# ASYMPTOTIC STABILITY OF A REPAIRABLE SYSTEM WITH IMPERFECT SWITCHING MECHANISM

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This paper studies the asymptotic stability of a repairable system with repair time of failed system that follows arbitrary distribution. We show that the system operator generates a positive $C_0$-semigroup of contraction in a Banach space, therefore there exists a unique, nonnegative, and time-dependant solution. By analyzing the spectrum of system operator, we deduce that all spectra lie in the left half-plane and 0 is the unique spectral point on imaginary axis. As a result, the time-dependant solution converges to the eigenvector of system operator corresponding to eigenvalue 0.

## 1. Introduction

Reliability of a system can be increased by using redundancy technique without changing the individual unit that forms the system. One of the common used forms of redundancy is cold standby system, which often finds application in various industrial or other types of setup.

In addition to the failure of individual units, some critical errors could cause the whole system to fail [4, 5, 6, 7, 9]. But, the switching mechanism is all assumed to be good enough in [4, 5, 6, 7, 9]. In reality, the failure of switching mechanism has to be considered [1, 6, 8]. The most general system (with repair facilities and multiple critical and noncritical errors and imperfect switching mechanism) was discussed in [6]. By probabilistic analysis, the author in [6] established the mathematic model of such system and obtained some desired results. However, all papers mentioned above are limited in applied field, and the results about reliability and availability which those papers had deduced are under the two following assumptions.

*Assumption 1.1.* The repairable system has unique and nonnegative solution.

*Assumption 1.2.* The solution of the repairable system is asymptotic stability.

Both assumptions hold obviously when the repair rate is constant (repair time following exponential distribution), however, whether they hold or not when repair rate is time dependant is still an open question. The purpose of this paper is to strictly provide mathematical proof for both assumptions.


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The rest of the paper is organized as follows. Section 2 describes the repairable system and introduces the mathematical model of system. Then, in Section 3, by $C_0$-semigroup theory, we obtain the unique and nonnegative solution of the system, that is to say that Assumption 1.1 holds. In Section 4, the asymptotic stability of the system is proved, so Assumption 1.2 holds. The paper is concluded in Section 5.

## 2. Model of system

**2.1. System description.** This paper presents such system consisting of $k$ ($\geq 1$) active, $N$ ($\geq 1$) cold standby units with $r$ ($\geq 1$) repair facilities, and $M$ ($\geq 0$) multiple noncritical and critical errors. The system require $k$ active units to operate and the switching mechanism is subjected to failure.

The following assumptions are associated with the model:

(1) multiple critical and noncritical errors can only occur in the system with more than one good unit;
(2) critical error and noncritical error rates are constant;
(3) the units failure rate are constant;
(4) all failures are statistically independent;
(5) the repair rate of noncritical errors is as constant as that of a failed active unit;
(6) the repair time of the failed system is arbitrarily distributed;
(7) the repaired unit is as good as new;
(8) the failure rate when $i$ units have failed is denoted by $a_i$ which is the product of $2^i k$ and [(failure rate of an active unit) plus (failure rates of any one of the multiple noncritical errors) and multiplied by (probability of a successful switching mechanism)];
(9) the units also fail simultaneously when one of the, say $j$, $M \geq j \geq 0$, critical errors hits the system with a failure rate denoted by $d_{i,j}$, $i = 0,1,\ldots,N$;
(10) the system is said to be in one of failed states if $(N+1)$ units have failed or if any one of the $M$ critical errors has occurred.

So, the transition diagram of the system can be depicted as in Figure 2.1.

The following symbols are associated with the model under study:

(1) 0: initial state (i.e., at $t = 0$, all $k$ units are in operation with $N$ cold standby units);
(2) $i$: number of failed units, $i = 1,\ldots,N$;
(3) $j$: failed state of the system, $j = N+1$ means failure of the system, $j = N+1+n$, $n = 1,\ldots,M$ means failure of the system corresponding to the $n$th critical error, $j = M+N+2$ means failure of switching mechanism from cold standby to active unit;
(4) $p_i(t)$: probability that the system is in state $i$, $i = 0,1,\ldots,N+M+2$, at time $t$;
(5) $\mu_j(x)$: repair rate when the system is in state $j$ and has elapsed repair time of $x$, and $0 < \int_0^T \mu_j(x)dx < \infty$, for any $T < \infty$, $\int_0^\infty \mu_j(x)dx = \infty$;
(6) $p_j(x,t)$: probability that the failed system is in state $j$ and has an elapsed repair time of $x$;
(7) $X_j$: random variables representing repair time when the system is in state $j$;
(8) $G_j(\cdot)$: distributed function of $X_j$;
(9) $g_j(\cdot)$: probability density function of $X_j$;

