FUZZY n-NORMED LINEAR SPACE

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Received 24 March 2005 and in revised form 29 September 2005

The primary purpose of this paper is to introduce the notion of fuzzy n-normed linear space as a generalization of n-normed space. Ascending family of α-n-norms corresponding to fuzzy n-norm is introduced. Best approximation sets in α-n-norms are defined. We also provide some results on best approximation sets in α-n-normed space.

1. Introduction

A satisfactory theory of 2-norm and n-norm on a linear space has been introduced and developed by Gähler in [9, 10]. Following Misiak [16], Kim and Cho [13] and Malčeski [15] developed the theory of n-normed space. In [11], Gunawan and Mashadi gave a simple way to derive an (n-1)-norm from the n-norm and realized that any n-normed space is an (n-1)-normed space. Best approximation theory in 2-normed space can be viewed in the papers [3, 4, 5, 9]. Different authors introduced the definitions of fuzzy norms on a linear space. For reference, one may see [2, 6, 7, 8, 12, 14, 17]. Following Cheng and Mordeson [2], Bag and Samanta [1] introduced the concept of fuzzy norm on a linear space.

In the present paper, we introduce the concept of fuzzy n-normed linear space as a generalization of n-normed space by Gunawan and Mashadi [11]. Bag and Samanta [1] introduced α-norms on a linear space corresponding to the fuzzy norm on a linear space. As an analogue of Bag and Samanta [1], we introduce the notion of α-n-norm on a linear space corresponding to the fuzzy n-norm on a linear space. Based on Elumalai et al. [3] and Elumalai and Souruparani [5], we introduce the notion of best approximation sets in α-n-norms and establish some results on it.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gähler [9], Gunawan and Mashadi [11], Elumalai et al. [3], and Bag and Samanta [1].

International Journal of Mathematics and Mathematical Sciences 2005:24 (2005) 3963–3977
DOI: 10.1155/IJMMS.2005.3963

Definition 2.1 [9]. Let X be a real vector space of dimension greater than 1 and let $\|\bullet,\bullet\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(1) $\|x,y\| = 0$ if any only if x and y are linearly dependent,
(2) $\|x,y\| = \|y,x\|$,
(3) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$, where α is real,
(4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

$\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X,\|\bullet,\bullet\|)$ is called a linear 2-normed space.

Definition 2.2 [11]. Let $n \in \mathbb{N}$ (natural numbers) and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\bullet,\ldots,\bullet\|$ on $\underbrace{X \times \cdots \times X}_{n}$ satisfying the following four properties,

(1) $\|x_1,x_2,\ldots,x_n\| = 0$ if any only if $x_1,x_2,\ldots,x_n$ are linearly dependent,
(2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under any permutation,
(3) $\|x_1,x_2,\ldots,\alpha x_n\| = |\alpha|\,\|x_1,x_2,\ldots,x_n\|$ for any $\alpha \in \mathbb{R}$ (real),
(4) $\|x_1,x_2,\ldots,x_{n-1},y+z\| \leq \|x_1,x_2,\ldots,x_{n-1},y\| + \|x_1,x_2,\ldots,x_{n-1},z\|$,

is called an n-norm on X and the pair $(X,\|\bullet,\ldots,\bullet\|)$ is called an n-normed space.

Definition 2.3 [3]. Let $(X,\|\bullet,\bullet\|)$ be a linear 2-normed space and let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then, for every $x \in X$ and for every $z \in X\backslash G$ which is independent of x and x_0, $d_z(x,G) \leq \|x-x_0,z\| + d_z(x_0,G)$, where $d_z(x,G) = \inf_{g\in G}\|x-g,z\|$. For each $G \subset X$ and $x_0 \in X$, define $D_z(x_0,G) = \{x \in X : d_z(x,G) = \|x-x_0,z\| + d_z(x_0,G)\}$ for any $z \in X\backslash G$ which is independent of x and x_0.

Also $P_{G,z}(x) = \{g_0 \in G : \|x-g_0,z\| = d_z(x,G)\}$ and $P^{-1}_{G,z}(x_0) = \{x \in X : \|x-x_0,z\| = d_z(x,G)\}$, where $x_0 \in G$.

Definition 2.4 [1]. Let X be a linear space over F (field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy norm on X if and only if for all $x,u \in X$ and $c \in F$,

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$,
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x,t) = 1$ if and only if $x = \underline{0}$,
(N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx,t) = N(x,t/|c|)$, if $c \neq 0$,
(N4) for all $s,t \in \mathbb{R}$, $x,u \in X$, $N(x+u,s+t) \geq \min\{N(x,s),N(u,t)\}$,
(N5) $N(x,\circ)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X,N) will be referred to as a fuzzy normed linear space.

Theorem 2.5 [1]. *Let (X,N) be a fuzzy normed linear space. Assume further that*
(N6) $N(x,t) > 0$ *for all* $t > 0$ *implies* $x = \underline{0}$.
Define $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$.

Then $\{\|\bullet\|_\alpha : \alpha \in (0,1)\}$ *is an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.*

3. Fuzzy n-normed linear space

By generalizing Definition 2.2, we obtain a satisfactory notion of fuzzy n-normed linear space as follows.

