MEROMORPHIC FUNCTIONS SHARING ONE VALUE

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We discuss the uniqueness problem of meromorphic functions sharing one value and obtain two theorems which improve a result of Xu and Qu and supplement some other results earlier given by Yang, Hua, and Lahiri.

1. Introduction, definitions, and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities, then f and g are said to share the value a IM (ignoring multiplicities). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside a set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Due to Nevanlinna [9], it is well known that if f and g share four distinct values CM, then f is a Möbius transformation of g.

Yang and Hua showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem 1.1 [12]. *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1, and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Corresponding to entire functions, Xu and Qu proved the following result.

Theorem 1.2 [10]. *Let f and g be two nonconstant entire functions, $n \geq 12$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.*

To state the next result, we require the following definition.

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Definition 1.3 [4, 5]. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity m $(\le k)$ if and only if it is an a-point of g with multiplicity m $(\le k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Since $E_k(a;f) = E_k(a;g)$ implies $E_p(a;f) = E_p(a;g)$ for any integer p $(0 \le p < k)$, clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞), respectively.

With the notion of weighted sharing of values improving Theorem 1.1 the following result is proved in [5].

Theorem 1.4 [5]. *Let f and g be two nonconstant meromorphic functions, $n \ge 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $(a,2)$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

Now one may ask the following questions which are the motivations of the paper.

(i) What happens if in Theorem 1.2 the two *nonconstant entire functions f and g* are replaced by *two nonconstant meromorphic functions*?

(ii) In Theorem 1.4, can the nature of sharing the value a be further relaxed? In the paper, we investigate the solutions of the above questions. We now state the following two theorems which are the main results of the paper.

Theorem 1.5. *Let f and g be two nonconstant meromorphic functions such that $n > 22 - [5\Theta(\infty;f) + 5\Theta(\infty;g) + \min\{\Theta(\infty;f), \Theta(\infty;g)\}]$, where n is an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a,0)$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

Theorem 1.6. *Let f and g be two nonconstant meromorphic functions and $n > \max\{8, 12 - \{3\Theta(\infty;f) + 3\Theta(\infty;g)\}\}$ an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g'$ share $(a,1)$, then either $f = dg$ for some $(n+1)$th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

Remark 1.7. In Theorem 1.5 if we take f and g to be two nonconstant entire functions, then the theorem is true for an integer $n \ge 12$. So Theorem 1.5 improves Theorem 1.2.

Remark 1.8. In Theorem 1.6 if we take f and g to be two nonconstant entire functions, then the theorem is true for an integer $n \ge 7$.

Through the standard definitions and notations of the value distribution theory available in [2], we explain some definitions and notations which are used in the paper.

Definition 1.9 [3]. For $a \in \mathbb{C} \cup \{\infty\}$, denote by $N(r,a;f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m, denote by $N(r,a;f \mid\le m)$ $(N(r,a;f \mid\ge m))$

the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)$ ($\overline{N}(r,a;f\mid\geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid< m)$, $N(r,a;f\mid> m)$, $\overline{N}(r,a;f\mid< m)$ and $\overline{N}(r,a;f\mid> m)$ are defined analogously.

Definition 1.10 [5]. Denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)$.

Definition 1.11 [1, 15, 16]. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. Denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, denote by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$, and denote by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way, one can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.12 (cf. [1]). Let k be a positive integer. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, and a 1-point of g with multiplicity q. Denote by $\overline{N}_{f>k}(r,1;g)$ the reduced counting function of those 1-points of f and g such that $p>q=k$. $\overline{N}_{g>k}(r,1;f)$ is defined analogously.

Definition 1.13 [4, 5]. Let f, g share a value IM. Denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.14 [6]. Let $a,b\in\mathbb{C}\cup\{\infty\}$. Denote by $N(r,a;f\mid g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are the b-points of g.