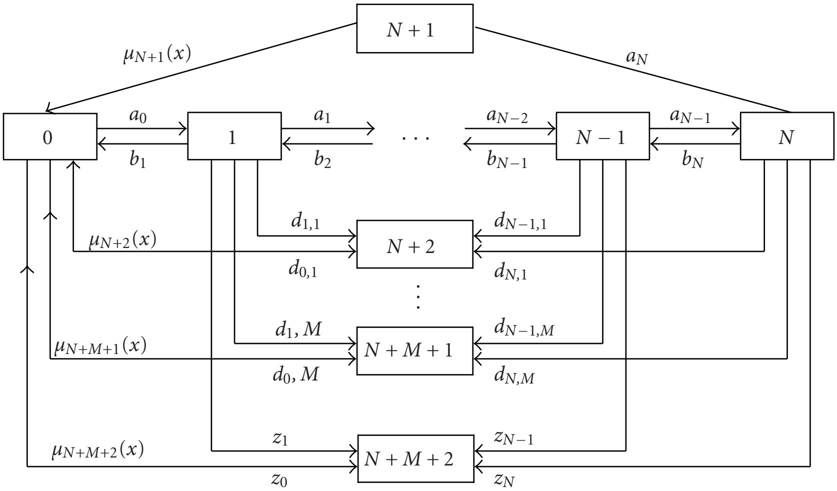


Figure 2.1. Transition diagram of the system.

(10) $E_j(x)$: the mean time to repair that the system is in state $j$ and has an elapsed repair time $x$;
(11) $x_i$: the chance that a system is successful when it is switched from a cold standby unit to an active unit, when it is at state $i$, $1 \geq x_i \geq 0$;
(12) $a$: constant failure rate of an active unit;
(13) $c_i$: constant failure rate of the $i$th noncritical error, $i = 0, 1, \ldots, M$;
(14) $a_i$: failure rate of $i$ unit failing, $a_i = 2^i k(a + c_0 + c_1 + \cdots + c_M)x_i$;
(15) $b$: constant repair rate of a unit;
(16) $b_i$: $\min(i, r)b$;
(17) $d_{i,j}$: constant critical error rate of the system from state $i$ to state $(N + 1 + j)$, $i = 0, 1, \ldots, N$; $j + 1, 2, \ldots, M$;
(18) $z_i$: constant failure rate of the system from state $i$ to state $(N + M + 2)$, $z_i = a^i k(a + c_0 + c_1 + \cdots + c_M)(1 - x_i)$.

**2.2. Mathematical model.** The mathematical model associated with Figure 2.1 can be expressed as follows [6]:

$$\frac{dp_0(t)}{dt} = -h_0 p_0(t) + b_1 p_1(t) + \sum_{i=0}^{M} \int_0^\infty p_{N+1+i}(x,t)\mu_{N+1+i}(x)dx, \tag{2.1}$$

$$\frac{dp_i(t)}{dt} = a_{i-1} p_{i-1}(t) - h_i p_i(t) + b_{i+1} p_{i+1}(t) \quad (i = 1, \ldots, N - 1), \tag{2.2}$$

$$\frac{dp_N(t)}{dt} = a_{N-1} p_{N-1}(t) - h_N p_N(t), \tag{2.3}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x) p_j(x,t) \quad (j = N + 1, \ldots, N + M + 2), \tag{2.4}$$

here, $h_0 = a_0 + z_0 + \sum_{j=1}^{M} d_{0,j}$; $h_n = a_n + b_n + z_n + \sum_{j=1}^{M} d_{n,j}$, $(n = 1, \ldots, N)$.

Boundary conditions: $p_{N+1}(0,t) = a_N p_N(t)$, $p_{N+M+2}(0,t) = \sum_{i=0}^{N} z_i p_i(t)$,

$$P_{N+1+n}(0,t) = \sum_{i=0}^{N} d_{i,n} p_i(t), \quad (n = 1,\dots,M). \tag{2.5}$$

Initial value: $p_0(0) = 1$, $p_i(0) = 0$, $p_j(x,0) = 0$, $i = 1,\dots,N$; $j = N+1,\dots,N+M+2$.

We describe it by abstract Cauchy problem in Banach space. For simplicity, we introduce notations as

$$A = \operatorname{diag}\left( -h_0, -h_1, \dots, -h_{N-1}, -h_N, -\frac{d}{dx} - \mu_{N+1}(x), \dots, -\frac{d}{dx} - \mu_{N+M+2}(x)\right). \tag{2.6}$$

Take state space $\mathbf{X}$ as

$$\mathbf{X} = \left\{ \vec{y} \in \mathbb{C}^{N+1} \times L^1[0,\infty) \times \cdots \times L^1[0,\infty) \mid \|\vec{y}\| = \sum_{i=0}^{N} |y_i| + \sum_{j=0}^{M+1} \|y_{N+1+j}(x)\|_{L^1[0,\infty)} \right\}. \tag{2.7}$$

It is obvious that $(\mathbf{X}, \|\cdot\|)$ is a Banach space. The domain of operator $A$ is $D(A) = \{\vec{p} \in \mathbf{X} \mid dp_j(x)/dx + \mu_j(x)p_j(x) \in L^1[0,\infty),\ p_j(x)$ are absolutely continuous functions, $j = N+1,\dots,N+M+2$, and satisfy $p_{N+1}(0) = a_N p_N,\ p_{N+1+n}(0) = \sum_{i=0}^{N} d_{i,n} p_i (n = 1,\dots,M)$, $p_{N+M+2}(0) = \sum_{i=0}^{N} z_i p_i\}$.

