NICELY SEMIRAMIFIED DIVISION ALGEBRAS OVER HENSELIAN FIELDS

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Received 5 July 2004

This paper deals with the structure of nicely semiramified valued division algebras. We prove that any defectless finite-dimensional central division algebra over a Henselian field E with an inertial maximal subfield and a totally ramified maximal subfield (not necessarily of radical type) (resp., split by inertial and totally ramified field extensions of E) is nicely semiramified.

1. Introduction

We recall that a nicely semiramified division algebra is defined to be a defectless finite-dimensional valued central division algebra D over a field E with inertial and totally ramified radical-type (TRRT) maximal subfields [7, Definition, page 149]. Equivalent statements to this definition were given in [7, Theorem 4.4] when the field E is Henselian. These division algebras, as claimed in [7, page 128], appeared in [10] as examples of division algebras with nonzero SK_1. The main purpose of this paper is to prove that over a Henselian field E, any central division algebra with inertial and totally ramified maximal subfields (resp., split by inertial and totally ramified field extensions of E) is nicely semiramified.

We precise that all rings considered in this work are assumed to be associative with a unit and all free modules are assumed to be finite-dimensional. A valued division algebra D over a field E—we adopt the same valuative definitions as in [7]—is called defectless (over E) if $[D:E]=[\bar{D}:\bar{E}](\Gamma_D:\Gamma_E)$, where $[\bar{D}:\bar{E}]$ (resp., $(\Gamma_D:\Gamma_E)$) is the residue degree (resp., ramification index) of D over E.

We recall that for any valued central division algebra D over a field E, the center $Z(\bar{D})$ of $\bar{D}$ is a normal field extension of $\bar{E}$ and the mapping

$$\begin{aligned} \theta_D:\Gamma_D/\Gamma_E &\longrightarrow \operatorname{Gal}(Z(\bar{D})/\bar{E}),\\ \gamma+\Gamma_E &\longmapsto \theta_D(\gamma+\Gamma_E):\bar{a}\longmapsto \overline{dad^{-1}}, \end{aligned} \tag{1.1}$$

where d is an arbitrary element of D such that $v(d)=\gamma$ (v being the valuation of D), is a surjective group homomorphism [7, Proposition 1.7]. We say that D is tame (over E) if

International Journal of Mathematics and Mathematical Sciences 2005:4 (2005) 571–577
DOI: 10.1155/IJMMS.2005.571

it is defectless over E, $Z(\bar{D})$ is separable (hence Galois) over $\bar{E}$, and the characteristic of $\bar{E}$ does not divide the cardinal of the kernel of θ_D. A subfield $K(\supseteq E)$ of D is called inertial over E if $[K:E]=[\bar{K}:\bar{E}]$ and $\bar{K}$ is separable over $\bar{E}$.

Now, let F be an associative ring with a unit, and Γ a totally ordered abelian group. We say that F is a graded ring of type Γ if there are subgroups F_γ ($\gamma\in\Gamma$) of F such that $F=\bigoplus_{\gamma\in\Gamma}F_\gamma$ and $F_\gamma F_\delta\subseteq F_{\gamma+\delta}$, for all $\gamma,\delta\in\Gamma$. In this case, the set $\Gamma_F=\{\gamma\in\Gamma\mid F_\gamma\neq 0\}$ is called the support of F. If $x\in F_\gamma$ for some $\gamma\in\Gamma_F$, we say that x is a homogeneous element of F. In particular, if x is a nonzero element of F_γ, we say that x has grade γ and we write $\mathrm{gr}(x)=\gamma$. We denote by F^h (resp., F^*) the set of homogeneous (resp., nonzero homogeneous) elements of F. A graded ring F which is commutative and for which all nonzero homogeneous elements are invertible is called a graded field.

