SEMIDISCRETE CENTRAL DIFFERENCE METHOD IN TIME FOR DETERMINING SURFACE TEMPERATURES

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We consider an inverse heat conduction problem (IHCP) in a quarter plane. We want to know the distribution of surface temperature in a body from a measured temperature history at a fixed location inside the body. This is a severely ill-posed problem in the sense that the solution (if exists) does not depend continuously on the data. Eldén (1995) has used a difference method for solving this problem, but he did not obtain the convergence at $x = 0$. In this paper, we gave a logarithmic stability of the approximation solution at $x = 0$ under a stronger a priori assumption $\|u(0,t)\|_p \leq E$ with $p > 1/2$. A numerical example shows that the computational effect of this method is satisfactory.

1. Introduction

In several engineering contexts, it is sometimes necessary to determine the surface temperature in a body from a measured temperature history at a fixed location inside the body [1]. This problem is called the inverse heat conduction problem (IHCP). IHCP is a severely ill-posed problem: a small perturbation in the data may cause dramatically large errors in the solution. As a model problem, we will consider the following sideways heat equation:

$$\begin{gathered} u_{xx} = u_t, \quad x > 0,\ t > 0, \\ u(x,0) = 0, \quad x \geq 0, \\ u(1,t) = g(t), \quad t \geq 0, \quad u(x,t)|_{x\to\infty} \text{ bounded}, \end{gathered} \tag{1.1}$$

and want to know $u(x,t)$ for $0 \leq x < 1$.

Some valid regularizing methods and error estimates for above problem have appeared [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], but most of them only consider the case when $x \in (0,1)$ and cannot obtain the convergence of approximation solution at $x = 0$. For example, Carasso utilized a particular Tikhonov regularization method in [2], and Eldén applied the difference schemes in time in [3]. In this paper, we specially deal with the convergence of an approximate solution at $x = 0$ by a central difference scheme in time which itself has a regularization effect. An error estimate is obtained and the estimate gives information

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about how to choose the step length in the time discretization. A numerical example is also given.

As we consider the problem (1.1) in $L^2(\mathbb{R})$ with respect to the variable t, we extend the domain of definition of the function $u(x,\cdot)$, $g(\cdot) := u(1,\cdot)$, $f(\cdot) := u(0,\cdot)$ and other functions appearing in the paper to the whole real t-axis by defining them to be zero for $t < 0$. The notation $\|\cdot\|$ denotes L^2-norm, and

$$\hat{h}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} h(t)dt \tag{1.2}$$

is the Fourier transform of function $h(t)$. We assume that there exists a priori bound for $f(t) := u(0,t)$:

$$\|f\|_p \leq E, \quad p \geq 0, \tag{1.3}$$

where $\|\cdot\|_p$ denotes the norm in $H^p(\mathbb{R})$ defined by

$$\|f\|_p := \left(\int_{-\infty}^{\infty} (1+\xi^2)^p \left| \hat{f}(\xi) \right|^2 d\xi \right)^{1/2}. \tag{1.4}$$

Let $g(t)$ and $g_\delta(t)$ denote the exact and measured data at $x = 1$ of the solution $u(x,t)$, respectively, which satisfy

$$\|g(t) - g_\delta(t)\| \leq \delta, \tag{1.5}$$

where δ is the measurement error. The solution of problem (1.1) has been given in [2] by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} e^{(1-x)\theta(\xi)} \hat{g}(\xi) d\xi, \quad 0 \leq x < 1, \tag{1.6}$$

or, equivalently,

$$\hat{u}(x,\xi) = e^{(1-x)\theta(\xi)} \hat{g}(\xi), \quad 0 \leq x < 1, \tag{1.7}$$

where $\theta(\xi)$ is the principal value of $\sqrt{i\xi}$:

$$\theta(\xi) = (1+\sigma i)\sqrt{\frac{|\xi|}{2}}, \quad \sigma = \operatorname{sign}(\xi),\ \xi \in \mathbb{R}. \tag{1.8}$$

It is easy to see from (1.7) that

$$\hat{f}(\xi) = e^{\theta(\xi)} \hat{g}(\xi). \tag{1.9}$$

Since the real part of $\theta(\xi)$ is nonnegative, $\hat{u}(x,\xi)$ is in $L^2(\mathbb{R})$, so from (1.7) we know that $\hat{g}(\xi)$ must decay rapidly as $\xi \to \infty$. Small errors in high-frequency components can blow up and completely destroy the solution for $0 \leq x < 1$. As the measured data $g_\delta(t)$, its Fourier transform $\hat{g}_\delta(\xi)$ is merely in $L^2(\mathbb{R})$. In order to obtain the stability of the solution, a central difference scheme in time, which we learned from Eldén [3], is considered in the next section and an error estimate is obtained.