We define operator $E$ as

$$E\vec{p} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & 0 & \int_0^\infty p_{N+1}(x)\mu_{N+1}(x)dx & \cdots & \int_0^\infty p_{N+M+2}(x)\mu_{N+M+2}(x)dx \\ a_0 & 0 & b_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & \vdots & & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & b_N & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & a_{N-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \cdots & & & & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

Then, the above equations (2.1)–(2.5) can be written as an abstract Cauchy problem in the Banach space $\mathbf{X}$,

$$\begin{gathered} \frac{d\vec{p}(t)}{dt} = (A+E)\vec{p}(t), \quad t > 0, \\ \vec{p}(0) = (1,0,\dots,0), \\ \vec{p}(t) = \big(p_0(t), p_1(t), \dots, p_N(t), p_{N+1}(x,t), \dots, p_{N+M+2}(x,t)\big). \end{gathered} \tag{2.9}$$

## 3. Unique and nonnegative solution of (2.1)–(2.5)

In this section, we will prove the existence of the unique and nonnegative solution of the repairable system. We begin with proving the following propositions.

THEOREM 3.1. (1) $\gamma \in \rho(A)$ *and* $\|(\gamma I - A)^{-1}\| < 1/\gamma$ *when* $\gamma > 0$;
(2) $D(A)$ *is dense in* $\mathbf{X}$;
(3) $C_0$*-semigroup* $T(t)$ *generated by* $A+E$ *is a positive* $C_0$*-semigroup;*
(4) $T(t)$ *is a positive* $C_0$*-semigroup of contraction.*

*Proof.* (1) $\gamma \in \rho(A)$ and $\|(\gamma I - A)^{-1}\| < 1/\gamma$ when $\gamma > 0$.

For any $\vec{y} = (y_0, \dots, y_N, y_{N+1}(x), \dots, y_{N+M+2}(x)) \in \mathbf{X}$, consider the equation $(\gamma I - A)\vec{p} = \vec{y}$, that is,

$$(\gamma + h_i) p_i = y_i \quad (i = 0, 1, \dots, N), \tag{3.1}$$

$$\frac{dp_j(x)}{dx} = -(\gamma + \mu_j(x)) p_j(x) + y_j(x), \quad j = N+1, \dots, N+M+2, \tag{3.2}$$

$$p_{N+1}(0) = a_N p_N, \qquad p_{N+M+2}(0) = \sum_{i=0}^{N} z_i p_i, \qquad p_{N+1+n}(0) = \sum_{i=0}^{N} d_{i,n} p_i \quad (n = 1, \dots, M). \tag{3.3}$$

Solving (3.1)-(3.2) with the help of (3.3), we can obtain that

$$\begin{gathered} p_i = \frac{y_i}{\gamma + h_i}, \quad i = 0, 1, \dots, N, \\ p_j(x) = p_j(0) e^{-\int_0^x (\gamma + \mu_j(\xi)) d\xi} + \int_0^x e^{-\int_\tau^x (\gamma + \mu_j(\xi)) d\xi} y_j(\tau) d\tau, \quad j = N+1, \dots, N+M+2. \end{gathered} \tag{3.4}$$

Combining the above equations with *Fubini* theorem, it follows that

$$\begin{aligned} \|\vec{p}\| &= \sum_{i=0}^{N} |p_i| + \sum_{j=N+1}^{N+M+2} \|p_j(x)\|_{L^1[0,\infty)} \\ &< \sum_{i=0}^{N} |p_i| + \sum_{j=N+1}^{N+M+2} \left\{ p_j(0) \int_0^\infty e^{-\gamma x} dx + \int_0^\infty |y_j(\tau)| d\tau \int_\tau^\infty e^{-\gamma(x-\tau)} dx \right\} \\ &= \sum_{i=0}^{N} |p_i| + \frac{1}{\gamma} \left\{ a_N |p_N| + \sum_{i=0}^{N} \sum_{n=1}^{M} d_{i,n} p_i + \sum_{i=0}^{N} z_i p_i \right\} + \frac{1}{\gamma} \sum_{j=N+1}^{N+M+2} \|y_j(x)\| \\ &\le \frac{1}{\gamma} \left\{ \sum_{i=0}^{N} |y_i| + \sum_{j=N+1}^{N+M+2} \|y_j(x)\|_{L^1[0,\infty)} \right\} = \frac{1}{\gamma} \|\vec{y}\|. \end{aligned} \tag{3.5}$$

Equation (3.5) shows that $(\gamma I - A)^{-1} : \mathbf{X} \to \mathbf{X}$ exists and $\|(\gamma I - A)^{-1}\| < 1/\gamma$ when $\gamma > 0$.