Let F be a commutative graded ring of type Γ. A (left) F-module (resp., F-algebra) A is called a graded F-module (resp., a graded F-algebra) (of type Γ) if $A=\bigoplus_{\delta\in\Gamma}A_\delta$ and $F_\gamma A_\delta\subseteq A_{\gamma+\delta}$, for all $\gamma,\delta\in\Gamma$ (resp., if A is a graded ring of type Γ and $F_\gamma\subseteq A_\gamma$, for all $\gamma\in\Gamma$). In particular, if F is a graded field, then graded F-algebras (resp., commutative graded F-algebras) for which homogeneous elements are invertible are called graded division F-algebras (resp., graded field extensions of F).

A graded F-module (resp., a graded F-algebra) is also called a graded module (resp., a graded algebra) over F.

Let F be a graded field of type Γ. Since Γ is totally ordered, then any graded division algebra A over F is a domain. We may then consider the algebra of central quotients of A that we denote by $\mathrm{Cq}(A)$. Remark that if A is a graded field extension of F, then $\mathrm{Cq}(A)$ coincides with the fraction field $\mathrm{Frac}(A)$ of A.

One can easily see that if F is a graded field, then any graded F-module M is free over F. Indeed, by [1, Theorem 3, page 29] any maximal E-linearly independent subset of homogeneous elements of M is a basis of M over E.

A graded field extension K of a graded field F is called totally ramified over F if $[K:F]=(\Gamma_K:\Gamma_F)$. It is called unramified over F if $[K:F]=[K_0:F_0]$ and K_0 is separable over F_0. Finally, K is called tame over F if K_0 is separable over F_0 and Γ_K/Γ_F has no p-torsion, where $p=\mathrm{char}(F)$.

If F is the center of a graded division algebra D, then D is called a graded central division algebra (GCDA) over F. We recall that in the same way as for (ungraded) fields, we can define the graded Brauer group $\mathrm{GBr}(F)$ of F, where GCDAs play the same role as central division algebras (CDAs) over (ungraded) fields (see [2, Section 5] or [6, Section 3]).

Let F be a graded field of type Γ and D a GCDA over F. Then a natural valuation v can be defined on $\mathrm{Cq}(D)$ by setting $v(d)=\mathrm{gr}(d)$ for any $d\in D^*$ (see [3, Section 4] or [6, Section 4]). Throughout the rest of the paper, such v will be called the canonical valuation of $\mathrm{Cq}(D)$. We recall that $\mathrm{Cq}(D)$, with respect to v, is a tame CDA over $\mathrm{Frac}(F)$ [3, Proposition 4.1(3)].

Conversely, for any defectless valued division algebra D with valuation v over a field E, we define $E^\gamma=\{x\in E\mid v(x)\geq\gamma\}$, $E^{>\gamma}=\{x\in E\mid v(x)>\gamma\}$, $D^\gamma=\{x\in D\mid v(x)\geq\gamma\}$, and $D^{>\gamma}=\{x\in D\mid v(x)>\gamma\}$. Obviously, $E^{>\gamma}$ (resp., $D^{>\gamma}$) is a subgroup of the additive group E^γ (resp., D^γ). So, we can define the quotient groups $GE_\gamma=E^\gamma/E^{>\gamma}$ and $GD_\gamma=D^\gamma/D^{>\gamma}$. For $x\in E\setminus\{0\}$ (resp., $x\in D\setminus\{0\}$), we denote by $\widetilde{x}$ the element $x+E^{>v(x)}$ of $GE_{v(x)}$

(resp., $x + D^{>v(x)}$ of $GD_{v(x)}$). One can easily check that the additive group $GE = \bigoplus_{\gamma\in\Gamma} GE_\gamma$ endowed with the multiplication law defined by $\widetilde{x}\widetilde{y} = \widetilde{xy}$ is a graded field. Analogously, $GD = \bigoplus_{\gamma\in\Gamma} GD_\gamma$ is a graded division algebra over GE [2, page 4281]. We recall that D is tame over E if and only if GD is a GCDA over GE [2, Corollary 4.4(3)] (remark that since D is defectless over E, then $[D:E] = [GD:GE]$).

