect to the maximum norm and then it is bounded with respect to the Euclidian norm.

(ii) As the sequence $\{x^{p_k}\}_{k\in\mathbb{N}}$ is bounded ($\{p_k\}_{k\in\mathbb{N}}$ is defined by (h_0)), it contains a subsequence noted also as $\{x^{p_k}\}_{k\in\mathbb{N}}$ which is convergent in $\mathbb{R}^n$ to an x^*. We show that x^* is a fixed point of F. For this, we consider the sequence $\{y^p = x^p - F(x^p)\}_{p\in\mathbb{N}}$ and prove that $\lim_{k\to\infty} y^{p_k} = 0$,

$$\begin{aligned} \|x^{p_k} - u\|^2 &= \|y^{p_k} + F(x^{p_k}) - u\|^2 \\ &= \|y^{p_k}\|^2 + \|F(x^{p_k}) - u\|^2 + 2\langle F(x^{p_k}) - u, y^{p_k}\rangle, \end{aligned} \tag{3.7}$$

however

$$\begin{aligned} \langle F(x^{p_k}) - u, y^{p_k}\rangle &= \langle F(x^{p_k}) - F(u), x^{p_k} - F(x^{p_k})\rangle \\ &= \langle F(x^{p_k}) - F(u), [x^{p_k} - F(u)] - [F(x^{p_k}) - F(u)]\rangle \\ &= \langle F(x^{p_k}) - F(u), x^{p_k} - u\rangle - \|F(x^{p_k}) - F(u)\|^2 \\ &\ge 0 \quad (\text{by } (h_3)), \end{aligned} \tag{3.8}$$

so

$$\begin{aligned} \|y^{p_k}\|^2 &\le \|x^{p_k} - u\|^2 - \|F(x^{p_k}) - u\|^2 \\ &= \|x^{p_k} - u\|^2 - \|x^{p_k+1} - u\|^2 \quad (\text{by } (h_0)). \end{aligned} \tag{3.9}$$

However, in (i) we have shown in particular that the sequence $\{\|x^p - u\|_\infty\}_{p\in\mathbb{N}}$ is decreasing (and it is positive), it is therefore convergent, then the sequence $\{\|x^p - u\|\}_{p\in\mathbb{N}}$ is also convergent, so

$$\lim_{p\to\infty} \|x^p - u\| = \lim_{k\to\infty} \|x^{p_k} - u\| = \lim_{k\to\infty} \|x^{p_k+1} - u\| = \|x^* - u\|, \tag{3.10}$$

and so

$$\lim_{k\to\infty} ||y^{p_k}|| = 0, \tag{3.11}$$

which implies that

$$x^* - F(x^*) = 0, \tag{3.12}$$

that is, x^* is a fixed point of F.

(iii) We prove as in (i) that the sequence $\{\|x^p - x^*\|_\infty\}_{p\in\mathbb{N}}$ is convergent, so

$$\lim_{p\to\infty} ||x^p - x^*||_\infty = \lim_{k\to\infty} ||x^{p_k} - x^*||_\infty = 0, \tag{3.13}$$

which proves that $x^p \to x^*$ with respect to the uniform norm $\|\cdot\cdot\|_\infty$. □

Remark 3.2. We have used the hypothesis (h_2) to prove that the sequence $\{x^p\}_{p\in\mathbb{N}}$ is bounded. In the case of the parallel algorithm of Jacobi, where $J(p) = \{1,\dots,\alpha\}$ for all $p \in \mathbb{N}$, we do not need this hypothesis, since in this case $x^{p+1} = F(x^p)$ for all $p \in \mathbb{N}$, and use (h_3) to obtain

$$||x^{p+1} - u|| = ||F(x^p) - F(u)|| \le ||x^p - u||, \tag{3.14}$$

hence we have the following corollary.

Corollary 3.3. *Under the hypotheses (h_1), (h_3), and (h_0') for all $p \in \mathbb{N}$, $J(p) = \{1,\dots,\alpha\}$. The parallel Jacobi algorithm defined by*

$$\begin{gathered} x^0 = (x_1^0,\dots,x_\alpha^0) \in \mathbb{R}^n, \\ x_i^{p+1} = F_i(x_1^p,\dots,x_\alpha^p), \\ i = 1,\dots,\alpha, \\ p = 1,2\dots \end{gathered} \tag{3.15}$$

converges in $\mathbb{R}^n$ to an x^* fixed point of F.