Definition 3.1. Let X be a linear space over a real field F. A fuzzy subset N of $\underbrace{X\times\cdots\times X}_{n}\times\mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a fuzzy n-norm on X if and only if

(N1) for all $t\in\mathbb{R}$ with $t\le 0$, $N(x_1,x_2,\dots,x_n,t)=0$,

(N2) for all $t\in\mathbb{R}$ with $t>0$, $N(x_1,x_2,\dots,x_n,t)=1$ if and only if $x_1,x_2,\dots,x_n$ are linearly dependent,

(N3) $N(x_1,x_2,\dots,x_n,t)$ is invariant under any permutation of $x_1,x_2,\dots,x_n$,

(N4) for all $t\in\mathbb{R}$ with $t>0$,

$$N(x_1,x_2,\dots,cx_n,t)=N\left(x_1,x_2,\dots,x_n,\frac{t}{|c|}\right),\quad \text{if } c\neq 0,\ c\in F \text{ (field)}, \tag{3.1}$$

(N5) for all $s,t\in\mathbb{R}$,

$$N(x_1,x_2,\dots,x_n+x_n',s+t)\ge\min\{N(x_1,x_2,\dots,x_n,s),N(x_1,x_2,\dots,x_n',t)\}, \tag{3.2}$$

(N6) $N(x_1,x_2,\dots,x_n,\circ)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x_1,x_2,\dots,x_n,t)=1$.

Then (X,N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 3.2. From (N3), it follows that in an f-n-NLS,

(N4) for all $t\in\mathbb{R}$ with $t>0$,

$$N(x_1,x_2,\dots,cx_i,\dots,x_n,t)=N\left(x_1,x_2,\dots,x_i,\dots,x_n,\frac{t}{|c|}\right),\quad \text{if } c\neq 0, \tag{3.3}$$

(N5) for all $s,t\in\mathbb{R}$,

$$\begin{aligned}&N(x_1,x_2,\dots,x_i+x_i',\dots,x_n,s+t)\\&\qquad\ge\min\{N(x_1,x_2,\dots,x_i,\dots,x_n,s),N(x_1,x_2,\dots,x_i',\dots,x_n,t)\}.\end{aligned} \tag{3.4}$$

The following example agrees with our notion of f-n-NLS.

Example 3.3. Let $(X,\|\bullet,\bullet,\dots,\bullet\|)$ be an n-normed space as in Definition 2.2. Define

$$\begin{aligned}&N(x_1,x_2,\dots,x_n,t)\\&\quad=\begin{cases}\dfrac{t}{t+\|x_1,x_2,\dots,x_n\|}, & \text{when } t>0,\ t\in\mathbb{R},\ (x_1,x_2,\dots,x_n)\in\underbrace{X\times\cdots\times X}_{n},\\[2mm] 0, & \text{when } t\le 0.\end{cases}\end{aligned} \tag{3.5}$$

Then (X,N) is an f-n-NLS.

Proof. (N1) for all $t\in\mathbb{R}$ with $t\le 0$, we have by our definition

$$N(x_1,x_2,\dots,x_n,t)=0. \tag{3.6}$$

(N2) for all $t \in \mathbb{R}$ with $t > 0$, we have $N(x_1, x_2, \ldots, x_n, t) = 1$

(i) if and only if $t/(t + \|x_1, x_2, \ldots, x_n\|) = 1$,

(ii) if and only if $t = t + \|x_1, x_2, \ldots, x_n\|$,

(iii) if and only if $\|x_1, x_2, \ldots, x_n\| = 0$,

(iv) if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

(N3) for all $t \in \mathbb{R}$ with $t > 0$,

$$\begin{aligned} N(x_1, x_2, \ldots, x_n, t) &= \frac{t}{t + \|x_1, x_2, \ldots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \ldots, x_n, x_{n-1}\|} = N(x_1, x_2, \ldots, x_n, x_{n-1}, t) = \ldots. \end{aligned} \tag{3.7}$$

(N4) For all $t \in \mathbb{R}$ with $t > 0$ and $c \in F$, $c \neq 0$,

$$\begin{aligned} N\left(x_1, x_2, \ldots, x_n, \frac{t}{|c|}\right) &= \frac{t/|c|}{(t/|c|) + \|x_1, x_2, \ldots, x_n\|} \\ &= \frac{t/|c|}{(t + |c|\|x_1, x_2, \ldots, x_n\|)/|c|} \\ &= \frac{t}{t + |c|\|x_1, x_2, \ldots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \ldots, cx_n\|} = N(x_1, x_2, \ldots, cx_n, t). \end{aligned} \tag{3.8}$$

(N5) We have to prove

$$N(x_1, x_2, \ldots, x_n + x_n', s + t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x_n', t)\}. \tag{3.9}$$

If

(a) $s + t < 0$,

(b) $s = t = 0$,

(c) $s + t > 0$; $s > 0$, $t < 0$; $s < 0$, $t > 0$, then the above relation is obvious. If