Now, let F be a graded field and let D be a GCDA over F, and denote by $H\,\mathrm{Frac}(F)$ the Henselization of $\mathrm{Frac}(F)$ with respect to the canonical valuation of $\mathrm{Frac}(F)$ [4, Section 16], and let $H\,\mathrm{Cq}(D) = \mathrm{Cq}(D) \otimes_{\mathrm{Frac}(F)} H\,\mathrm{Frac}(F)$. Then, D is graded isomorphic to $GH\,\mathrm{Cq}(D)$ by means of the mapping $x \mapsto \widetilde{x}$, where x is an arbitrary homogeneous element of D. Indeed, we have $D_0 = (GH\,\mathrm{Cq}(D))_0$ and $\Gamma_D = \Gamma_{GH\,\mathrm{Cq}(D)}$. We write $D \cong_g GH\,\mathrm{Cq}(D)$. We also have $F \cong_g GH\,\mathrm{Frac}(F)$.

We recall that if E is Henselian, then the tame part $\mathrm{TBr}(E)$ of the Brauer group $\mathrm{Br}(E)$ of E (i.e., $\mathrm{TBr}(E) = \{[D] \in \mathrm{Br}(E) \mid D$ is a tame CDA over $E\}$) is isomorphic to $\mathrm{GBr}(GE)$ [6, Theorem 5.3]. Also, for any graded field F, $\mathrm{GBr}(F)$ is isomorphic to $\mathrm{TBr}(H\,\mathrm{Frac}(F))$ [6, Theorem 5.1]. These isomorphisms are index-preserving. We call them the canonical isomorphisms.

2. Nicely semiramified division algebras over Henselian fields

Let F be a graded field and K a finite-dimensional graded field extension of F. For an arbitrary abelian group A—namely for $A = \Gamma_K/\Gamma_F$—and a family $a_1, a_2, \dots, a_r$ of elements of A, we say that $a_1, a_2, \dots, a_r$ are independent if the subgroup $\langle a_1, a_2, \dots, a_r\rangle$ of A, generated by $a_1, a_2, \dots, a_r$, equals $\bigoplus_{i=1}^r \langle a_i\rangle$. We recall that K is called totally ramified of radical type (TRRT) over F if there are homogeneous elements $t_1, \dots, t_r \in F^*$ and nonnegative integers $n_1, \dots, n_r$ such that the following conditions are satisfied:

(1) $K = F[t_1^{1/n_1}, \dots, t_r^{1/n_r}]$,

(2) $\mathrm{gr}(t_i^{1/n_i}) + \Gamma_F$ $(1 \le i \le r)$ are independent elements of Γ_K/Γ_F, with order n_i, respectively.

One can see that in the same way as for TRRT valued field extensions (see [7, Lemma 4.1]), a totally ramified finite-dimensional graded field extension K of F is TRRT over F if and only if there is a subgroup G of K^*/F^* such that the mapping $G \to \Gamma_K/\Gamma_F$, defined by $xF^* \mapsto \mathrm{gr}(x) + \Gamma_F$, is a group isomorphism.

Lemma 2.1. *Let F be a graded field and K a totally ramified finite-dimensional graded field extension of F. Then, K is TRRT over F.*

Proof. Let $(x_i)_{i=1}^r$ be a family of nonzero homogeneous elements of K such that $(\mathrm{gr}(x_i) + \Gamma_F)_{i=1}^r$ is a basis of Γ_K/Γ_F (i.e., $\Gamma_K/\Gamma_F = \bigoplus_{i=1}^r \langle \mathrm{gr}(x_i) + \Gamma_F\rangle$). Let n_i be the order of $\mathrm{gr}(x_i) + \Gamma_F$ $(1 \le i \le r)$ and set $I = \{\bar{m} = (m_1, \dots, m_r) \in \mathbb{N}^r \mid 0 \le m_i < n_i$ for all $1 \le i \le r\}$. Let G be the subgroup of K^*/F^*, generated by the elements x_iF^* $(1 \le i \le r)$. For $\bar{m} = (m_1, \dots, m_r) \in I$, let $x^{\bar{m}} = x_1^{m_1} \cdots x_r^{m_r}$. Since K/F is totally ramified, then for any $1 \le i \le r$, $x_i^{n_i} \in F^*$ (indeed, we have $K_0 = F_0$ and $\mathrm{gr}(x_i^{n_i}) \in \Gamma_F$). Clearly, the elements $x^{\bar{m}}$ $(\bar{m} \in I)$ are pairwise distinct. Hence, $G = \{x^{\bar{m}}F^* \mid \bar{m} \in I\}$ and the map $\phi : G \to \Gamma_K/\Gamma_F$, $x^{\bar{m}}F^* \mapsto \mathrm{gr}(x^{\bar{m}}) + \Gamma_F$, is a group isomorphism. □