4. Applications

4.1. Solutions of maximal monotone operators. In this section, we apply the parallel Jacobi algorithm to the proximal mapping $F = (I + T)^{-1}$ associated with the maximal monotone operator T. We give first a general result concerning the maximal monotone operators. Such operators have been studied extensively because of their role in convex analysis (minimization of functionals, min-max problems, convex programming, etc.) and certain partial differential equations (see Rockafellar [10]).

Let T be a multivalued maximal monotone operator defined from $\mathbb{R}^n$ to $\mathbb{R}^n$. A fundamental problem is to determine an x^* in $\mathbb{R}^n$ satisfying $0 \in Tx^*$ which will be called a solution of the operator T.

Theorem 4.1. *Let T be a multivalued maximal monotone operator such that $T^{-1}0 \neq \phi$. Then every parallel Jacobi algorithm associated with the single-valued mapping $F = (I + T)^{-1}$ converges in $\mathbb{R}^n$ to an x^* solution of the problem $0 \in Tx$.*

Proof.

$$\begin{aligned} 0 \in Tx &\iff x \in (I+T)x \\ &\iff x = (I+T)^{-1}x \\ &\iff x = Fx. \end{aligned} \tag{4.1}$$

Thus, the solutions of T are the fixed points of F, so the condition $T^{-1}0 \neq \phi$ implies the existence of a fixed point u of $\mathbb{R}^n$. It remains to show that F verifies condition (h_3) and Corollary 3.3. Considering $x^i \in \mathbb{R}^n$ $(i = 1,2)$ and putting $y^i = Fx^i$, then $x^i \in y^i + Ty^i$ or $x^i - y^i \in Ty^i$. As T is monotone, we have $\langle (x^1 - y^1) - (x^2 - y^2), y^1 - y^2 \rangle \geq 0$, and therefore $\langle x^1 - x^2, y^1 - y^2 \rangle - \| y^1 - y^2 \|^2 \geq 0$, which implies that $\|Fx^1 - Fx^2\|^2 \leq \langle Fx^1 - Fx^2, x^1 - x^2 \rangle$. □

4.2. Minimization of functional

Corollary 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous convex function which is proper (i.e, not identically $+\infty$). Suppose that the minimization problem $\min_{\mathbb{R}^n} f(x)$ has a solution. Then any parallel Jacobi algorithm associated with the single-valued mapping $F = (I + \partial f)^{-1}$ converges to a minimizer of f in $\mathbb{R}^n$.*

Proof. Since in this case the subdifferential ∂f is maximal monotone. However, the minimizers of f are the solutions of ∂f; we then apply Theorem 4.1 to ∂f. □

4.3. Saddle point. In this paragraph, we apply Theorem 4.1 to calculate a saddle point of functional $L : \mathbb{R}^n \times \mathbb{R}^p \to [-\infty, +\infty]$. Recall that a saddle point of L is an element (x^*, y^*) of $\mathbb{R}^n \times \mathbb{R}^p$ satisfying

$$L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*), \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{4.2}$$

which is equivalent to

$$L(x^*, y^*) = \inf_{x \in \mathbb{R}^n} L(x, y^*) = \sup_{y \in \mathbb{R}^p} L(x^*, y). \tag{4.3}$$

Suppose that $L(x, y)$ is convex lower semicontinuous in $x \in \mathbb{R}^n$ and concave upper semicontinuous in $y \in \mathbb{R}^p$. Such functionals are called saddle functions in the terminology of Rockafellar [6]. Let T_L be a multifunction defined in $\mathbb{R}^n \times \mathbb{R}^p$ by

$$\begin{gathered} (u, v) \in T_L(x, y) \iff L(x, y') + \langle y' - y, v \rangle \leq L(x, y) \leq L(x', y) - \langle x' - x, u \rangle \\ \forall (x', y') \in \mathbb{R}^n \times \mathbb{R}^p. \end{gathered} \tag{4.4}$$

If L is proper and closed in a certain general sense, then T_L is maximal monotone; see Rockafellar [6, 7]. In this case, the global saddle points of L (with respect to minimizing in

x and maximizing in y) are the elements (x,y) solutions of the problem $(0,0) \in T_L(x,y)$. That is,

$$(0,0) \in T_L(x^*,y^*) \iff (x^*,y^*) = \arg\min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^p} L(x,y). \tag{4.5}$$

We can then apply Theorem 4.1 to the operator T_L, so we have the following corollary.