(d) $s > 0$, $t > 0$, $s + t > 0$, then

$$\begin{aligned} N(x_1, x_2, \ldots, x_n + x_n', s + t) &= \frac{s + t}{s + t + \|x_1, x_2, \ldots, x_n + x_n'\|} \\ &\geq \frac{s + t}{s + t + \|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x_n'\|}. \end{aligned} \tag{3.10}$$

If

$$\frac{s}{s + \|x_1, x_2, \ldots, x_n\|} \geq \frac{t}{t + \|x_1, x_2, \ldots, x_n'\|}, \tag{3.11}$$

then

$$\frac{s}{s + \|x_1, x_2, \ldots, x_n\|} - \frac{t}{t + \|x_1, x_2, \ldots, x_n'\|} \geq 0, \tag{3.12}$$

which implies

$$s(t+||x_1,x_2,\dots,x_n'||)-t(s+||x_1,x_2,\dots,x_n||)\geq 0, \tag{3.13}$$

which in turn implies

$$s||x_1,x_2,\dots,x_n'||-t||x_1,x_2,\dots,x_n||\geq 0. \tag{3.14}$$

So

$$\begin{aligned}&\frac{s+t}{s+t+||x_1,x_2,\dots,x_n||+||x_1,x_2,\dots,x_n'||}-\frac{t}{t+||x_1,x_2,\dots,x_n'||}\\&\quad=\frac{s||x_1,x_2,\dots,x_n'||-t||x_1,x_2,\dots,x_n||}{(s+t+||x_1,x_2,\dots,x_n||+||x_1,x_2,\dots,x_n'||)(t+||x_1,x_2,\dots,x_n'||)}.\end{aligned} \tag{3.15}$$

By (3.14),

$$\frac{s+t}{s+t+||x_1,x_2,\dots,x_n||+||x_1,x_2,\dots,x_n'||}-\frac{t}{t+||x_1,x_2,\dots,x_n'||}\geq 0, \tag{3.16}$$

which implies

$$\frac{s+t}{s+t+||x_1,x_2,\dots,x_n||+||x_1,x_2,\dots,x_n'||}\geq\frac{t}{t+||x_1,x_2,\dots,x_n'||}. \tag{3.17}$$

Similarly, if

$$\frac{t}{t+||x_1,x_2,\dots,x_n'||}\geq\frac{s}{s+||x_1,x_2,\dots,x_n||}, \tag{3.18}$$

then

$$\frac{s+t}{s+t+||x_1,x_2,\dots,x_n||+||x_1,x_2,\dots,x_n'||}\geq\frac{s}{s+||x_1,x_2,\dots,x_n||}. \tag{3.19}$$

Thus,

$$N(x_1,x_2,\dots,x_n+x_n',s+t)\geq\min\{N(x_1,x_2,\dots,x_n,s),N(x_1,x_2,\dots,x_n',t)\}. \tag{3.20}$$

(N6) For all $t_1,t_2\in\mathbb{R}$, if $t_1<t_2\leq 0$, then, by our definition,

$$N(x_1,x_2,\dots,x_n,t_1)=N(x_1,x_2,\dots,x_n,t_2)=0. \tag{3.21}$$

Suppose $t_2 > t_1 > 0$, then

$$\begin{aligned}&\frac{t_2}{t_2+||x_1,x_2,\dots,x_n||}-\frac{t_1}{t_1+||x_1,x_2,\dots,x_n||}\\&\qquad=\frac{||x_1,x_2,\dots,x_n||(t_2-t_1)}{(t_2+||x_1,x_2,\dots,x_n||)(t_1+||x_1,x_2,\dots,x_n||)}\geq 0,\end{aligned}\tag{3.22}$$

for all $(x_1,x_2,\dots,x_n)\in\underbrace{X\times\cdots\times X}_{n}$, implies

$$\frac{t_2}{t_2+||x_1,x_2,\dots,x_n||}\geq\frac{t_1}{t_1+||x_1,x_2,\dots,x_n||},\tag{3.23}$$

which in turn implies $N(x_1,x_2,\dots,x_n,t_2)\geq N(x_1,x_2,\dots,x_n,t_1)$.

Thus $N(x_1,x_2,\dots,x_n,t)$ is a nondecreasing function.

Also,

$$\begin{aligned}\lim_{t\to\infty}N(x_1,x_2,\dots,x_n,t)&=\lim_{t\to\infty}\frac{t}{t+||x_1,x_2,\dots,x_n||}\\&=\lim_{t\to\infty}\frac{t}{t(1+(1/t)||x_1,x_2,\dots,x_n||)}=1.\end{aligned}\tag{3.24}$$

Thus (X,N) is an f-n-NLS. □

As a consequence of Theorem 2.5, we introduce an interesting notion of ascending family of α-n-norms corresponding to the fuzzy n-norm in the following theorem.

THEOREM 3.4. *Let (X,N) be an f-n-NLS. Assume the condition that*

(N7) $N(x_1,x_2,\dots,x_n,t)>0$ *for all* $t>0$ *implies* $x_1,x_2,\dots,x_n$ *are linearly dependent.*

Define $\|x_1,x_2,\dots,x_n\|_\alpha=\inf\{t:N(x_1,x_2,\dots,x_n,t)\geq\alpha\}$, $\alpha\in(0,1)$.