Let F be a graded field and D a GCDA over F. We recall that in the same way as for valued central division algebras, D is called nicely semiramified (NSR) (over F) if it has unramified and totally ramified radical-type maximal graded subfields. We recall also that D is NSR (over F) if and only if $\mathrm{Cq}(D)$ is NSR (over $\mathrm{Frac}(F)$) (see [3, Proposition 6.4]). Moreover, $\mathrm{Cq}(D)$ is NSR (over $\mathrm{Frac}(F)$) if and only if $H\,\mathrm{Cq}(D)$ is NSR (over $H\,\mathrm{Frac}(F)$). Indeed, assume that $\mathrm{Cq}(D)$ is NSR and let K (resp., L) be an inertial (resp., a TRRT) maximal subfield of $\mathrm{Cq}(D)$. Then, by [8, Theorem 1], HK $(= K \otimes_{\mathrm{Frac}(F)} H\,\mathrm{Frac}(F))$ (resp., HL $(= L \otimes_{\mathrm{Frac}(F)} H\,\mathrm{Frac}(F))$) is an inertial (resp., a TRRT) maximal subfield of $H\,\mathrm{Cq}(D)$. Conversely, if $H\,\mathrm{Cq}(D)$ is NSR, then it has an inertial (resp., a TRRT) maximal subfield K' (resp., L'). So GK' (resp., GL') is an unramified (resp., a totally ramified) maximal graded subfield of D $(\cong_g GH\,\mathrm{Cq}(D))$. Hence D is NSR. Therefore, by the above, $\mathrm{Cq}(D)$ is NSR.

The following lemma is analogous to [7, Theorem 4.4]. It gives equivalent statements for a GCDA over a graded field to be NSR. In condition (3)(i) of this lemma, the graded field extensions $L^{(i)}$ are said to be linearly disjoint if the graded ring $L^{(1)} \otimes_F \cdots \otimes_F L^{(k)}$ is a graded field.

Lemma 2.2. *Let F be a graded field and D a GCDA over F. Then the following statements are equivalent:*

(1) *D is NSR;*
(2) *D is split by unramified and totally ramified graded field extensions of F;*
(3) *$D \cong_g (L^{(1)}/F, \sigma_1, t_1) \otimes_F \cdots \otimes_F (L^{(k)}/F, \sigma_k, t_k)$, where $L^{(i)}$, σ_i, and t_i satisfy the following conditions:*
 (i) *$L^{(i)}$ are linearly disjoint cyclic unramified graded field extensions of F with dimension $[L^{(i)} : F] = n_i$ and with Galois group generated by σ_i $(1 \le i \le k)$,*
 (ii) *t_i are nonzero homogeneous elements of F such that $\mathrm{gr}(t_i^{n/n_i}) + n\Gamma_F$ are independent elements of $\Gamma_F/n\Gamma_F$, with order n_i $(1 \le i \le k)$, respectively.*

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Let K (resp., K') be an unramified (resp., a totally ramified—hence TRRT) graded field extension of F splitting D. It follows from the commutative diagrams

$$\begin{array}{ccc} \mathrm{GBr}(F) & \xrightarrow{\ \mathrm{Ext}\ } & \mathrm{GBr}(K) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{TBr}\,(H\,\mathrm{Frac}(F)) & \xrightarrow{\ \mathrm{Ext}\ } & \mathrm{TBr}\,(H\,\mathrm{Frac}(K)) \end{array}$$