Corollary 4.3. *Let L be a proper saddle function from $\mathbb{R}^n \times \mathbb{R}^p$ into $[-\infty,+\infty]$ having a saddle point. Then any parallel Jacobi algorithm associated with the single-valued mapping $F = (I+T_L)^{-1}$ from $\mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^n \times \mathbb{R}^p$ converges to a saddle point of L.*

4.4. Convex programming. We consider now the convex programming problem:
(P)

$$\begin{aligned} &\min f_0(x), \quad x \in \mathbb{R}^n, \\ &f_i(x) \le 0 \quad (1 \le i \le m), \end{aligned} \tag{4.6}$$

where f_i are lower semicontinuous convex functions. This problem can be reduced to an unconstrained one by means of the Lagrangian

$$L(x,y) = f_0(x) + \sum_{i=1}^{m} y_i f_i(x), \tag{4.7}$$

where $x \in \mathbb{R}^n$ and $y \in (\mathbb{R}_+)^m$. We observe that L is a saddle function in the sense of [6, page 363], due to the assumptions of convexity and continuity. The dual problem associated with (P) is
(D)

$$\max \left\{ g_0(y) = \inf_{x\in\mathbb{R}^n} L(x,y) \right\} \quad y \in (\mathbb{R}_+)^m. \tag{4.8}$$

If (x^*,y^*) is a saddle point of the Lagrangian L, then x^* is an optimal solution of the primal problem (P) and y^* is an optimal solution of the dual problem (D).

Let $\partial L(x,y)$ the subdifferential of L at $(x,y) \in \mathbb{R}^n \times \mathbb{R}^p$ be defined as the set of vectors $(u,v) \in \mathbb{R}^n \times \mathbb{R}^p$ satisfying

$$\forall (x',y') \in \mathbb{R}^n \times \mathbb{R}^p, \quad L(x,y') - \langle y'-y, v\rangle \le L(x,y) \le L(x',y) - \langle x'-x, u\rangle \tag{4.9}$$

(see Luque [5] and Rockafellar [6]). Then the operator $T_L : (x,y) \to \{(u,v) : (u,-v) \in \partial L(x,y)\}$ is maximal monotone (see Rockafellar [6, Corollary 37.5.2]), so we apply Theorem 4.1 to T_L.

Corollary 4.4. *Suppose that the convex programming (P) defined by (4.6) has a solution. Then any parallel Jacobi algorithm associated with the single-valued mapping $F = (I+T_L)^{-1}$ from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}^n \times \mathbb{R}^p$ converges to a saddle point (x^*,y^*) of L, and so x^* is a solution of the primal (P) and y^* a solution of the dual (D).*

4.5. Variational inequality. A simple formulation of the variational inequality problem is to find an $x^* \in \mathbb{R}^n$ satisfying

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in \mathbb{R}^n, \tag{4.10}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued monotone and maximal operator. (In fact, it is sufficient that A is monotone and hemicontinuous, i.e, verifying $\lim_{t \to 0^+} \langle A(x+ty), h\rangle = \langle Ax, h\rangle \ \forall x, y, h \in \mathbb{R}^n$.) This is equivalent to finding an $x^* \in \mathbb{R}^n$ such that

$$0 \in Ax^* + N(x^*), \tag{4.11}$$

where $N(x)$ is the normal cone to $\mathbb{R}^n$ at x defined by (see Rockafellar [6, 10])

$$N(x) = \{ y \in \mathbb{R}^n : \langle y, x - z\rangle \geq 0 \ \forall z \in \mathbb{R}^n \}. \tag{4.12}$$

Rockafellar [10] has considered the multifunction T defined in $\mathbb{R}^n$ by

$$Tx = Ax + N(x) \tag{4.13}$$

and shown in [8] that T is maximal monotone. The relation $0 \in Tx^*$ is reduced to $-Ax^* \in N(x^*)$ or $\langle -Ax^*, x^* - z\rangle \geq 0$ for all $z \in \mathbb{R}^n$ which is the variational inequality (4.10). Therefore, the solutions of the operator T (defined by (4.13)) are exactly the solutions of the variational inequality (4.10). By using Theorem 4.1, we can write the following corollary.

Corollary 4.5. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued monotone and hemicontinuous operator such that the problem (4.10) has a solution, then any parallel Jacobi algorithm associated with the single-valued mapping $F = (I+T)^{-1}$, where T is defined by (4.13), converges to x^* solution of the problem (4.10).*

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