(2) $D(A)$ is dense in $\mathbf{X}$.

If we set $L = \{(p_0, p_1, \dots, p_N, p_{N+1}(x), \dots, p_{N+M+2}(x)) \mid p_j(x) \in C_0^\infty[0,\infty)$, and there exist numbers $c_j$ such that $p_j(x) = 0$, $x \in [0, c_j]$, $j = N+1, \dots, N+M+2\}$. It is obvious that $L$ is dense in $\mathbf{X}$. So it suffices to prove that $D(A)$ is dense in $L$.

Take $\vec{p} \in L$, then there are $c_j > 0$, such that $p_j(x) = 0$, $x \in [0, c_j]$, $j = N+1, \dots, N+M+2$. It follows that $p_j(x) = 0$, $x \in [0, 2s]$, where $0 < 2s < \min\{c_j\}$.

Set

$$\begin{aligned}
f^s(0) &= \left(p_0, p_1, \dots, p_N, f^s_{N+1}(0), \dots, f^s_{N+M+2}(0)\right)\\
&= \left(p_0, p_1, \dots, p_N, a_N p_N, \sum_{i=0}^{N} d_{i,1} p_i, \dots, \sum_{i=0}^{N} d_{i,M} p_i, \sum_{i=0}^{N} z_i p_i\right),\\
f^s(x) &= \left(p_0, p_1, \dots, p_N, f^s_{N+1}(x), \dots, f^s_{N+M+2}(x)\right),\\
f^s_j(x) &= \begin{cases} f^s_j(0)\left(1-\dfrac{x}{s}\right)^2, & x \in [0,s),\\ -\mu_j (x-s)^2(x-2s)^2, & x \in [s,2s),\\ p_j(x), & x \in [2s,\infty), \end{cases} \quad j = N+1, \dots, N+M+2,
\end{aligned} \tag{3.6}$$

where $\mu_j = f^s_j(0)\int_0^s \mu_j(x)(1-x/s)^2 dx / \int_s^{2s} \mu_j(x)(x-s)^2(x-2s)^2 dx$.

Then, it is easy to verify that $f^s(x) \in D(A)$, moreover,

$$\begin{aligned}
\|\vec{p} - f^s(x)\| &= \sum_{j=N+1}^{N+M+2} \int_0^\infty \left| p_j(x) - f^s_j(x) \right| dx = \sum_{j=N+1}^{N+M+2} \int_0^{2s} \left| p_j(x) - f^s_j(x) \right| dx\\
&= \sum_{j=N+1}^{N+M+2} \left( \int_0^s \left| f^s_j(0) \right| \left(1-\frac{x}{s}\right)^2 dx + \int_s^{2s} \left|\mu_j\right| (x-s)^2(x-2s)^2 dx \right)\\
&= \sum_{j=N+1}^{N+M+2} \left( \left| f^s_j(0) \right| \frac{s}{3} + \left|\mu_j\right| \frac{s^5}{30} \right) \longrightarrow 0, \quad \text{when } s \longrightarrow 0.
\end{aligned} \tag{3.7}$$

This shows that $D(A)$ is dense in $L$. In other words, $D(A)$ is dense in $\mathbf{X}$. From (1), (2), and *Hille Yosida* theory [11], we know that $A$ generates a $C_0$-semigroup. It is easy to check that

$$E : \mathbf{X} \longrightarrow \mathbf{X}, \quad \|E\| \le \max\{a_0, a_i + b_i, b_N, W\} \ (i = 1, \dots, N-1) \tag{3.8}$$

is a bounded linear operator (here, $W = \sup_{x\in\mathbb{R}^+} \mu_j(x)$, $j = N+1, \dots, N+M+2$). Thus, by the Perturbation theory of $C_0$-semigroup [11] we deduce that $A+E$ generates a $C_0$-semigroup $T(t)$.

(3) $T(t)$, generated by $A+E$, is a positive $C_0$-semigroup.