Then $\{\|\bullet,\bullet,\dots,\bullet\|_\alpha:\alpha\in(0,1)\}$ *is an ascending family of n-norms on X. These n-norms are called α-n-norms on X corresponding to the fuzzy n-norm on X.*

Proof. (1) $\|x_1,x_2,\dots,x_n\|_\alpha=0$. This

(i) implies $\inf\{t:N(x_1,x_2,\dots,x_n,t)\geq\alpha\}=0$,

(ii) implies, for all $t\in\mathbb{R}$, $t>0$, $N(x_1,x_2,\dots,x_n,t)\geq\alpha>0$, $\alpha\in(0,1)$,

(iii) implies, by (N7), $x_1,x_2,\dots,x_n$ are linearly dependent.

Conversely assume that $x_1,x_2,\dots,x_n$ are linearly dependent. This

(i) implies, by (N2), $N(x_1,x_2,\dots,x_n,t)=1$ for all $t>0$,

(ii) implies, for all $\alpha\in(0,1)$, $\inf\{t:N(x_1,x_2,\dots,x_n,t)\geq\alpha\}=0$,

(iii) implies $\|x_1,x_2,\dots,x_n\|_\alpha=0$.

(2) As $N(x_1,x_2,\dots,x_n,t)$ is invariant under any permutation, it follows that $\|x_1,x_2,\dots,x_n\|_\alpha$ is invariant under any permutation.

(3) If $c\neq 0$, then

$$\begin{aligned}||x_1,x_2,\dots,cx_n||_\alpha&=\inf\{s:N(x_1,x_2,\dots,cx_n,s)\geq\alpha\}\\&=\inf\left\{s:N\left(x_1,x_2,\dots,x_n,\frac{s}{|c|}\right)\geq\alpha\right\}.\end{aligned}\tag{3.25}$$

Let $t = s/|c|$, then

$$\begin{aligned} ||x_1,x_2,\dots,cx_n||_\alpha &= \inf\{|c|t : N(x_1,x_2,\dots,x_n,t) \geq \alpha\} \\ &= |c|\inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha\} \\ &= |c|\,||x_1,x_2,\dots,x_n||_\alpha. \end{aligned} \tag{3.26}$$

If $c = 0$, then

$$\begin{aligned} ||x_1,x_2,\dots,cx_n||_\alpha &= ||x_1,x_2,\dots,0||_\alpha \\ &= 0 = 0||x_1,x_2,\dots,x_n||_\alpha \\ &= |c|\,||x_1,x_2,\dots,x_n|_\alpha, \quad \forall c \in F \text{ (field)}. \end{aligned} \tag{3.27}$$

(4)

$$\begin{aligned} &||x_1,x_2,\dots,x_n||_\alpha + ||x_1,x_2,\dots,x_n'||_\alpha \\ &\quad = \inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha\} + \inf\{s : N(x_1,x_2,\dots,x_n',s) \geq \alpha\} \\ &\quad = \inf\{t+s : N(x_1,x_2,\dots,x_n,t) \geq \alpha,\ N(x_1,x_2,\dots,x_n',s) \geq \alpha\} \\ &\quad \geq \inf\{t+s : N(x_1,x_2,\dots,x_n+x_n',t+s) \geq \alpha\} \\ &\quad \geq \inf\{r : N(x_1,x_2,\dots,x_n+x_n',r) \geq \alpha\}, \quad r = t+s \\ &\quad = ||x_1,x_2,\dots,x_n+x_n'||_\alpha. \end{aligned} \tag{3.28}$$

Therefore, $\|x_1,x_2,\dots,x_n+x_n'\|_\alpha \leq \|x_1,x_2,\dots,x_n\|_\alpha + \|x_1,x_2,\dots,x_n'\|_\alpha$.

Thus $\{\|\bullet,\bullet,\dots,\bullet\|_\alpha : \alpha \in (0,1)\}$ is an α-n-norm on X.

Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned} ||x_1,x_2,\dots,x_n||_{\alpha_1} &= \inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_1\}, \\ ||x_1,x_2,\dots,x_n||_{\alpha_2} &= \inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_2\}. \end{aligned} \tag{3.29}$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_2\} \subset \{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_1\} \tag{3.30}$$

implies

$$\inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_2\} \geq \inf\{t : N(x_1,x_2,\dots,x_n,t) \geq \alpha_1\} \tag{3.31}$$

which implies

$$||x_1,x_2,\dots,x_n|_{\alpha_2} \geq ||x_1,x_2,\dots,x_n||_{\alpha_1}. \tag{3.32}$$

Hence, $\{\|\bullet,\bullet,\dots,\bullet\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-n-norms on X corresponding to the fuzzy n-norm on X. □

4. Best approximation sets in α-n-normed space

Inspired by this α-n-norm on X, we introduce the notion of two subsets of X, namely, $D_{x_2,x_3,\dots,x_n}(x_0,G)$ and $P_{G,x_2,x_3,\dots,x_n}(x)$.