2. The central difference schemes and error estimate

In this section, we will first consider discretization in time by central difference and then discuss the error estimate. As an approximation of problem (1.1) we now consider the following problem:

$$\begin{aligned} v_{xx}(x,t) &= \frac{1}{2k}\big(v(x,t+k)-v(x,t-k)\big), \quad x>0,\ t>0,\\ v(x,0) &= 0, \quad x\geq 0,\\ v(1,t) &= g_\delta(t), \quad t\geq 0,\\ v(x,t)|_{x\to\infty} &\text{ bounded}, \end{aligned} \tag{2.1}$$

where we have replaced the time derivative by a central difference with step length k. The advantage of not discretizing in the space variable is that we can use Fourier transform techniques.

By taking the Fourier transform for variable t in (2.1) we have

$$\begin{aligned} \hat{v}_{xx}(x,\xi) &= i\frac{\sin k\xi}{k}\hat{v}(x,\xi),\\ \hat{v}(1,\xi) &= \hat{g}_\delta(\xi),\\ \hat{v}(x,\xi)|_{x\to\infty} &\text{ bounded}. \end{aligned} \tag{2.2}$$

The solution of (2.2) has been given in [3]:

$$\hat{v}(x,\xi) = e^{(1-x)\rho(k,\xi)}\hat{g}_\delta(\xi), \tag{2.3}$$

or, equivalently,

$$v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi t}e^{(1-x)\rho(k,\xi)}\hat{g}_\delta(\xi)d\xi, \quad 0\leq x<1, \tag{2.4}$$

where $\rho(k,\xi)$ is the principal value of $\sqrt{i((\sin k\xi)/k)}$:

$$\rho(k,\xi) = (1+\nu i)\sqrt{\frac{|\sin k\xi|}{2k}}, \quad \nu = \operatorname{sign}(\sin k\xi),\ \xi\in\mathbb{R}. \tag{2.5}$$

We will discuss the convergence and error estimate of approximation solution $v(x,t)$ at $x=0$.

Theorem 2.1. *If $p>1/2$, and conditions (1.3), (1.5) hold, functions $u(x,t)$ and $v(x,t)$ are given by (1.6), (2.4), respectively. Let*

$$k = \frac{1}{2\left(\ln\left(\left(\frac{E}{\delta}\right)\left(\ln\left(\frac{E}{\delta}\right)\right)^{-2p}\right)\right)^2}. \tag{2.6}$$

Then, there holds

$$\|u(0,\cdot)-v(0,\cdot)\| \le E\left(\left(\ln\left(\frac{E}{\delta}\right)\right)^{-2p}+\varepsilon\right), \tag{2.7}$$

where $\varepsilon = \max\{2k^p,(\sqrt{2}/6)k^{p-1/2},(\sqrt{2}/6)k^2\}$.

Proof. By (1.7), (2.3), and Parseval formula, we have

$$\|u(0,\cdot)-v(0,\cdot)\| = \|\hat{u}(0,\cdot)-\hat{v}(0,\cdot)\| = \|e^{\theta(\xi)}\hat{g}(\xi)-e^{\rho(k,\xi)}\hat{g}_\delta(\xi)\|. \tag{2.8}$$

For abbreviation, we denote, for example,

$$\theta := \theta(\xi), \qquad \rho := \rho(k,\xi), \qquad \hat{g} := \hat{g}(\xi), \tag{2.9}$$

Then, (1.9), (1.3), and (1.5) lead to

$$\begin{aligned}
\|u(0,\cdot)-v(0,\cdot)\| &= \|e^{\theta}\hat{g}-e^{\rho}\hat{g}_\delta\| \\
&= \|e^{\theta}\hat{g}-e^{\rho}\hat{g}+e^{\rho}\hat{g}-e^{\rho}\hat{g}_\delta\| \\
&\le \|(1-e^{\rho-\theta})\hat{f}\|+\|e^{\rho}(\hat{g}-\hat{g}_\delta)\| \\
&= \|(1-e^{\rho-\theta})(1+\xi^2)^{-p/2}(1+\xi^2)^{p/2}\hat{f}\|+\|e^{\rho}(\hat{g}-\hat{g}_\delta)\| \\
&\le \sup_{\xi\in\mathbb{R}} A(\xi)E+\sup_{\xi\in\mathbb{R}} B(\xi)\delta,
\end{aligned} \tag{2.10}$$

where

$$A(\xi) := \left|(1-e^{\rho-\theta})(1+\xi^2)^{-p/2}\right|, \qquad B(\xi) := |e^{\rho}|. \tag{2.11}$$