Definition 1.15 [6]. Let $a,b\in\mathbb{C}\cup\{\infty\}$. Denote by $N(r,a;f\mid g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth, we will denote by h and H the following two functions:

$$\begin{aligned} h&=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right),\\ H&=\left(\frac{F'''}{F''}-\frac{2F''}{F'-1}\right)-\left(\frac{G'''}{G''}-\frac{2G''}{G'-1}\right). \end{aligned}\tag{2.1}$$

Lemma 2.1 [15, 16]. *If f, g are two nonconstant meromorphic functions such that they share $(1,0)$ and $h\not\equiv 0$, then*

$$N_E^{1)}(r,1;f)\leq N(r,h)+S(r,f)+S(r,g).\tag{2.2}$$

Lemma 2.2 [7]. *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid < k) + k\overline{N}(r,0;f \mid \geq k) + S(r,f). \quad (2.3)$$

Lemma 2.3. *Let f and g be two nonconstant meromorphic functions sharing $(1,0)$. Then*

$$\begin{aligned} &\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ &\quad \leq N(r,1;g) - \overline{N}(r,1;g). \end{aligned} \quad (2.4)$$

Proof. Let z_0 be a 1-point of f of multiplicity p a 1-point of g of multiplicity q. We denote by $N_1(r)$, $N_2(r)$, and $N_3(r)$ the counting functions of those 1-points of f and g when $1 \leq q < p$, $2 \leq q = p$ and $p < q$, respectively, where in the first counting function each point is counted $q-1$ times and in the remaining two counting functions each point is counted $q-2$ times.

Since f, g share $(1,0)$, we note that a simple 1-point of g is either a simple 1-point of f or a 1-point of f with multiplicity ≥ 2. So we can write

$$N(r,1;g) - \overline{N}(r,1;g) = \overline{N}_E^{(2}(r,1;f) + \overline{N}_L(r,1;g) + N_1(r) + N_2(r) + N_3(r). \quad (2.5)$$

Also we note that

$$N_1(r) \geq \overline{N}_L(r,1;f) - \overline{N}_{f>1}(r,1;g), \quad (2.6)$$

$$N_2(r) \geq \overline{N}_E^{(2}(r,1;f) - \overline{N}(r,1;f,g \mid = 2), \quad (2.7)$$

$$N_3(r) \geq \overline{N}_L(r,1;g) - \overline{N}_{g>1}(r,1;f), \quad (2.8)$$

where by $\overline{N}(r,1;f,g \mid = 2)$ we mean the reduced counting functions of 1-points of f and g with multiplicities two for each one.

Using (2.6)–(2.8) in (2.5), we deduce that

$$\begin{aligned} &N(r,1;g) - \overline{N}(r,1;g) \\ &\quad \geq \overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + 2\overline{N}_E^{(2}(r,1;f) \\ &\qquad - \overline{N}(r,1;f,g \mid = 2) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f). \end{aligned} \quad (2.9)$$

Now (i) follows from (2.9). This proves the lemma. □

Lemma 2.4 [1]. *If f, g are two nonconstant meromorphic functions such that they share $(1,1)$, then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g). \quad (2.10)$$

Lemma 2.5. *Let f, g share $(1,0)$ and $h \not\equiv 0$, then*

$$N(r,h) \le \overline{N}(r,0;f \mid\ge 2) + \overline{N}(r,0;g \mid\ge 2) + \overline{N}(r,\infty;f \mid\ge 2) + \overline{N}(r,\infty;g \mid\ge 2) + \overline{N}_*(r,1;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \tag{2.11}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) those 1-points of f and g whose multiplicities are distinct from the multiplicities of the corresponding 1-points of g and f, respectively, (iv) zeros of f' which are not the zeros of $f(f-1)$ and (v) zeros of g' which are not zeros of $g(g-1)$.

Since h has only simple poles, the lemma follows from above. This proves the lemma. □

Lemma 2.6 [15]. *Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r). \tag{2.12}$$

Lemma 2.7. *Let f, g share $(1,0)$. Then*
(i) $\overline{N}_{f>1}(r,1;g) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f)$,
(ii) $\overline{N}_{g>1}(r,1;f) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;f') + S(r,g)$.

Proof. We prove (i) because (ii) can be proved in a similar manner.