$$\begin{array}{ccc} \mathrm{GBr}(F) & \xrightarrow{\ \mathrm{Ext}\ } & \mathrm{GBr}(K') \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathrm{TBr}\,(H\,\mathrm{Frac}(F)) & \xrightarrow{\ \mathrm{Ext}\ } & \mathrm{TBr}\,(H\,\mathrm{Frac}(K')) \end{array} \tag{2.1}$$

where the horizontal maps are scalar extension homomorphisms and the vertical ones are canonical isomorphisms (see [6, Remark 5.10]), that $H\,\mathrm{Frac}(K)$ (resp., $H\,\mathrm{Frac}(K')$ $(= K' \otimes_{\mathrm{Frac}(F)} H\,\mathrm{Frac}(F))$) is an inertial (resp., a TRRT) field extension of $H\,\mathrm{Frac}(F)$ splitting $H\,\mathrm{Cq}(D)$. So, by [7, Theorem 4.4] $H\,\mathrm{Cq}(D)$ is NSR and we can write $H\,\mathrm{Cq}(D) = (M^{(1)}/H\,\mathrm{Frac}(F), \tau_1, s_1) \otimes_{H\,\mathrm{Frac}(F)} \cdots \otimes_{H\,\mathrm{Frac}(F)} (M^{(k)}/H\,\mathrm{Frac}(F), \tau_k, s_k)$, where $M^{(i)}$, τ_i, and

s_i satisfy the following conditions:

(i) $M^{(i)}$ are linearly disjoint cyclic inertial field extensions of $H\operatorname{Frac}(F)$ with dimension n_i and with Galois group generated by τ_i, respectively $(1 \le i \le k)$,

(ii) s_i $(1 \le i \le k)$ are elements of $H\operatorname{Frac}(F)^*$ such that, if we denote by v the canonical valuation of $H\operatorname{Frac}(F)$, then $(n/n_i)v(s_i) + n\Gamma_F$ are independent elements of $\Gamma_F/n\Gamma_F$, with order n_i, respectively.

Therefore,

$$\begin{aligned} D &\cong_g GH\operatorname{Cq}(D) \\ &\cong_g (GM^{(1)}/GH\operatorname{Frac}(F), \tau_1, \widetilde{s}_1) \otimes_{GH\operatorname{Frac}(F)} \cdots \otimes_{GH\operatorname{Frac}(F)} (GM^{(k)}/GH\operatorname{Frac}(F), \tau_k, \widetilde{s}_k). \end{aligned} \tag{2.2}$$

Indeed, let $D_i = (M^{(i)}/H\operatorname{Frac}(F), \tau_i, s_i) = \bigoplus_{j=1}^{n_i} M^{(i)} x_i^j$, where $x_i^{n_i} = s_i$ and $x_i a = \tau_i(a)x_i$ for all $a \in M^{(i)}$ $(1 \le i \le k)$. Since $M^{(i)}/H\operatorname{Frac}(F)$ is cyclic inertial, then by [5, Remark 3.1] the unramified graded field extension $GM^{(i)}/GH\operatorname{Frac}(F)$ is cyclic and, up to a group isomorphism, $\operatorname{Gal}(GM^{(i)}/GH\operatorname{Frac}(F)) = \langle \tau_i \rangle$. Moreover, it is clear that for any $d \in M^{(i)}$, we have $\widetilde{x}_i \widetilde{d} \widetilde{x}_i^{-1} = \widetilde{\tau_i(d)} = \tau_i(\widetilde{d})$. Thus, $GD_i = (GM^{(i)}/GH\operatorname{Frac}(F), \tau_i, \widetilde{s}_i)$. It follows from the canonical isomorphism $\operatorname{TBr}(H\operatorname{Frac}(F)) \cong \operatorname{GBr}(GH\operatorname{Frac}(F))$ that

$$\begin{aligned} D\ (\cong_g GH\operatorname{Cq}(D)) &\cong_g GD_1 \otimes_{GH\operatorname{Frac}(F)} \cdots \otimes_{GH\operatorname{Frac}(F)} GD_k \\ &= (GM^{(1)}/GH\operatorname{Frac}(F), \tau_1, \widetilde{s}_1) \otimes_{GH\operatorname{Frac}(F)} \cdots \otimes_{GH\operatorname{Frac}(F)} (GM^{(k)}/GH\operatorname{Frac}(F), \tau_k, \widetilde{s}_k). \end{aligned} \tag{2.3}$$

The conditions of (3) in this lemma are then satisfied.