Definition 4.1. Let $(X,\|\bullet,\bullet,\dots,\bullet\|_\alpha)$ be an α-n-normed space corresponding to the fuzzy n-norm N on X. Let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every $x_2,x_3,\dots,x_n \in X\backslash G$ which is independent of x and x_0,

$$d_{x_2,x_3,\dots,x_n}(x,G) \le ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \tag{4.1}$$

where

$$d_{x_2,x_3,\dots,x_n}(x,G) = \inf_{g\in G} ||x-g,x_2,x_3,\dots,x_n||_\alpha. \tag{4.2}$$

For each $G \subset X$ and $x_0 \in X$,we define

$$\begin{aligned} &D_{x_2,x_3,\dots,x_n}(x_0,G) \\ &\quad = \{x \in X : d_{x_2,x_3,\dots,x_n}(x,G) = ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G)\} \end{aligned} \tag{4.3}$$

for any $x_2,x_3,\dots,x_n \in X\backslash G$ which is independent of x and x_0.

We denote

$$\begin{aligned} P_{G,x_2,x_3,\dots,x_n}(x) &= \{g_0 \in G : ||x-g_0,x_2,x_3,\dots,x_n||_\alpha = d_{x_2,x_3,\dots,x_n}(x,G)\}, \\ P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0) &= \{x \in X : ||x-x_0,x_2,x_3,\dots,x_n||_\alpha = d_{x_2,x_3,\dots,x_n}(x,G)\}, \end{aligned} \tag{4.4}$$

where $x_0 \in G$.

We give the following examples in the α-2-normed linear space and α-n-normed linear space for the sets $D_{x_2,x_3,\dots,x_n}(x_0,G)$ and $P_{G,x_2,x_3,\dots,x_n}(x)$. It is easy to find the set $P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0)$.

Example 4.2. Let $X = \mathbb{R}^3$ be a linear space over $\mathbb{R}$.

Define $\|\bullet,\bullet\| : X \times X \to \mathbb{R}$ by

$$\begin{aligned} ||x_1,x_2||_1 &= \max\{|a_1b_2-a_2b_1|, |b_1c_2-b_2c_1|, |a_1c_2-a_2c_1|\}, \\ ||x_1,x_2||_2 &= \frac{1}{2}\{\max\{|a_1b_2-a_2b_1|, |b_1c_2-b_2c_1|, |a_1c_2-a_2c_1|\}\}, \end{aligned} \tag{4.5}$$

where $x_i = (a_i,b_i,c_i) \in \mathbb{R}^3$, $i = 1,2$. Then $(X,\|\bullet,\bullet\|_1)$ and $(X,\|\bullet,\bullet\|_2)$ are 2-normed linear spaces.

Define $N : X \times X \times \mathbb{R} \to [0,1]$ by

$$N(x_1,x_2,t) = \begin{cases} 1, & \text{if } t > ||x_1,x_2||_1, \\ 0.5, & \text{if } ||x_1,x_2||_2 < t \le ||x_1,x_2||_1, \\ 0, & \text{if } t \le ||x_1,x_2||_2. \end{cases} \tag{4.6}$$

Then (X,N) is a fuzzy 2-normed linear space. Define $\|x_1,x_2\|_\alpha = \inf\{t : N(x_1,x_2,t) \ge \alpha\}$, $\alpha \in (0,1)$.

The α-2-norms are given by

$$\begin{aligned} ||x_1,x_2||_\alpha &= ||x_1,x_2||_1, \quad \text{when } 1>\alpha>0.5,\\ &= ||x_1,x_2||_2, \quad \text{when } 0<\alpha\le 0.5. \end{aligned} \tag{4.7}$$

Let $G=\{(a,0,0):a\in\mathbb{R}\}$ be a subset of X.

Choose $x_0=(0,1,1)$ and $x_2\in K=\{(0,0,k):k\in\mathbb{R}\backslash\{0\}\}$.

Then

$$\begin{gathered} D_{x_2}(x_0,G)=\{x=(0,b,0),\ b\in R^+\backslash\{0\}:d_{x_2}(x,G)=||x-x_0,x_2||_\alpha+d_{x_2}(x_0,G)\},\\ P_{G,x_2}(x)=\{g'=(a,0,0):-1\le a\le 1\}. \end{gathered} \tag{4.8}$$

Example 4.3. Let $X=\mathbb{R}^{n+1}$ be a linear space over $\mathbb{R}$.

Define $\|\bullet,\bullet,\dots,\bullet\|:\underbrace{X\times\cdots\times X}_{n}\to\mathbb{R}$ by

$$\begin{aligned} ||x_1,x_2,\dots,x_n||_1 &= \max\{\Delta_1,\Delta_2,\dots,\Delta_n\},\\ ||x_1,x_2,\dots,x_n|_2 &= \frac{1}{2}\{\max\{\Delta_1,\Delta_2,\dots,\Delta_n\}\}, \end{aligned} \tag{4.9}$$

where

$$\Delta_1=\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n+1)}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n2} & a_{n3} & \cdots & a_{n(n+1)} \end{vmatrix},\quad \Delta_2=\begin{vmatrix} a_{13} & \cdots & a_{1(n+1)} & a_{11}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n3} & \cdots & a_{n(n+1)} & a_{n1} \end{vmatrix},\quad \dots,\quad \Delta_n=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{4.10}$$

and $x_i=(a_{i1},a_{i2},\dots,a_{i(n+1)})\in\mathbb{R}^{n+1}$, $i=1,2,\dots,n$.