We start by estimating the second term of the right side of (2.10). From (2.5) and (2.6) we know

$$\sup_{\xi\in\mathbb{R}} B(\xi)\delta = \sup_{\xi\in\mathbb{R}} e^{\mathrm{Re}(\rho)}\delta = \sup_{\xi\in\mathbb{R}} e^{\sqrt{|\sin k\xi|/2k}}\delta \le e^{\sqrt{1/2k}}\delta = \left(\ln\frac{E}{\delta}\right)^{-2p}E. \tag{2.12}$$

To estimate the first term of the right side of (2.10), we rewrite $A(\xi)$ as

$$A(\xi) = |1-e^{-\tau}|(1+\xi^2)^{-p/2}, \tag{2.13}$$

where

$$\tau = \theta - \rho = \frac{1+\sigma i}{\sqrt{2}}\sqrt{|\xi|} - \frac{1+\nu i}{\sqrt{2}}\left(\frac{|\sin k\xi|}{k}\right)^{1/2}. \tag{2.14}$$

Denote

$$\xi_0 := \frac{1}{k} = 2\left(\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-2p}\right)\right)^2. \tag{2.15}$$

We now estimate $A(\xi)$ for large values of ξ, that is, for $|\xi| \geq \xi_0$. Note that $\mathrm{Re}(\tau) \geq 0$ and by (2.13), we have

$$A(\xi) \leq 2|\xi|^{-p} \leq 2\xi_0^{-p} = 2k^p \tag{2.16}$$

so that

$$A(\xi)E \leq 2k^p E. \tag{2.17}$$

It remains to estimate $A(\xi)$ for $|\xi| < \xi_0$, that is, $|k\xi| < 1$. We now observe that for ξ in this interval, $\sigma = \mathrm{sign}(\xi) = \mathrm{sign}(\sin k\xi) = \nu$, which means that we can rewrite (2.14) as

$$\tau = \tau_1(1+\sigma i), \quad \tau_1 = \frac{1}{\sqrt{2k}}\left(\sqrt{|k\xi|} - \sqrt{|\sin k\xi|}\right). \tag{2.18}$$

Since $\tau_1 \geq 0$, using inequalities $\sqrt{\sin a^2} \geq \sqrt{a^2 - a^6/6} \geq a(1 - a^4/6)$ $(0 \leq a < 1)$ and $1 - e^{-y} \leq y$ $(y \geq 0)$, we get

$$\begin{aligned} |1-e^{-\tau}| &= |1-e^{-i\sigma\tau_1} + e^{-i\sigma\tau_1} - e^{-(\tau_1+i\sigma\tau_1)}| \leq |1-e^{-i\sigma\tau_1}| + |1-e^{-\tau_1}| \\ &= 2\left|\sin\left(\frac{\sigma\tau_1}{2}\right)\right| + |1-e^{-\tau_1}| \leq 2\tau_1 \leq \frac{\sqrt{2}}{6}k^{-1/2}|k\xi|^{5/2}. \end{aligned} \tag{2.19}$$

Combining this estimate with (2.13), we know

$$A(\xi) \leq \frac{\sqrt{2}}{6}k^{-1/2}|k\xi|^{5/2}(1+\xi^2)^{-p/2}. \tag{2.20}$$

If $1/2 < p < 5/2$, from (2.20) we have

$$\begin{aligned} A(\xi)E &\leq \frac{\sqrt{2}}{6}k^{-1/2}|k\xi|^{5/2}|\xi|^{-p}E = \frac{\sqrt{2}}{6}k^{p-1/2}|k\xi|^{5/2-p}E \\ &\leq \frac{\sqrt{2}}{6}k^{p-1/2}E \quad \text{for } |k\xi| < 1. \end{aligned} \tag{2.21}$$

If $p \geq 5/2$, for $|\xi| \geq 1$, from (2.20) we have

$$\begin{aligned} A(\xi)E &\leq \frac{\sqrt{2}}{6}k^{-1/2}|k\xi|^{5/2}|\xi|^{-p}E = \frac{\sqrt{2}}{6}k^2|\xi|^{5/2-p}E \\ &\leq \frac{\sqrt{2}}{6}k^2E \quad \text{for } |k\xi| < 1,\ |\xi| \geq 1. \end{aligned} \tag{2.22}$$

For $|\xi| < 1$, from (2.20) we have

$$A(\xi)E \le \frac{\sqrt{2}}{6}k^{-1/2}|k\xi|^{5/2}E \le \frac{\sqrt{2}}{6}k^2 E \quad \text{for } |k\xi| < 1,\ |\xi| < 1. \tag{2.23}$$