By the solution of (3.1)–(3.3), we know that $\vec{p}$ is a nonnegative vector if $\vec{y}$ is a nonnegative vector ($y_i \ge 0$, $i = 0, 1, \dots, N$, and $y_j(x) \ge 0$ $j = N+1, \dots, N+M+2$). In other words, $(\gamma I - A)^{-1}$ is a positive operator [2]. By the expression of $E$, it can be easily verified that $E$ is a positive operator. We note that

$$(\gamma I - A - E)^{-1} = \left[I - (\gamma I - A)^{-1}E\right]^{-1}(\gamma I - A)^{-1}. \tag{3.9}$$

When $\gamma > \max\{a_0, a_i + b_i, b_N, W\}$, by (3.5), it is easy to see that $\|(\gamma I - A)^{-1}E\| < 1$, that is to say $[I - (\gamma I - A)^{-1}E]^{-1}$ exists and is bounded, and

$$\left[I - (\gamma I - A)^{-1}E\right]^{-1} = \sum_{k=0}^{\infty} \left[(\gamma I - A)^{-1}E\right]^k. \tag{3.10}$$

Therefore, $[I-(\gamma I-A)^{-1}E]^{-1}$ is a positive operator. By (3.9) and (3.10), we get that $(\gamma I-A-E)^{-1}$ is a positive operator. By [2], we know that $A+E$ generates a positive $C_0$-semigroup.

(4) $T(t)$ is a positive $C_0$-semigroup of contraction.

For any $\vec{p}\in D(A)$, we take

$$Q_p=\left(\frac{[p_0]^+}{p_0},\frac{[p_1]^+}{p_1},\dots,\frac{[p_N]^+}{p_N},\frac{[p_{N+1}(x)]^+}{p_{N+1}(x)},\dots,\frac{[p_{N+M+2}(x)]^+}{p_{N+M+2}(x)}\right); \tag{3.11}$$

here,

$$\begin{gathered}
[p_i]^+=\begin{cases}p_i, & p_i>0,\\ 0, & p_i\le 0,\end{cases}\quad i=0,1,\dots,N,\\
[p_j(x)]^+=\begin{cases}p_j(x), & p_j(x)>0,\\ 0, & p_j(x)\le 0,\end{cases}\quad j=N+1,\dots,N+M+2.
\end{gathered} \tag{3.12}$$

For any $\vec{p}\in D(A)$ and $Q_p$, we have

$$\begin{aligned}
\langle (A+E)\vec{p},Q_p\rangle &= \left\{-h_0p_0+b_1p_1+\sum_{j=N+1}^{N+M+2}\int_0^\infty p_j(x)\mu_j(x)dx\right\}\frac{[p_0]^+}{p_0}\\
&\quad+\sum_{i=1}^{N-1}\{a_{i-1}p_{i-1}-h_ip_i+b_{i+1}p_{i+1}\}\frac{[p_i]^+}{p_i}+\{a_{N-1}p_{N-1}-h_Np_N\}\frac{[p_N]^+}{p_N}\\
&\quad-\sum_{j=N+1}^{N+M+2}\int_0^\infty\left\{\frac{dp_j(x)}{dx}+\mu_j(x)p_j(x)\right\}\frac{[p_j(x)]^+}{p_j(x)}dx\\
&\le -h_0[p_0]^++b_1[p_1]^++\sum_{j=N+1}^{N+M+2}\int_0^\infty\mu_j(x)[p_j(x)]^+dx\\
&\quad+\sum_{i=1}^{N-1}\left\{a_{i-1}[p_{i-1}]^+-h_i[p_i]^++b_{i+1}[p_{i+1}]^+\right\}+a_{N-1}[p_{N-1}]^+-h_N[p_N]^+\\
&\quad-\sum_{j=N+1}^{N+M+2}\int_0^\infty\frac{dp_j(x)}{dx}\frac{[p_j(x)]^+}{p_j(x)}dx-\sum_{j=N+1}^{N+M+2}\int_0^\infty\mu_j(x)[p_j(x)]^+dx\\
&\le -h_0[p_0]^++b_1[p_1]^++\sum_{i=1}^{N-1}\left\{a_{i-1}[p_{i-1}]^+-h_i[p_i]^++b_{i+1}[p_{i+1}]^+\right\}\\
&\quad+a_{N-1}[p_{N-1}]^+-h_N[p_N]^+-a_N[p_N]^+-\sum_{i=0}^{N}\sum_{n=1}^{M}d_{i,n}[p_i]^+-\sum_{i=0}^{N}z_i[p_i]^+=0.
\end{aligned} \tag{3.13}$$

From the definition of dispersive operator and (3.13), we know that $A+E$ is a dispersive operator. Combining (1), (2), (3) with the *Philips* theory [11], we derive that $A+E$

generates a positive $C_0$-semigroup of contraction. Because $C_0$-semigroup is unique [2], this positive $C_0$-semigroup of contraction is just $T(t)$. Thus, we complete the proof of Theorem 3.1. □

THEOREM 3.2. *The repairable system (2.1)–(2.5) has a unique, nonnegative, and time-dependant solution* $\vec{p}(x,t)$, *which satisfies* $\|\vec{p}(\cdot,t)\| = 1$, $t \in [0,\infty)$.