Using Lemma 2.2, we obtain

$$\begin{aligned}\overline{N}_{f>1}(r,1;g) &\le \overline{N}(r,1;f \mid\ge 2)\\ &\le N(r,0;f' \mid f=1)\\ &\le N(r,0;f' \mid f\ne 0) - N_0(r,0;f')\\ &\le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f).\end{aligned} \tag{2.13}$$

□

Lemma 2.8. *Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \le \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f). \tag{2.14}$$

Proof. Using Lemma 2.2, we get

$$\begin{aligned}\overline{N}_{f>2}(r,1;g) &\le \overline{N}(r,1;f \mid\ge 3)\\ &\le \frac{1}{2}N(r,0;f' \mid f=1)\\ &\le \frac{1}{2}N(r,0;f' \mid f\ne 0) - \frac{1}{2}N_0(r,0;f')\\ &\le \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f).\end{aligned} \tag{2.15}$$

□

Lemma 2.9 [14]. *If $h \equiv 0$ and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)}{T(r)} < 1, \quad r \in I, \tag{2.16}$$

where $T(r) = \max\{T(r,f), T(r,g)\}$, then $f \equiv g$ or $f \cdot g \equiv 1$.

Lemma 2.10 (cf. [8, 11]). *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 2.11. *Let f be a nonconstant meromorphic function and $F = f^{n+1}/a(n+1)$, n being a positive integer. Then*

$$T(r,F) \le T(r,F') + N(r,0;f) - N(r,0;f') + S(r,f). \tag{2.17}$$

Proof. By the first fundamental theorem and Milloux theorem, we get

$$m\left(r,\frac{1}{F}\right) \le m\left(r,\frac{F'}{F}\right) + m\left(r,\frac{1}{F'}\right), \tag{2.18}$$

that is,

$$N(r,0;F) + m(r,0;F) \le N(r,0;F) + m(r,0;F') + S(r,F), \tag{2.19}$$

that is,

$$T(r,F) \le T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F). \tag{2.20}$$

Since $N(r,0;F) = (n+1)N(r,0;f)$ and $N(r,0;F') = nN(r,0;f) + N(r,0;f')$ and by Lemma 2.10, $S(r,F) = S(r,f)$, then the lemma follows from (2.20). This proves the lemma. □

Lemma 2.12. *Let f, g be two nonconstant meromorphic functions and $F = f^{n+1}/a(n+1)$, $G = g^{n+1}/a(n+1)$, where n (> 2) is an integer. Then $F' \equiv G'$ implies $F \equiv G$.*

Proof. $F' \equiv G'$ then $F = G + c$ where c is a constant. If possible, let $c \neq 0$. Then by the second fundamental theorem and Lemma 2.10, we get

$$\begin{aligned}(n+1)T(r,f) &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,c;F) + S(r,F)\\ &= \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,0;g) + S(r,f)\\ &\le 2T(r,f) + T(r,g) + S(r,f)\\ &\le 3T(r) + S(r).\end{aligned} \tag{2.21}$$

In a similar manner, we get

$$(n+1)T(r,g) \le 3T(r) + S(r). \tag{2.22}$$

This shows that

$$(n-2)T(r) \leq S(r), \tag{2.23}$$

which is a contradiction for $n > 2$. This proves the lemma. □

LEMMA 2.13 [12]. *Let f, g be two nonconstant meromorphic functions and $n > 6$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1, c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

LEMMA 2.14. *Let f, g be two nonconstant meromorphic functions such that they share $(1,0)$ and $h \not\equiv 0$. Then*

$$\begin{aligned} T(r,f) \leq{} & N_2(r,0;f) + N_2(r,\infty;f) + N_2(r,0;g) + N_2(r,\infty;g) + 2\overline{N}(r,0;f) \\ & + 2\overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned} \tag{2.24}$$

Proof. By the second fundamental theorem, we get

$$\begin{aligned} & T(r,f) + T(r,g) \\ & \quad \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;f) \\ & \qquad + \overline{N}(r,1;g) - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g). \end{aligned} \tag{2.25}$$