Then $(X,\|\bullet,\bullet,\dots,\bullet\|_1)$ and $(X,\|\bullet,\dots,\bullet,\bullet\|_2)$ are n-normed linear spaces. Define $N:\underbrace{X\times\cdots\times X}_{n}\times\mathbb{R}\to[0,1]$ by

$$N(x_1,x_2,\dots,x_n,t)=\begin{cases} 1, & \text{if } t>||x_1,x_2,\dots,x_n||_1,\\ 0.5, & \text{if } ||x_1,x_2,\dots,x_n||_2<t\le ||x_1,x_2,\dots,x_n||_1,\\ 0, & \text{if } t\le ||x_1,x_2,\dots,x_n||_2. \end{cases} \tag{4.11}$$

Then (X,N) is a fuzzy n-normed linear space. Define $\|x_1,x_2,\dots,x_n\|_\alpha=\inf\{t:N(x_1,x_2,\dots,x_n,t)\ge\alpha\}$, $\alpha\in(0,1)$.

The α-n-norms are given by

$$\begin{aligned}
&\|x_1,x_2,\dots,x_n\|_\alpha \\
&\quad= \|x_1,x_2,\dots,x_n\|_1, \quad \text{when } 1>\alpha>0.5, \\
&\quad= \|x_1,x_2,\dots,x_n|_2, \quad \text{when } 0<\alpha\le 0.5.
\end{aligned} \tag{4.12}$$

Let $G=\{(a,0,0,\dots,n \text{ times } 0): a\in\mathbb{R}\}$ be a subset of X.

Choose $x_0=(0,1,1,\dots,n \text{ times } 1)$ and

$$x_2,x_3,\dots,x_n\in K=\left\{\left(0,0,k_3^{(i)},\dots,k_{n+1}^{(i)}\right): k_3^{(i)},\dots,k_{n+1}^{(i)}\in\mathbb{R}\setminus\{0\}\right\}. \tag{4.13}$$

That is,

$$\begin{aligned}
x_2 &= \left(0,0,k_3^{(2)},\dots,k_{n+1}^{(2)}\right),\\
x_3 &= \left(0,0,k_3^{(3)},\dots,k_{n+1}^{(3)}\right),\\
&\ \ \vdots\\
x_n &= \left(0,0,k_3^{(n)},\dots,k_{n+1}^{(n)}\right).
\end{aligned} \tag{4.14}$$

Then

$$\begin{aligned}
&D_{x_2,x_3,\dots,x_n}(x_0,G)\\
&\quad=\{x=(0,b,0,\dots,(\mathrm{n}-1) \text{ times } 0),\ b\in\mathbb{R}^+\setminus\{0\}:\\
&\qquad d_{x_2,x_3,\dots,x_n}(x,G)=\|x-x_0,x_2,x_3,\dots,x_n\|_\alpha+d_{x_2,x_3,\dots,x_n}(x_0,G)\},
\end{aligned} \tag{4.15}$$

where $d_{x_2,x_3,\dots,x_n}(x,G)=\max\{|b|\Delta,|a|\Delta\}$,

$$\begin{aligned}
\Delta &= \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \cdots & k_{n+1}^{(2)}\\ k_3^{(3)} & k_4^{(3)} & \cdots & k_{n+1}^{(3)}\\ \vdots & \vdots & \ddots & \vdots\\ k_3^{(n)} & k_4^{(n)} & \cdots & k_{n+1}^{(n)} \end{vmatrix},\\
\|x-x_0,x_2,\dots,x_n\|_\alpha &= |b-1|\Delta, \qquad d_{x_2,x_3,\dots,x_n}(x_0,G)=\max\{\Delta,|a|\Delta\}
\end{aligned} \tag{4.16}$$

and also $P_{G,x_2,x_3,\dots,x_n}(x)=\{g'=(a,0,\dots,n \text{ times } 0): -1\le a\le 1\}$.

By routine calculation the following theorems are validate from the Examples 4.2 and 4.3.

Theorem 4.4. *For $x\in D_{x_2,x_3,\dots,x_n}(x_0,G)$ and $y\in D_{x_2,x_3,\dots,x_n}(x,G)$,*

(i) $\|y-x_0,x_2,x_3,\dots,x_n\|_\alpha=\|y-x,x_2,x_3,\dots,x_n\|_\alpha+\|x-x_0,x_2,x_3,\dots,x_n\|_\alpha$,

(ii) $y-x+x_0\in D_{x_2,x_3,\dots,x_n}(x_0,G)$.