*Proof.* From Theorem 3.1 and [11], we know that the system (2.1)–(2.5) has a unique nonnegative solution $\vec{p}(x,t)$ and it can be expressed as

$$\vec{p}(x,t) = T(t)(1,0,\dots,0). \tag{3.14}$$

By Theorem 3.1 and (3.14), we obtain that

$$||\vec{p}(\cdot,t)|| = ||T(t)(1,0,\dots,0)|| \le ||(1,0,\dots,0)|| = 1, \quad t \in [0,\infty). \tag{3.15}$$

On the other hand, since $(1,0,\dots,0) \in D(A+E)$, so $\vec{p}(x,t) \in D(A+E)$, and $p_j(x,t)$, $j = N+1,\dots,N+M+2$ satisfy system (2.1)–(2.5). Then, we have

$$\frac{d}{dt}||\vec{p}(\cdot,t)|| = \sum_{i=0}^{N}\frac{dp_i(t)}{dt} + \sum_{j=N+1}^{N+M+2}\frac{d}{dt}\int_0^\infty p_j(x,t)dx = 0. \tag{3.16}$$

Hence, $\|\vec{p}(\cdot,t)\| = \|\vec{p}(0)\| = 1$. This just reflects the physical meaning of $\vec{p}(x,t)$. □

## 4. Asymptotic stability of system (2.1)–(2.5)

In this section, we will study the asymptotic stability of the repairable system. We will prove that there exists a nonnegative steady solution of the system, and the dynamic solution converges to the steady solution when time $t$ tends to infinity. Therefore, the system is asymptotic stability.

LEMMA 4.1. $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx = \int_0^\infty xg_j(x)dx$, *for* $j = N+1,\dots,N+M+2$.

*Proof.* From [3, pages 11 and 8], we know that

$$\begin{aligned}\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx &= \int_0^\infty [1-G_j(x)]dx, \quad (G(0)=0),\\ \int_0^\infty xg_j(x)dx &= \int_0^\infty [1-G_j(x)]dx, \quad (G(\infty)=1).\end{aligned} \tag{4.1}$$

So, $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx = \int_0^\infty xg_j(x)dx$, we complete the proof of Lemma 4.1. □

LEMMA 4.2. *There exists* $K \in \mathbb{R}$, *such that* $\int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx \le K$ *for any* $t \ge 0$.

*Proof.* Let $G_j^t(x) = \Pr\{X_j - t \le x \mid X_j > t\} = (G_j(x+t) - G_j(t))/(1 - G_j(t))$, $x \ge 0$.
So, $1 - G_j^t(x) = (1 - G_j(x+t))/(1 - G_j(t))$, then

$$\begin{aligned} E_j(t) = E\{X_j - t \mid X_j > t\} &= \int_0^\infty x\,dG_j^t(x) = \int_0^\infty [1 - G_j^t(x)]dx \\ &= \int_0^\infty \frac{1 - G_j(x+t)}{1 - G_j(t)}dx = \int_t^\infty \frac{1 - G_j(x)}{1 - G_j(t)}dx \\ &= \int_t^\infty e^{-\int_0^x \mu_j(\xi)d\xi} \cdot e^{\int_0^t \mu_j(\xi)d\xi}dx \\ &= \int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx. \end{aligned} \tag{4.2}$$

Because the failed unit is repairable, and the expectation of repair time of any failed unit is less than $\infty$, then, there exist $K_j \in \mathbb{R}$, such that $E_j(t) \le K_j$. Let $K = \max\{K_j,\ j = 1,2,\dots,N+M+2\}$, then for any $t \ge 0$, $E_j(t) \le K$, that is, $\int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}dx \le K$.
As a special case, let $t = 0$; we have $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}dx \le K$. □

Theorem 4.3. *0 is the simple eigenvalue of $A + E$.*

*Proof.* Consider $(A + E)\vec{p} = 0$ as the following equations:

$$-h_0 p_0 + b_1 p_1 + \sum_{j=N+1}^{N+M+2} \int_0^\infty \mu_j(x) p_j(x) dx = 0, \tag{4.3}$$

$$a_{i-1} p_{i-1} - h_i p_i + b_{i+1} p_{i+1} = 0 \quad (i = 1,\dots,N-1), \tag{4.4}$$

$$a_{N-1} p_{N-1} - h_N p_N = 0, \tag{4.5}$$

$$-\frac{dp_j(x)}{dx} - \mu_j(x) p_j(x) = 0 \quad (j = N+1,\dots,N+M+2), \tag{4.6}$$

$$p_{N+1}(0) = a_N p_N, \qquad p_{N+M+2}(0) = \sum_{i=0}^{N} z_i p_i, \qquad P_{N+1+n}(0) = \sum_{i=0}^{N} d_{i,n} p_i \quad (n = 1,\dots,M). \tag{4.7}$$

Solving (4.6) with the help of (4.7), we obtain that

$$p_j(x) = p_j(0) e^{-\int_0^x \mu_j(\xi)d\xi} \quad (j = N+1,\dots,N+M+2). \tag{4.8}$$