By Lemmas 2.1, 2.3, and 2.5, we get

$$\begin{aligned} & \overline{N}(r,1;f) + \overline{N}(r,1;g) \\ & \quad \leq N_E^{1)}(r,1;f) + \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) + \overline{N}(r,1;g) \\ & \quad \leq N_E^{1)}(r,1;f) + N(r,1;g) - \overline{N}_L(r,1;g) + \overline{N}_{f>1}(r,1;g) + \overline{N}_{g>1}(r,1;f) \\ & \quad \leq \overline{N}(r,0;f \mid \geq 2) + \overline{N}(r,0;g \mid \geq 2) + \overline{N}(r,\infty;f \mid \geq 2) + \overline{N}(r,\infty;g \mid \geq 2) \\ & \qquad + \overline{N}_*(r,1;f,g) + T(r,g) - m(r,1;g) + O(1) - \overline{N}_L(r,1;g) + \overline{N}_{f>1}(r,1;g) \\ & \qquad + \overline{N}_{g>1}(r,1;f) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') + S(r,f) + S(r,g). \end{aligned} \tag{2.26}$$

Since $\overline{N}_*(r,1;f,g) = \overline{N}_L(r,1;f) + \overline{N}_L(r,1;g)$, by Lemmas 2.6 and 2.7, we get from (2.25) and (2.26) in view of Definition 1.10 that

$$\begin{aligned} T(r,f) \leq{} & N_2(r,0;f) + N_2(r,\infty;f) + N_2(r,0;g) + N_2(r,\infty;g) + 2\overline{N}(r,0;f) \\ & + 2\overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \end{aligned} \tag{2.27}$$

□

LEMMA 2.15. *Let f, g be two nonconstant meromorphic functions such that they share $(1,1)$ and $h \not\equiv 0$. Then*

$$\begin{aligned} T(r,f) \leq{} & N_2(r,0;f) + N_2(r,\infty;f) + N_2(r,0;g) + N_2(r,\infty;g) \\ & + \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) + S(r,f) + S(r,g). \end{aligned} \tag{2.28}$$

Proof. By the second fundamental theorem, we get

$$\begin{aligned} T(r,f)+T(r,g) &\\ &\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;f) \\ &\quad+\overline{N}(r,1;g)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g). \end{aligned} \tag{2.29}$$

Since f, g share $(1,1)$, $N_E^{1)}(r,1;f)=N(r,1;f\mid=1)$. So using Lemmas 2.1, 2.4, 2.5, and 2.8, we get

$$\begin{aligned} &\overline{N}(r,1;f)+\overline{N}(r,1;g) \\ &\quad\le N(r,1;f\mid=1)+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)+\overline{N}(r,1;g) \\ &\quad\le N(r,1;f\mid=1)+N(r,1;g)-\overline{N}_L(r,1;f)-\overline{N}_L(r,1;g)+\overline{N}_{f>2}(r,1;g) \\ &\quad\le \overline{N}(r,0;f\mid\ge 2)+\overline{N}(r,0;g\mid\ge 2)+\overline{N}(r,\infty;f\mid\ge 2)+\overline{N}(r,\infty;g\mid\ge 2) \\ &\qquad+\overline{N}_*(r,1;f,g)+T(r,g)-m(r,1;g)+O(1)-\overline{N}_L(r,1;f)-\overline{N}_L(r,1;g) \\ &\qquad+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \end{aligned} \tag{2.30}$$

From (2.29) and (2.30), we obtain in view of Definition 1.10 that

$$\begin{aligned} T(r,f) &\le N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g) \\ &\quad+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)+S(r,f)+S(r,g). \end{aligned} \tag{2.31}$$

This proves the lemma. □

Lemma 2.16 [13]. *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)})\le k\overline{N}(r,\infty;f)+N(r,0;f)+S(r,f). \tag{2.32}$$

3. Proofs of the theorems

Proof of Theorem 1.5. Let $F=f^{n+1}/a(n+1)$ and $G=g^{n+1}/a(n+1)$. Then $F'=f^nf'/a$ and $G'=g^ng'/a$. Since f^nf' and g^ng' share $(a,0)$, it follows that F', G' share $(1,0)$. If possible, we suppose that $H\not\equiv 0$. Then by Lemma 2.14, we obtain

$$\begin{aligned} T(r,F') &\le N_2(r,0;F')+N_2(r,\infty;F')+N_2(r,0;G')+N_2(r,\infty;G')+2\overline{N}(r,0;F') \\ &\quad+2\overline{N}(r,\infty;F')+\overline{N}(r,0;G')+\overline{N}(r,\infty;G')+S(r,F')+S(r,G'). \end{aligned} \tag{3.1}$$