Proof. (i) Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$ and $y \in D_{x_2,x_3,\dots,x_n}(x,G)$.
Then by (4.3) we have

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(x,G) &= \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \\ d_{x_2,x_3,\dots,x_n}(y,G) &= \|y-x,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x,G). \end{aligned} \tag{4.17}$$

Consider

$$\begin{aligned} &\|y-x_0,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \|y-x_0-x+x,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \|(y-x)+(x-x_0),x_2,x_3,\dots,x_n\|_\alpha \\ &\quad \le \|y-x,x_2,x_3,\dots,x_n\|_\alpha + \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \left(d_{x_2,x_3,\dots,x_n}(y,G) - d_{x_2,x_3,\dots,x_n}(x,G)\right) \\ &\qquad + \left(d_{x_2,x_3,\dots,x_n}(x,G) - d_{x_2,x_3,\dots,x_n}(x_0,G)\right) \\ &\quad = d_{x_2,x_3,\dots,x_n}(y,G) - d_{x_2,x_3,\dots,x_n}(x_0,G) \\ &\quad \le \|y-x_0,x_2,x_3,\dots,x_n\|_\alpha. \end{aligned} \tag{4.18}$$

Therefore,

$$\|y-x_0,x_2,x_3,\dots,x_n\|_\alpha = \|y-x,x_2,x_3,\dots,x_n\|_\alpha + \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha. \tag{4.19}$$

(ii) By (4.2), we have

$$\begin{aligned} &d_{x_2,x_3,\dots,x_n}(y-x+x_0,G) \\ &\quad \ge d_{x_2,x_3,\dots,x_n}(y,G) - \|y-(y-x+x_0),x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = d_{x_2,x_3,\dots,x_n}(y,G) - \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \left(\|y-x,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x,G)\right) - \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \|y-x,x_2,x_3,\dots,x_n\|_\alpha + \left(\|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G)\right) \\ &\qquad - \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha \\ &\quad = \|y-x,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G) \\ &\quad = \|(y-x+x_0)-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G). \end{aligned} \tag{4.20}$$

Again by (4.2), it follows that

$$d_{x_2,x_3,\dots,x_n}(y-x+x_0,G) = \|(y-x+x_0)-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \tag{4.21}$$

which implies $y-x+x_0 \in D_{x_2,x_3,\dots,x_n}(x_0,G)$. □

Theorem 4.5. *Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$. Then*
(i) $[x_0,x] = \{\lambda x_0 + (1-\lambda)x : 0 \le \lambda \le 1\} \subset D_{x_2,x_3,\dots,x_n}(x_0,G)$,
(ii) $D_{x_2,x_3,\dots,x_n}(x,G) \subset D_{x_2,x_3,\dots,x_n}(x_0,G)$.

Proof. (i) Let $y = \lambda x_0 + (1-\lambda)x$ such that $0 \le \lambda \le 1$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(y,G) &\ge d_{x_2,x_3,\dots,x_n}(x,G) - \|x-y,x_2,x_3,\dots,x_n\|_\alpha \\ &= \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G) - \|x-y,x_2,x_3,\dots,x_n\|_\alpha \\ &= \|y-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G). \end{aligned} \tag{4.22}$$

By (4.2), we have

$$d_{x_2,x_3,\dots,x_n}(y,G) = \|y-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \tag{4.23}$$

which implies $y \in D_{x_2,x_3,\dots,x_n}(x_0,G)$.

(ii) Let $y \in D_{x_2,x_3,\dots,x_n}(x,G)$. Then, by (4.3) and Theorem 4.4(i),

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(y,G) &= \|y-x,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x,G) \\ &= \|y-x,x_2,x_3,\dots,x_n\|_\alpha + (\|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G)) \\ &= \|y-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \end{aligned} \tag{4.24}$$

which implies $y \in D_{x_2,x_3,\dots,x_n}(x_0,G)$.

Therefore, $D_{x_2,x_3,\dots,x_n}(x,G) \subset D_{x_2,x_3,\dots,x_n}(x_0,G)$. □

Theorem 4.6. *Let $x_0, y_0 \in X$ and $\lambda \neq 0$. Then*

(i) $D_{x_2,x_3,\dots,x_n}(x_0,G) + y_0 = D_{x_2,x_3,\dots,x_n}(x_0+y_0,G+y_0)$,

(ii) $D_{x_2,x_3,\dots,x_n}(x_0,\lambda G) = \lambda D_{x_2,x_3,\dots,x_n}(x_0/\lambda,G)$.

Proof. (i) Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(x+y_0,G+y_0) &= d_{x_2,x_3,\dots,x_n}(x,G) \\ &= \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G) \\ &= \|x+y_0-(x_0+y_0),x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G). \end{aligned} \tag{4.25}$$

Therefore, $x+y_0 \in D_{x_2,x_3,\dots,x_n}(x_0+y_0,G+y_0)$.

Conversely, let $y \in D_{x_2,x_3,\dots,x_n}(x_0+y_0,G+y_0)$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(y-y_0,G) &= d_{x_2,x_3,\dots,x_n}(y,G+y_0) \\ &= \|y-y_0-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0+y_0,G+y_0) \\ &= \|(y-y_0)-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G). \end{aligned} \tag{4.26}$$

Therefore, $y-y_0 \in D_{x_2,x_3,\dots,x_n}(x_0,G)$, and so

$$D_{x_2,x_3,\dots,x_n}(x_0,G) + y_0 = D_{x_2,x_3,\dots,x_n}(x_0+y_0,G+y_0). \tag{4.27}$$

(ii) Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,\lambda G)$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}\left(\frac{x}{\lambda},G\right) &= \frac{1}{|\lambda|}d_{x_2,x_3,\dots,x_n}(x,\lambda G) \\ &= \frac{1}{|\lambda|}\left(||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,\lambda G)\right) \\ &= \left|\left|\frac{x}{\lambda}-\frac{x_0}{\lambda},x_2,x_3,\dots,x_n\right|\right|_\alpha + d_{x_2,x_3,\dots,x_n}\left(\frac{x_0}{\lambda},G\right). \end{aligned} \tag{4.28}$$

Therefore, $x/\lambda \in D_{x_2,x_3,\dots,x_n}(x_0/\lambda,G)$.