Substitution of (4.8) into (4.3) with the help of (4.4)–(4.7) yields that

$$\begin{gathered} \left(-h_0 + z_0 + \sum_{n=1}^{M} d_{0,n}\right) p_0 + \left(b_1 + z_0 + \sum_{n=1}^{M} d_{1,n}\right) p_1 \\ + \sum_{i=2}^{N-1} \left\{z_i + \sum_{n=1}^{M} d_{i,n}\right\} p_i + \left(a_N + z_N + \sum_{n=1}^{M} d_{N,n}\right) p_N = 0, \\ a_{i-1} p_{i-1} - h_i p_i + b_{i+1} p_{i+1} = 0 \quad (i = 1,\dots,N-1), \\ a_{N-1} p_{N-1} - h_N p_N = 0. \end{gathered} \tag{4.9}$$

It is easy to check that the determinant coefficient matrix of the above equations equals 0. Moreover, if $p_0 > 0$, then $p_i > 0$, $(i = 1,\dots,N)$, and

$$p_j(x) = p_j(0)e^{-\int_0^x \mu_j(\xi)d\xi} > 0 \quad (j = N+1,\dots,N+M+2). \tag{4.10}$$

Using Lemma 4.2, we can deduce that $p_j(x) \in L^1[0,+\infty)$. So, the vector

$$\vec{p} = \left(p_0, p_1,\dots, p_N, p_{N+1}(x),\dots, p_{N+M+2}(x)\right) \tag{4.11}$$

is the eigenvector corresponding to 0 of $A+E$. Taking $Q = (1,1,\dots,1)$, we have

$$\langle \vec{p}, Q\rangle = \sum_{i=0}^{N} p_i + \sum_{j=N+1}^{N+M+2} \int_0^\infty p_j(x)dx > 0. \tag{4.12}$$

For any $\vec{q} = (q_0, q_1,\dots, q_N, q_{N+1}(x),\dots, q_{N+M+2}(x)) \in D(A+E)$,

$$(A+E)\vec{q} = \begin{pmatrix} -h_0q_0 + b_1q_1 + \displaystyle\sum_{j=N+1}^{N+M+2}\int_0^\infty \mu_j(x)q_j(x)dx \\ a_0q_0 - h_1q_1 + b_2q_2 \\ \vdots \\ a_{N-2}q_{N-2} - h_{N-1}q_{N-1} + b_Nq_N \\ a_{N-1}q_{N-1} - h_Nq_N \\ -\dfrac{dp_{N+1}(x)}{dx} - \mu_{N+1}(x)p_{N+1}(x) \\ \vdots \\ -\dfrac{dp_{N+M+2}(x)}{dx} - \mu_{N+M+2}(x)p_{N+M+2}(x) \end{pmatrix}. \tag{4.13}$$

By (2.5), it is easy to deduce that $\langle (A+E)\vec{q}, Q\rangle = 0$. So, 0 is the simple eigenvalue of $A+E$. □

Theorem 4.4. $\{r \in \mathbb{C} \mid \mathrm{Re}\, r > 0$, *or* $r = ia$, $a \in \mathbb{R}$, $a \neq 0\}$ *belong to the resolvent set of* $A+E$.

*Proof.* For any $r \in \mathbb{C}$, $\mathrm{Re}\, r > 0$, or $r = ia$, $a \in \mathbb{R}$, $a \neq 0$, and for any $\vec{y} \in \mathbf{X}$, solve $(rI - (A+E))\vec{p} = \vec{y}$:

$$(r+h_0)p_0 - b_1p_1 - \sum_{j=N+1}^{N+M+2}\int_0^\infty \mu_j(x)p_j(x)dx = y_0, \tag{4.14}$$

$$-a_{i-1}p_{i-1} + (r+h_i)p_i - b_{i+1}p_{i+1} = y_i \quad (i = 1,\dots,N-1), \tag{4.15}$$

$$-a_{N-1}p_{N-1} + (r+h_N)p_N = y_N, \tag{4.16}$$

$$\frac{dp_j(x)}{dx} + (r+\mu_j(x))p_j(x) = y_j(x) \quad (j = N+1,\dots,N+M+2), \tag{4.17}$$

$$p_{N+1}(0) = a_Np_N, \qquad p_{N+M+2}(0) = \sum_{i=0}^{N} z_ip_i, \qquad P_{N+1+n}(0) = \sum_{i=0}^{N} d_{i,n}p_i \quad (n = 1,\dots,M). \tag{4.18}$$

Solving (4.14)–(4.17), with the help of (4.18), we can obtain that

$$p_j(x) = p_j(0)e^{-\int_0^x (r+\mu_j(\xi))d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau. \tag{4.19}$$