We see that

$$\begin{aligned} N_2(r,0;F')+N_2(r,\infty;F') &\le 2\overline{N}(r,0;f)+N(r,0;f')+2\overline{N}(r,\infty;f), \\ N_2(r,0;G')+N_2(r,\infty;G') &\le 2\overline{N}(r,0;g)+N(r,0;g')+2\overline{N}(r,\infty;g), \\ 2\overline{N}(r,0;F')+2\overline{N}(r,\infty;F') &\le 2\overline{N}(r,0;f)+2N(r,0;f')+2\overline{N}(r,\infty;f), \\ \overline{N}(r,0;G')+\overline{N}(r,\infty;G') &\le \overline{N}(r,0;g)+N(r,0;g')+\overline{N}(r,\infty;g). \end{aligned} \tag{3.2}$$

Also by Lemma 2.10, we get

$$\begin{aligned} T(r,F') &\le 2T(r,F)+S(r,F) = 2(n+1)T(r,f)+S(r,f),\\ T(r,G') &\le 2T(r,G)+S(r,G) = 2(n+1)T(r,g)+S(r,g). \end{aligned} \tag{3.3}$$

So $S(r,F') = S(r,f)$ and $S(r,G') = S(r,g)$. So by Lemmas 2.11 and 2.16, we get from (3.1) for $\varepsilon\ (>0)$ that

$$\begin{aligned} T(r,F) &\le T(r,F')+N(r,0;f)-N(r,0;f')+S(r,f)\\ &\le 4\overline{N}(r,0;f)+N(r,0;f)+3\overline{N}(r,0;g)+4\overline{N}(r,\infty;f)\\ &\quad +3\overline{N}(r,\infty;g)+2N(r,0;f')+2N(r,0;g')+S(r,f)+S(r,g)\\ &\le 7T(r,f)+5T(r,g)+(6-6\Theta(\infty;f)+\varepsilon)T(r,f)\\ &\quad +(5-5\Theta(\infty;g)+\varepsilon)T(r,g)+S(r,f)+S(r,g)\\ &\le \{23-6\Theta(\infty;f)-5\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \end{aligned} \tag{3.4}$$

So using Lemma 2.10, we get

$$(n+1)T(r,f) \le \{23-6\Theta(\infty;f)-5\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \tag{3.5}$$

In a similar manner, we obtain

$$(n+1)T(r,g) \le \{23-5\Theta(\infty;f)-6\Theta(\infty;g)+2\varepsilon\}T(r)+S(r). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$[n-22+5\Theta(\infty;f)+5\Theta(\infty;g)+\min\{\Theta(\infty;f),\Theta(\infty;g)\}-2\varepsilon]T(r) \le S(r). \tag{3.7}$$

Since $\varepsilon\ (>0)$ is arbitrary, (3.7) implies a contradiction. Hence $H\equiv 0$.

Since

$$\overline{N}(r,0;f') \le T(r,f')-m\left(r,\frac{1}{f'}\right) \le 2T(r,f)-m\left(r,\frac{1}{f'}\right)+S(r,f), \tag{3.8}$$

we note that

$$\begin{aligned} &\overline{N}(r,0;F')+\overline{N}(r,\infty;F')+\overline{N}(r,0;G')+\overline{N}(r,\infty;G')\\ &\quad\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f')+\overline{N}(r,0;g')\\ &\quad\le 4T(r,f)+4T(r,g)-m(r,0;f')-m(r,0;g')+S(r)\\ &\quad\le 8T(r)-m(r,0;f')-m(r,0;g')+S(r). \end{aligned} \tag{3.9}$$

Also using Lemma 2.10, we get

$$\begin{aligned} T(r,F') + m\left(r,\frac{1}{f'}\right) &= m\left(r,\frac{f^n f'}{a}\right) + m\left(r,\frac{1}{f'}\right) + N\left(r,\infty;\frac{f^n f'}{a}\right) \\ &\geq m\left(r,\frac{f^n}{a}\right) + N(r,\infty;f^n) \\ &= T(r,f^n) + O(1) \\ &= nT(r,f) + O(1). \end{aligned} \tag{3.10}$$

Similarly

$$T(r,G') + m\left(r,\frac{1}{g'}\right) \geq nT(r,g) + O(1). \tag{3.11}$$

From (3.10) and (3.11), we get

$$\max\{T(r,F'),T(r,G')\} \geq nT(r) - m\left(r,\frac{1}{f'}\right) - m\left(r,\frac{1}{g'}\right) + O(1). \tag{3.12}$$

By (3.9) and (3.12) applying Lemma 2.9, we get either $F' \equiv G'$ or $F'G' \equiv 1$.