(3)⇒(1). By [5, Theorem 3.11(b)] $\operatorname{Frac}(L^{(i)})/\operatorname{Frac}(F)$ are cyclic with $\operatorname{Gal}(\operatorname{Frac}(L^{(i)})/\operatorname{Frac}(F)) = \langle \sigma_i \rangle$. Thus, applying [8, Theorem 1], one can easily see that $H\operatorname{Frac}(L^{(i)})/H\operatorname{Frac}(F)$ are cyclic with $\operatorname{Gal}(H\operatorname{Frac}(L^{(i)})/H\operatorname{Frac}(F)) \cong \operatorname{Gal}(\operatorname{Frac}(L^{(i)})/\operatorname{Frac}(F))$. Moreover, since $\overline{H\operatorname{Frac}(L^{(i)})} = L_0^{(i)}$, $\Gamma_{H\operatorname{Frac}(L^{(i)})} = \Gamma_{L^{(i)}}$, $\overline{H\operatorname{Frac}(F)} = F_0$, and $\Gamma_{H\operatorname{Frac}(F)} = \Gamma_F$, it follows that $H\operatorname{Cq}(D)$ $(\cong D \otimes_F H\operatorname{Frac}(F) \cong (H\operatorname{Frac}(L^{(1)})/H\operatorname{Frac}(F), \sigma_1, t_1) \otimes_{H\operatorname{Frac}(F)} \cdots \otimes_{H\operatorname{Frac}(F)} (H\operatorname{Frac}(L^{(k)})/H\operatorname{Frac}(F), \sigma_k, t_k))$ is NSR (by [7, Theorem 4.4]). So, by the arguments preceding this lemma, D is NSR. □

Lemma 2.3. *Let E be a Henselian valued field and D a defectless CDA over E. Then the following statements are equivalent:*

(1) *D is* NSR*;*

(2) *GD is* NSR.

Proof. (1)⇒(2). Since D is nicely semiramified—hence tame—over E, then GD is a graded central division algebra over GE (see [2, Corollary 4.4(3)]—remark that since D is defectless over E, then $[D:E] = [GD:GE]$). Let L (resp., M) be an inertial (resp., a TRRT) maximal subfield of D. Then, GL (resp., GM) is an unramified (resp., a totally ramified) maximal graded subfield of GD. Hence, GD is nicely semiramified.

(2)⇒(1). Since GD is nicely semiramified over GE, then by Lemma 2.2, $GD \cong_g (L^{(1)}/GE, \sigma_1, t_1) \otimes_{GE} \cdots \otimes_{GE} (L^{(k)}/GE, \sigma_k, t_k)$, where $L^{(i)}/GE$, σ_i, and t_i satisfy the conditions of