For $y_j(x) \in L^1[0,\infty)$, combining Lemma 4.2, we can derive that

$$\begin{aligned}\int_0^\infty \left| \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau \right| dx &\le \int_0^\infty dx \int_0^x e^{-\int_\tau^x \mu_j(\xi)d\xi} |y_j(\tau)| d\tau \\ &= \int_0^\infty |y_j(\tau)| d(\tau) \int_\tau^\infty e^{-\int_\tau^x \mu_j(\xi)d\xi} dx \\ &\le ||y_j||_{L^1[0,\infty)} \cdot K.\end{aligned} \tag{4.20}$$

So, $p_j(x) \in L^1[0,\infty)$, $j = N+1,\ldots,N+M+2$. Substituting them into (4.14) with the help of (4.15)–(4.16) yields that

$$\begin{aligned}&\left(r + h_0 - z_0 - \sum_{n=1}^{M} d_{0,n} g_{N+1+n}\right) p_0 - \left(b_1 + z_1 + \sum_{n=1}^{M} d_{1,n} g_{N+1+n}\right) p_1 \\ &\quad - \sum_{i=2}^{N-1} \left\{ z_i + \sum_{n=1}^{M} d_{i,n} g_{N+1+n} \right\} p_i - \left( a_N g_{N+1} + z_N + \sum_{n=1}^{M} d_{N,n} g_{N+1+n} \right) p_N = y_0 + \sum_{j=N+1}^{N+M+2} G_j,\end{aligned} \tag{4.21}$$

$$-a_{i-1}p_{i-1} + (r+h_i)p_i - b_{i+1}p_{i+1} = y_i \quad (i = 1,\ldots,N-1), \tag{4.22}$$

$$-a_{N-1}p_{N-1} + (r+h_N)p_N = y_N, \tag{4.23}$$

where

$$\begin{aligned}g_j &= \int_0^\infty \mu_j(x) e^{-\int_0^x (r+\mu_j(\xi))d\xi} dx, \\ G_j &= \int_0^\infty \mu_j(x)dx \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau.\end{aligned} \tag{4.24}$$

When $\operatorname{Re} r > 0$, or $r = ia$, $a \in \mathbb{R}$, $a \neq 0$, we have $|g_j| \le 1$, then it follows that the coefficient matrix of (4.21)–(4.23) is a strictly diagonally dominant matrix. So, (4.21)–(4.23) has unique solution. Assuming that $\{\widehat{p}_0, \widehat{p}_1, \ldots, \widehat{p}_N\}$ is the unique solution of (4.21)–(4.23), then $\{\widehat{p}_0, \widehat{p}_1, \ldots, \widehat{p}_N\}$ and

$$\widehat{p}_j(x) = \widehat{p}_j(0)e^{-\int_0^x (r+\mu_j(\xi))d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))d\xi} y_j(\tau)d\tau \quad j = N+1,\ldots,N+M+2 \tag{4.25}$$

is the unique solution of (4.14)–(4.18). So, $R(rI - A - E) = \mathbf{X}$, and because $(rI - A - E)$ is a closed operator, we can deduce that $(rI - A - E)^{-1}$ exists and is bounded. In other words, $\{r \in \mathbb{C} \mid \operatorname{Re} r > 0, \text{ or } r = ia,\ a \in \mathbb{R},\ a \neq 0\}$ belongs to the resolvent set of $A + E$. This completes the proof of Theorem 4.4. □

Corollary 4.5. *The system (2.1)–(2.5) has a nonnegative stable solution.*

In Theorem 4.4, we proved that all spectra of $A + E$ lie in the left half-plane and there is no spectra on the imaginary axis except 0. Noticing that $\vec{p}$ in (4.11) is the eigenvector corresponding to 0 of $A + E$. It is obvious that $\vec{p}$ is nonnegative. Hence, $\vec{p}$ is the nonnegative stable solution of the system.

Theorem 4.6. *Let $\hat{p}$ be the nonnegative eigenvector corresponding to 0 and satisfy $\|\hat{p}\| = 1$, let $Q = (1, 1, \ldots, 1)$, then the time-dependant solution $\hat{p}(\cdot, t)$ of the system tends to the stable solution $\hat{p}$, that is, $\lim_{t\to\infty} \hat{p}(\cdot, t) = \langle \vec{p}_0, Q\rangle \hat{p} = \hat{p}$. Here, $\vec{p}_0$ is the initial value of the system.*

From [12], and [10, Theorem 14], we know that Theorem 4.6 is the direct result of the stability of the semigroup. Thus, we proved that $\hat{p}$, the eigenvector corresponding to 0 of $A + E$, is the unique and nonnegative stable solution of the repairable system, and $\lim_{t\to\infty} \hat{p}(\cdot, t) = \hat{p}$.

## 5. Conclusion

The problem of asymptotic stability of a general redundant repairable system with imperfect switching mechanism is studied in the paper from a theoretical standpoint. By $C_0$-semigroup theory, we firstly prove the existence of unique solution of a system; secondly, we prove the solution is asymptotic stability. Thus, we provide strictly mathematical proof for such general system. This is the main contribution of the paper.

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