If $F' \equiv G'$, then by Lemma 2.12 we obtain $F \equiv G$ or $f \equiv dg$, where d is some $(n+1)$th root of unity.

If $F'G' \equiv 1$, then $f^n f' g^n g' = a^2$. Set $f_1 = a^{-1/(n+1)} f$ and $g_1 = a^{-1/(n+1)} g$, then $f_1^n f_1' g_1^n g_1' = 1$. So using Lemma 2.13, we get $g = c_1 e^{cz}$, $f = c_2 e^{-cz}$, where c, c_1, and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$. This completes the proof of the theorem. □

Proof of Theorem 1.6. Let $F = f^{n+1}/a(n+1)$ and $G = g^{n+1}/a(n+1)$. Then $F' = f^n f'/a$ and $G' = g^n g'/a$. Since $f^n f'$ and $g^n g'$ share $(a,1)$, it follows that F', G' share $(1,1)$. Suppose that $H \not\equiv 0$. Then by Lemma 2.15, we obtain

$$\begin{aligned} T(r,F') \leq{}& N_2(r,0;F') + N_2(r,\infty;F') + N_2(r,0;G') + N_2(r,\infty;G') \\ &+ \frac{1}{2}\overline{N}(r,0;F') + \frac{1}{2}\overline{N}(r,\infty;F') + S(r,F') + S(r,G'). \end{aligned} \tag{3.13}$$

We see that

$$\begin{aligned} N_2(r,0;F') + N_2(r,\infty;F') &\leq 2\overline{N}(r,0;f) + N(r,0;f') + 2\overline{N}(r,\infty;f), \\ N_2(r,0;G') + N_2(r,\infty;G') &\leq 2\overline{N}(r,0;g) + N(r,0;g') + 2\overline{N}(r,\infty;g), \\ \frac{1}{2}\overline{N}(r,0;F') + \frac{1}{2}\overline{N}(r,\infty;F') &\leq \frac{1}{2}\left[\overline{N}(r,0;f) + N(r,0;f') + \overline{N}(r,\infty;f)\right]. \end{aligned} \tag{3.14}$$

Again using Lemma 2.10 and proceeding in the same way as done in the proof of Theorem 1.5, we can show that $S(r,F') = S(r,f)$ and $S(r,G') = S(r,g)$. So by Lemmas 2.11 and 2.16, we obtain from (3.13) for $\varepsilon > 0$ that

$$\begin{aligned} T(r,F) &\leq T(r,F') + N(r,0;f) - N(r,0;f') + S(r,f) \\ &\leq 2\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,0;f) + \frac{3}{2}N(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g) \\ &\quad + 3\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\ &\leq (7 - 3\Theta(\infty;f) + \varepsilon)T(r,f) + (6 - 3\Theta(\infty;g) + \varepsilon)T(r,g) + S(r) \\ &\leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \end{aligned} \tag{3.15}$$

So using Lemma 2.10, we get

$$(n+1)T(r,f) \leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \tag{3.16}$$

Similarly, we can obtain

$$(n+1)T(r,g) \leq \{13 - 3\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon\}T(r) + S(r). \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$[n - 12 + 3\Theta(\infty;f) + 3\Theta(\infty;g) - 2\varepsilon] \leq S(r). \tag{3.18}$$

Since ε (> 0) is arbitrary, we get a contradiction from (3.18). Hence $H \equiv 0$.

Now proceeding in the same way as in the proof of Theorem 1.5, we obtain either $F' \equiv G'$ or $F'G' \equiv 1$. Again proceeding in the same manner as in the proof of Theorem 1.5, we obtain the conclusion of Theorem 1.6. This proves the theorem. □

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