Lemma 2.2(3). Since $L^{(i)}/GE$ are unramified, then by [5, Theorem 5.2 (see the proof)], there are inertial cyclic field extensions $K^{(i)}$ of E such that—up to a graded isomorphism—$L^{(i)} = GK^{(i)}$ with $\mathrm{Gal}(L^{(i)}/GE) \cong \mathrm{Gal}(K^{(i)}/E)$. Let $D^{(i)} = (L^{(i)}/GE, \sigma_i, t_i)$. It follows from the canonical isomorphism $\mathrm{TBr}(E) \overset{\alpha}{\cong} \mathrm{GBr}(GE)$ that there is a unique—up to an algebra isomorphism—central division algebra $D_{(i)}$ over E such that $D^{(i)} \cong_g GD_{(i)}$. By [6, Corollary 5.8], $K^{(i)}$ splits $D_{(i)}$. Hence, up to an isomorphism, $K^{(i)}$ is a maximal subfield of $D_{(i)}$ (see [9, Corollary, page 241]). Write $D_{(i)} = (K^{(i)}/E, \sigma_i, s_i)$. Then, $(L^{(i)}/GE, \sigma_i, t_i) = D^{(i)} \cong_g GD_{(i)} = (GK^{(i)}/GE, \sigma_i, \widetilde{s}_i) \cong_g (L^{(i)}/GE, \sigma_i, \widetilde{s}_i)$. Accordingly, $(L^{(i)}/GE, \sigma_i, t_i) \otimes_{GE} H\,\mathrm{Frac}(GE) \cong (L^{(i)}/GE, \sigma_i, \widetilde{s}_i) \otimes_{GE} H\,\mathrm{Frac}(GE)$ or, equivalently, $(H\,\mathrm{Frac}(L^{(i)})/H\,\mathrm{Frac}(GE), \sigma_i, t_i) \cong (H\,\mathrm{Frac}(L^{(i)})/H\,\mathrm{Frac}(GE), \sigma_i, \widetilde{s}_i)$. Therefore, by [9, Lemma, page 278], there is $\alpha_i \in H\,\mathrm{Frac}(L^{(i)})$ such that $t_i = (\prod_{j=1}^{n_i} \sigma_i^j(\alpha_i))\widetilde{s}_i$. Let v be the canonical valuation of $H\,\mathrm{Frac}(L^{(i)})$. Since v is Henselian, then $\mathrm{gr}(t_i)\ (= v(t_i)) = n_i v(\alpha_i) + v(s_i)$. So, $(n/n_i)\,\mathrm{gr}(t_i) + n\Gamma_E = (n/n_i)v(s_i) + n\Gamma_E$ (since $\Gamma_{H\,\mathrm{Frac}(L^{(i)})} = \Gamma_E$). We have $[D] = \alpha^{-1}([GD])$, therefore $D \cong (K^{(1)}/E, \sigma_1, s_1) \otimes_E \cdots \otimes_E (K^{(k)}/E, \sigma_k, s_k)$. Hence, by [7, Example 4.3], D is NSR. □

Theorem 2.4. *Let E be a Henselian valued field and D a defectless CDA over E. Then the following statements are equivalent:*

(1) *D is NSR;*
(2) *D has inertial and totally ramified maximal subfields;*
(3) *D is split by inertial and by totally ramified field extensions of E.*

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Obvious.

(3)⇒(1). Since D is split by an inertial field extension of E, then D is tame over E (see [6, page 99]). Hence, GD is a GCDA over GE. Now, let K (resp., K') be an inertial (resp., a totally ramified) field extension of E splitting D. Then GK (resp., GK') is an unramified (resp., a totally ramified) graded field extension of GE that splits GD. So, by Lemmas 2.2 and 2.3, D is NSR. □

References

[1] M. Boulagouaz, *The graded and tame extensions*, Commutative Ring Theory (Fès, 1992), Lecture Notes in Pure and Appl. Math., vol. 153, Marcel Dekker, New York, 1994, pp. 27–40.

[2] ———, *Le gradué d'une algèbre à division valuée* [*The associated graded ring of a valued division algebra*], Comm. Algebra **23** (1995), no. 11, 4275–4300 (French).

[3] ———, *Algèbre à division graduée centrale* [*Central graded division algebra*], Comm. Algebra **26** (1998), no. 9, 2933–2947 (French).

[4] O. Endler, *Valuation Theory*, Springer, New York, 1972.

[5] Y.-S. Hwang and A. R. Wadsworth, *Algebraic extensions of graded and valued fields*, Comm. Algebra **27** (1999), no. 2, 821–840.

[6] ———, *Correspondences between valued division algebras and graded division algebras*, J. Algebra **220** (1999), no. 1, 73–114.

[7] B. Jacob and A. R. Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), no. 1, 126–179.

[8] P. Morandi, *The Henselization of a valued division algebra*, J. Algebra **122** (1989), no. 1, 232–243.

[9] R. S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics, vol. 88, Springer, New York, 1982.

[10] V. P. Platonov, *The Tannaka-Artin problem, and reduced K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 2, 227–261, 469, translated in Math. USSR Izv. **10** (1976), 211–243.

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