Summarizing (2.17), (2.20), (2.21), (2.22), and (2.23), we know

$$A(\xi)E \le \max\left\{2k^p, \frac{\sqrt{2}}{6}k^{p-1/2}, \frac{\sqrt{2}}{6}k^2\right\}E =: \varepsilon E \longrightarrow 0 \quad \text{for } \delta \longrightarrow 0,\ p > \frac{1}{2}. \tag{2.24}$$

Combining (2.24) with (2.12), we have

$$\|u(0,\cdot) - v(0,\cdot)\| \le E\left(\left(\ln\left(\frac{E}{\delta}\right)\right)^{-2p} + \varepsilon\right). \tag{2.25}$$

This is just the estimate (2.7). □

It is obvious that

$$\lim_{\delta\to 0}\|u(0,\cdot) - v(0,\cdot)\| = 0 \quad \text{for } p > \frac{1}{2}. \tag{2.26}$$

Theorem 2.1 solves the convergence of approximation solution $v(x,t)$ of problem (1.1) at $x = 0$, which is just the problem left over by Eldén in [3].

3. A numerical example

It is easy to verify that the function

$$u(x,t) = \begin{cases} \dfrac{x+1}{t^{3/2}}\exp\left\{-\dfrac{(x+1)^2}{4t}\right\}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{3.1}$$

is the exact solution of problem (1.1) with data

$$g(t) = \begin{cases} \dfrac{2}{t^{3/2}}\exp\left\{-\dfrac{1}{t}\right\}, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{3.2}$$

So

$$f(t) := u(0,t) = t^{-3/2}\exp\left\{-\frac{1}{4t}\right\}. \tag{3.3}$$

Figures 3.1 and 3.2 give a comparison of the exact solution $u(0,t)$ with its approximation $v(0,t)$ for $p = 2/3$ and $p = 1$, respectively. To obtain the solution $v(0,t)$ of problem (2.1) we applied the "method of lines" in [4]. The step length k is chosen according to (2.6), and we get the measured data $g_\delta(t)$ by adding random errors of amplitude δ to $g(t)$. It can be seen from these figures that the computation effect of the method given in this paper is satisfactory.

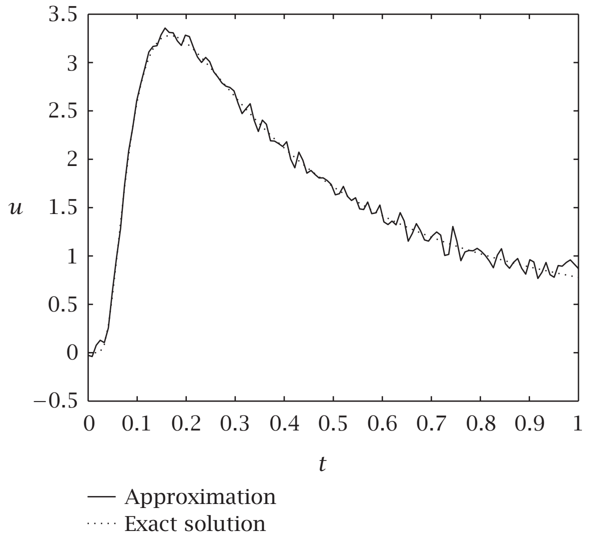

Figure 3.1. $x = 0$, $p = 2/3$, $\delta = 10^{-4}$, $E = 6$, $k = 1/122$.

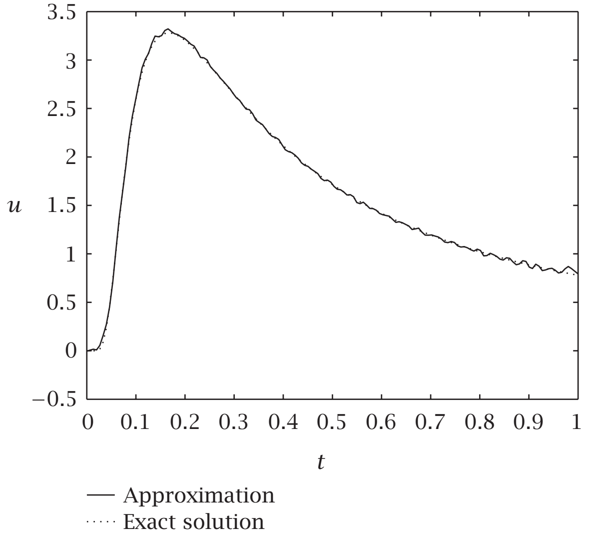

Figure 3.2. $x = 0$, $p = 1$, $\delta = 10^{-5}$, $E = 12$, $k = 1/152$.

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