Conversely, let $x \in D_{x_2,x_3,\dots,x_n}(x_0/\lambda,G)$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(\lambda x,\lambda G) &= |\lambda| d_{x_2,x_3,\dots,x_n}(x,G) \\ &= |\lambda|\left(\left|\left|x-\frac{x_0}{\lambda},x_2,x_3,\dots,x_n\right|\right|_\alpha + d_{x_2,x_3,\dots,x_n}\left(\frac{x_0}{\lambda},G\right)\right) \\ &= ||\lambda x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,\lambda G). \end{aligned} \tag{4.29}$$

Therefore, $\lambda x \in D_{x_2,x_3,\dots,x_n}(x_0,\lambda G)$.

Thus, $D_{x_2,x_3,\dots,x_n}(x_0,\lambda G) = \lambda D_{x_2,x_3,\dots,x_n}(x_0/\lambda,G)$. □

Theorem 4.7. *Let $G \subset G_1$ and $x_0 \in X$, where G_1 is a subset of X such that*

$$d_{x_2,x_3,\dots,x_n}(x_0,G) = d_{x_2,x_3,\dots,x_n}(x_0,G_1). \tag{4.30}$$

Then $D_{x_2,x_3,\dots,x_n}(x_0,G_1) \subset D_{x_2,x_3,\dots,x_n}(x_0,G)$.

Proof. Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,G_1)$. Then, by (4.30), we have

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}(x,G) &\geq d_{x_2,x_3,\dots,x_n}(x,G_1) \\ &= ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G_1) \\ &= ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G). \end{aligned} \tag{4.31}$$

By (4.2), it follows that

$$d_{x_2,x_3,\dots,x_n}(x,G) = ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \tag{4.32}$$

which implies $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$.

Hence, $D_{x_2,x_3,\dots,x_n}(x_0,G_1) \subset D_{x_2,x_3,\dots,x_n}(x_0,G)$. □

Theorem 4.8. (i) $P_{G,x_2,x_3,\dots,x_n}(x_0) \subset P_{G,x_2,x_3,\dots,x_n}(x)$ *for every* $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$,

(ii) $D_{x_2,x_3,\dots,x_n}(x_0,G) = P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0)$ *for every* $x_0 \in \overline{G}$.

Proof. (i) Let $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$ and $g \in P_{G,x_2,x_3,\dots,x_n}(x_0)$.

Now,

$$\begin{aligned} &d_{x_2,x_3,\dots,x_n}(x,G) \\ &\quad = ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G) \\ &\quad = ||x-x_0,x_2,x_3,\dots,x_n||_\alpha + ||x_0-g_0,x_2,x_3,\dots,x_n||_\alpha. \end{aligned} \tag{4.33}$$

By Theorem 4.4(i), we have

$$d_{x_2,x_3,\dots,x_n}(x,G) = \|x-g_0,x_2,x_3,\dots,x_n\|_\alpha, \tag{4.34}$$

which implies $g_0 \in P_{G,x_2,x_3,\dots,x_n}(x)$, which in turn implies $P_{G,x_2,x_3,\dots,x_n}(x_0) \subset P_{G,x_2,x_3,\dots,x_n}(x)$.

(ii) Let $x_0 \in \overline{G}$ and $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$. Then

$$\begin{aligned} d_{x_2,x_3,\dots,x_n}&(x,G) \\ &= \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G) \\ &= \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha, \quad \text{where } x_0 \in \overline{G}, \end{aligned} \tag{4.35}$$

which implies $x \in P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0)$. So

$$D_{x_2,x_3,\dots,x_n}(x_0,G) \subset P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0). \tag{4.36}$$

Conversely, let $x \in P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0)$.

Then $x_o \in P_{G,x_2,x_3,\dots,x_n}(x)$.

Since $x_0 \in \overline{G}, d_{x_2,x_3,\dots,x_n}(x_0,G) = 0$.

Hence, we have

$$d_{x_2,x_3,\dots,x_n}(x,G) = \|x-x_0,x_2,x_3,\dots,x_n\|_\alpha + d_{x_2,x_3,\dots,x_n}(x_0,G), \tag{4.37}$$

which implies $x \in D_{x_2,x_3,\dots,x_n}(x_0,G)$, which in turn implies

$$P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0) \subset D_{x_2,x_3,\dots,x_n}(x_0,G). \tag{4.38}$$

From (4.36) and (4.38), we have

$$D_{x_2,x_3,\dots,x_n}(x_0,G) = P^{-1}_{G,x_2,x_3,\dots,x_n}(x_0). \tag{4.39}$$

□

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